**ELECTRONIC SUPPLEMENTARY INFORMATION**

**Kinetic and thermodynamic determinants of trace metal partitioning at biointerphases: the role of intracellular speciation dynamics**

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**N.B.** If not explicitly indicated in this document, meanings of symbols are those defined in the glossary reported at the end of the main text. Here-listed references are given at the end of the document.

**I. 1. Detailed derivations of eqns (19), (27).**

**Derivation of eqn (19).** After differentiation of eqn (10) with respect to time and combination with eqns (10)-(11), we obtain

\[
\frac{dJ_u(t)}{dt} - \frac{dJ_M(t)}{dt} = k_c \left[ (1 + \lambda)J_M(t) - \lambda J_u(t) + k_d^r \phi_u^T(t) \right],
\]

where we used the definition \( \phi_u^T(t) = \phi_u^M(t) + \phi_u^C(t) \) for the total concentration of intracellular metal species (free and complexed) at \( t \). Substituting the expression of \( J_M(t) \) given by eqn (8) and further introducing the mass transfer resistant \( R_T = 1/\left(D_{M,\text{out}}J_{\text{el}}a^{-1}\right) \), eqn (S1) leads to

\[
\frac{dJ_u(t)}{dt} + k_c \left( \lambda J_u(t) - k_d^r \phi_u^T(t) \right) - R_T \left[ \frac{dc_M^*(t)}{dt} + k_c(1 + \lambda)c_M^*(t) - \beta_a^{-1} \left( \frac{dc_M^a(t)}{dt} + k_c(1 + \lambda)c_M^a(t) \right) \right] = 0.
\]

(S2)

In addition, after combining eqn (13) with eqn (11), it comes

\[
\int_a^t \xi^2 \left[ c_M(\xi, t) - c_M(\xi, 0) \right] d\xi = -a^2 \left[ \frac{\lambda}{1 + \lambda} \int_0^t J_u(\nu) d\nu + \frac{1}{1 + \lambda} \left( \phi_u^M(t) - \phi_u^M(0) \right) - \frac{k_d^r}{1 + \lambda} \int_0^t \phi_u^T(\nu) d\nu \right].
\]

(S3)

After substitution of eqn (17) into eqn (S3) and subsequent derivation with respect to time, we further obtain
\[
\dot{\xi} = \frac{1}{a} \left( \frac{dc_M(\xi, t)}{dt} + k_e (1+\lambda) [c_M(\xi, t) - c_M(\xi, 0)] \right) \quad \text{d}\xi = a^2 \left\{ k_e \Phi_u^{M,0} - J_u(t) - \int_0^t \lambda J_u(\nu) - k_d \Phi_u^T(\nu) \, d\nu \right\}
\]

(S4)

Using eqn (16), eqn (S4) provides

\[
\frac{J_u(t)}{J_u^*} - \Omega_1 \left( \frac{dc_M^*(t)}{dt} + k_e (1+\lambda) c_M^*(t) \right) - 2\beta_u^{-1} \Omega_2 \left( \frac{dc_M^*(t)}{dt} + k_e (1+\lambda) c_M^*(t) \right) +
\]

\[
\frac{k_e}{J_u^*} \int_0^t \lambda J_u(\nu) - k_d \Phi_u^T(\nu) \, d\nu + k_e \tau_o = 0
\]

(S5)

, where \( \tau_o \) depends on the initial conditions as specified by eqn (20).

Eliminating the term \( \frac{dc_M^*(t)}{dt} + k_e (1+\lambda) c_M^*(t) \) between eqn (S2) and eqn (S5), we finally obtain

\[
\Delta \tau \frac{dJ_u(t)}{dt} + \int_0^t \left( k_e \Delta \tau - 1 \right) - \frac{k_e k_d^* \Delta \tau \Phi_u^T(t) - \tau_L}{K_M} \left( \frac{dc_M^*(t)}{dt} + k_e (1+\lambda) c_M^*(t) \right) -
\]

\[
\frac{k_e}{J_u^*} \int_0^t \lambda J_u(\nu) - k_d \Phi_u^T(\nu) \, d\nu - k_e \tau_o = 0
\]

(S6)

, where we used the relationship \( \tau_L = -K_M \beta_u^{-1} (\Omega_1 + 2\Omega_2)^{-1} \) and \( \Delta \tau = \tau_L - \tau_E = J_u^* \Omega_1 R_T \) derived from eqn (22) in the main text. Introducing the dimensionless variable \( \Sigma(t) = \int_0^t \left( J_u(\nu) / J_u^* \right) d\nu \) and \( \Phi^T(t) = \int_0^t \Phi_u^T(\nu) \, d\nu \) defined in the main text, eqn (S6) can be rewritten in the form

\[
\Delta \tau \Sigma_{tt} + \int_0^t \left( k_e \Delta \tau - 1 \right) - k_d^* \Delta \tau \Phi^T_t - \tau_L \frac{\Sigma_t}{\left( 1-\Sigma_t \right)^2} - k_e \lambda \Sigma(t) + k_d^* \Phi^T(t) - k_e \tau_o = 0
\]

(S7)

, where we used the relationships \( J_u(t) / J_u^* = \Sigma_t \), \( c_M^*(t) = K_M \Sigma_t / (1-\Sigma_t) \) and \( \frac{dc_M^*(t)}{dt} = \frac{K_M \Sigma_{tt}}{(1-\Sigma_t)^2} \).

After rearrangements, eqn (S7) finally becomes

\[
\Sigma_{tt} = \frac{(1-\Sigma_t)^2 \left[ k_e \left( \tau_o + \lambda \Sigma(t) \right) - k_d \left( \Phi^T(t) - \Phi^T_0 \Delta \tau \right) \right] + \Sigma_t (1-\Sigma_t) \left[ (1-\Sigma_t)(1-\lambda k_e \Delta \tau) + k_e \tau_L (1+\lambda) \right]}{\Delta \tau (1-\Sigma_t)^2 - \tau_L}
\]

(S8)

, which is eqn (19) in the main text.
**Derivation of eqn (27).** Rewriting eqn (S2) in terms of $\Sigma(t) = \int_0^t \left( \frac{J_u(t)}{J_u^*} \right) d\nu$ and $\Phi^T(t) = \int_0^t \Phi^T_u(\nu) d\nu$, we obtain after arrangements

$$\frac{d\Sigma^*}{dt} = -k_c (1 + \lambda) \Sigma^* + \left[ \Sigma_{it} \left( Bn^{-1} + \frac{1}{(1 - \Sigma_t)^2} \right) + k_c \Sigma_l \left( \lambda Bn^{-1} + \frac{1 + \lambda}{1 - \Sigma_t} - Bn^{-1} k^*_M \Phi^T \right) \right] / x_o$$

(S9)

, where we used the relationship $R_{TJ^*} = \beta_a^{-1} Bn^{-1} K_M$ recalling that $Bn^{-1} = R_T / R_S$, $J_u^* = K_H k_{int} K_M$ and $R_S = 1 / (k_{int} K_H \beta_a)$. Equation (S9) identifies with eqn (27) in the main text.

**I. 2. Demonstration of eqns (23), (24).**

**Demonstration of eqn (23).** Under the electrostatic conditions detailed in the main text (\(\psi(a \leq r \leq r_o) \equiv <\psi>\) and \(\psi(r_o \leq r \leq r_c) = 0\)), the time constant \(\tau_L\) defined by eqn (21) reduces to\(^1\)\(^2\)

$$\tau_L = \frac{1}{S_o K_H k_{int}} \left( V_{soft} + \frac{V_p}{\beta_a \phi} \right).$$

(S10)

Further introducing the critical volume fraction \(\phi^* = k_c V_p / (S_o K_H k_{int} \beta_a)\) in eqn (S10), we obtain the dimensionless form of \(\tau_L\) provided by

$$k_c \tau_L = \phi^* \left( V + \phi^{-1} \right)$$

(S11)

, which is eqn (23) in the main text with \(V = \beta_a V_{soft} / V_p\).

**Demonstration of eqn (24).** The time constant \(\tau_E\) is provided by the expression\(^1\)

$$\tau_E = \left[ \frac{V_{soft} + V_p / (\phi \beta_a)}{k_{int} K_H a V_p / (\varepsilon \phi D_{M,out})} \left( \varepsilon \alpha (1 - \gamma_o) + (1 - \alpha) / \beta_a - \gamma_a \phi / 2 \right) \right] / (S_o k_{int} K_H)$$

(S12)

with \(\alpha = a / r_o, \gamma_o = r_o / r_c\) and \(\gamma_a = a / r_c\). Equations (S11) and (S12) hold at sufficiently low volume fractions \(\phi\) and for constant electrostatic potential inside the soft surface layer of the microorganism and zero potential at the electrolyte side of the interphase. Using eqn (S10), eqn (S12) can be rewritten

$$\tau_E = \tau_L \left[ 1 + Bn^{-1} \frac{\bar{J}_{el}}{(1 + \bar{V} \phi)} \left( \alpha (1 - \gamma_o) + (\varepsilon \beta_a)^{-1} (1 - \alpha) - \gamma_a / 2 \right) \right].$$

(S13)

Realizing that \(r_c = r_o \phi^{-1/3}\) and \(\gamma_a = \alpha \phi^{1/3}\), it comes

$$\tau_E = \tau_L \left[ 1 + Bn^{-1} \frac{\bar{J}_{el}}{(1 + \bar{V} \phi)} \left( \varepsilon \beta_a \right)^{-1} \left[ 1 - a / r_o \right] + a / r_o \left( 1 - 3 / 2 \phi^{1/3} \right) \right].$$

(S14)
In addition, with help of eqn (9) it can be shown that under the electrostatic conditions of interest in this work \( f_{cl} \) reduces to \( f_{cl} = \left( (\varepsilon \beta_a)^{-1} \left(1 - \frac{a}{r_o} + \frac{a}{r_o} (1 - \varphi^{1/3}) \right) \right)^{-1} \). Substituting into eqn (S14) and expanding the result for \( \varphi \ll 1 \) that holds for dilute suspensions of microorganisms (case treated in the main text), we obtain eqn (24) with \( \zeta \approx 1 - \frac{\varphi^{1/3}/2}{1 + (\varepsilon \beta_a)^{-1} (r_o / a - 1)} \). The value of \( \zeta \) is close to unity as the term \( 1 + (\varepsilon \beta_a)^{-1} (r_o / a - 1) \) satisfies \( 1 + (\varepsilon \beta_a)^{-1} (r_o / a - 1) \geq 1 \). Accordingly, for the sake of mathematical simplification the \( \varphi \)-dependent term \( \frac{\varphi^{1/3}/2}{1 + (\varepsilon \beta_a)^{-1} (r_o / a - 1)} \) involved in \( \zeta \) was discarded for the derivation of eqns (40) and (41) in the main text.

II. 1. Demonstration of eqns (29)-(31) and detailed expressions of the ratio \( \frac{c_{M}^{a,\infty} (K^*)}{c_{M}^{a,\infty} (K^* = 0)} \).

**Demonstration of eqns (29), (30).** In the equilibrium regime reached at \( t \to \infty \), eqn (25) provides \( \Sigma_l (\infty) - \Phi_l^T (\infty) + \omega_{u}^\infty (\infty) = 0 \) while eqn (26) yields \( \Phi_l^T (\infty) = (1 + K^{* -1}) \omega_{u}^\infty (\infty) \). Combination of these two equations further leads to

\[
\Phi_l^T (\infty) = (1 + K) \Sigma_l (\infty) .
\]   \( \text{(S15)} \)

Substituting eqn (S15) into eqn (27) in the limit \( t \to \infty \) where \( \Sigma_l (\infty) = 0 \) and \( d\bar{c}_{M}^{a,\infty} (\infty) / dt = 0 \), it comes after rearrangements

\[
c_{M}^{a,\infty} = \beta_a c_{M}^{a,\infty} 
\]   \( \text{(S16)} \)

, where we used the relationships \( \lambda k_c = k_{a,\rho_S} \frac{V_i}{k_d} \left( 1 + K^{* -1} \right) = k_{d} \left( 1 + K^* \right) \) derived from eqn (18) and from \( K^* = k_{a,\rho_S} V_i / k_d \), and the equality \( c_{M}^{a,\infty} = K_{M} \Sigma_l (\infty) / (1 - \Sigma_l (\infty)) \). As argued in the text, eqn (S16) legitimates the systematic applicability of the (thermodynamic) BLM formalism at \( t \to \infty \). Further substitution of eqn (S15) into eqn (19) taken at \( t \to \infty \) yields after simplification

\[
\left[ 1 - \Sigma_l (\infty) \right] \left[ k_c \Sigma_o + k_c \lambda \Sigma (\infty) - k_d \Phi_l^T (\infty) \right] \Sigma_l (\infty) + k_c \Sigma_o \left[ 1 - \Sigma_l (\infty) + k_c \lambda l (1 + \lambda) \right] = 0 .
\]   \( \text{(S17)} \)
In the following, we express $\Sigma(\infty)$ and $\Phi^T(\infty)$ as a function of $c^*_{M,\infty}$, recalling that surface and bulk M concentrations are interrelated via eqn (S16) in the here-examined equilibrium regime. The integration of eqn (11) between $t = 0$ and $t \rightarrow \infty$ leads to

$$\phi_u^M(\infty) - \phi_u^{M,0} = J_u^* \Sigma(\infty) - \left(k_c + k_d^* \rho_S \right) x_M^\infty + k_d^* x_c^\infty$$  \hspace{1cm} (S18)

where we introduced $x_M^\infty = \int_0^\infty \phi_u^M(t)dt$ and $x_c^\infty = \int_0^\infty \phi_u^c(t)dt$. Similarly, the integration of eqn (12) between $t = 0$ and $t \rightarrow \infty$ provides

$$\phi_u^c(\infty) - \phi_u^{c,0} = k_d^* \rho_S x_M^\infty - k_d^* x_c^\infty.$$  \hspace{1cm} (S19)

Solving eqn (S18)-(S19) in $x_M^\infty$, we obtain

$$\begin{cases} x_c^\infty = -\frac{\phi_u^M(\infty) - \phi_u^{M,0} - J_u^* \Sigma(\infty) + \left(1 + \frac{1}{\lambda} \right) \left(\phi_u^c(\infty) - \phi_u^{c,0}\right)}{k_c K^{*{-1}}} \\ x_M^\infty = -\frac{\phi_u^M(\infty) - \phi_u^{M,0} + \phi_u^c(\infty) - \phi_u^{c,0} - J_u^* \Sigma(\infty)}{k_c} \end{cases}.$$  \hspace{1cm} (S20a,b)

Using eqn (S20), it is straightforward to show that $\Phi^T(\infty) = k_c \left(x_M^\infty + x_c^\infty\right) / J_u^*$ is defined after reduction by

$$\Phi^T(\infty) = -\left(1 + K^*\right) \left[-\Sigma(\infty) + \frac{\phi_u^M(\infty) - \phi_u^{M,0} + \left(\phi_u^c(\infty) - \phi_u^{c,0}\right) \left(1 + \frac{1}{\lambda}\right)}{J_u^*}\right].$$  \hspace{1cm} (S21)

Equation (12) at equilibrium further yields $\overline{\phi_u^c} (\infty) / \overline{\phi_u^M} (\infty) = \overline{K}^*$ while eqn (10) in the equilibrium limit leads to $\phi_u^M(\infty) = J_u(\infty) / k_c = J_u^* \Sigma_l(\infty) / k_c$. Substitution of these expressions into eqn (S21) gives

$$\Phi^T(\infty) = \left(1 + K^*\right) \alpha_o + \left(1 + K^*\right) \Sigma_l(\infty) - k_c^{-1} \Sigma_l(\infty) \left[1 + K^* \left(1 + \frac{1}{\lambda}\right)\right]$$  \hspace{1cm} (S22)

where we introduced the constant $\alpha_o = \left[\phi_u^{M,0} + \phi_u^{c,0} \left(1 + \frac{1}{\lambda}\right)\right] / J_u^*$. From eqn (S22), we infer

$$\left(1 + K^*\right) \Sigma_l(\infty) - \Phi^T(\infty) = -\left(1 + K^*\right) \alpha_o + \left(1 + K^*\right) k_c^{-1} \Sigma_l(\infty) \left[1 + K^* \left(1 + \frac{1}{\lambda}\right)\right].$$  \hspace{1cm} (S23)

Realizing that the term $k_c \Sigma(\infty) - k_d^* \Phi^T(\infty)$ in eqn (S17) can be rewritten in the form

$$k_d^* \left[\left(1 + K^*\right) \Sigma(\infty) - \Phi^T(\infty)\right],$$  \hspace{1cm} it comes after combining eqn (S23) and eqn (S17)

$$\alpha_o \Sigma_l(\infty) + B \Sigma_l(\infty) + C = 0.$$  \hspace{1cm} (S24)
where we wrote $\Sigma_{t,\infty} = \Sigma_t(\infty)$ for shortening notations, and the coefficients $A$, $B$ and $C$ are here defined by

$$A = -\left[1 + \lambda + K^*(1 + \lambda)\right], \quad B = 1 + \lambda + K^*(1 + \lambda) + k_e \tau_L (1 + \lambda) - k_e (\tau_o - \lambda \alpha_o)$$

and

$$C = k_e (\tau_o - \lambda \alpha_o).$$

The second-order polynomial equation (S24) in $\Sigma_{t,\infty}$ can be transformed into a second-order polynomial expression in terms of the variable $c_{M,\infty}^a / K_M$ with proper use of the equivalence $c_{M,\infty}^a / K_M = \Sigma_{t,\infty} / (1 - \Sigma_{t,\infty})$. After algebraic arrangements, the final result reads as

$$\left(\frac{c_{M,\infty}^a}{K_M}\right)^2 + \left[1 + \frac{1 + K^*}{k_e \tau_L} + \frac{\tau_o - \lambda \alpha_o}{\tau_L (1 + \lambda)}\right] \left(\frac{c_{M,\infty}^a}{K_M}\right) + \frac{\tau_o - \lambda \alpha_o}{\tau_L (1 + \lambda)} = 0. \quad (S25)$$

As $\tau_o - \lambda \alpha_o$ is always strictly negative, the discriminant of the quadratic eqn (S25) is positive. After calculation, it is demonstrated that the physically-relevant solution $c_{M,\infty}^a / K_M$ of eqn (S25) is then expressed by

$$c_{M,\infty}^a / K_M = \frac{1}{2} \left\{-\left(1 + \frac{\mu_o}{\tau_L} + \frac{1 + K^*}{k_e \tau_L}\right) + \frac{1 + K^*}{k_e \tau_L} \left[1 + \frac{2 k_e \tau_L}{1 + K^*} \left(1 + \frac{\mu_o}{\tau_L}\right) + \frac{k_e \tau_L}{1 + K^*} \left(1 - \frac{\mu_o}{\tau_L}\right)\right]^2\right\}^{1/2} \quad (S26)$$

with $\mu_o = (\tau_o - \lambda \alpha_o) / (1 + \lambda)$ $(< 0)$ and $\tau_o$ defined by eqn (20). Equation (29) in the main text directly follows from eqn (S26) using the equilibrium expression (S16). The time constant $-\mu_o$ in eqn (S26) simplifies after some developments into

$$k_e \mu_o = k_e \left(\Omega_1 c_{M,0}^{*,0} + 2\beta_a^{-1} \Omega_2 c_{M,0}^{a,0}\right) - \bar{\phi}_u^{T,0}. \quad (S27)$$

Using the relationship $\tau_L = -K_M \beta_a^{-1} (\Omega_1 + 2\Omega_2),^1$ it comes

$$k_e \mu_o = \frac{k_e \Delta \tau}{J_u^* R_T} c_{M,0}^{*,0} \Delta \tau_o - \frac{k_e \tau_L}{K_M} c_{M,0}^{a,0} - \bar{\phi}_u^{T,0} \quad (S28)$$

Further using the equality $R_T J_u^* = \beta_a^{-1} B n^{-1} K_M$ and the definition $x_o = \left[K_M / (\beta_a c_{M,0}^{*,0})\right]^{-1}$, we finally obtain

$$k_e \mu_o = k_e \tau_L \left[\left(1 - \tau_E / \tau_L\right) \Delta \tau_o x_o B n - c_{M,0}^{a,0} / K_M\right] - \bar{\phi}_u^{T,0} \quad (S29)$$

which is eqn (30) in the main text.

**Demonstration of eqn (31).** Starting from eqn (29), the ratio $c_{M}^{*,\infty} (K^*) / c_{M}^{*,\infty} (K^* = 0)$ can be formulated according to

S6
Under the peculiar condition $1 + k_c\mu_0 + k_e\tau_L = 0$, eqn (S30) simplifies into

$$
\frac{c^*_M}{c^*_M(K^* = 0)} = \frac{-\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right] - K^* + \left(1 + K^*\right) \left[1 + \frac{2k_c\tau_L (1 + \mu_0 / \tau_L)}{1 + K^*} + \left(\frac{1 + 2k_c\mu_0}{1 + K^*}\right)^2\right]^{1/2}}{2\left[\frac{4k^2_c\mu_0\tau_L}{\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right]^2}\right]^{1/2}}
$$

(S31)

where we introduced $\sigma = \mp 1$ for $1 + k_c\tau_L (1 + \mu_0 / \tau_L) < 0$. The complex stability constant $K^*_{1/2}$ introduced in §II.5 corresponds to the value taken by $K^*$ such that the equation $c^*_M(K^* = K^*_{1/2}) = c^*_M(K^* = 0) / 2$ is verified. For situations where $1 + k_c\mu_0 + k_e\tau_L = 0$, this equation can be solved with use of eqn (S31) and, after developments, we obtain the following solution

$$
K^*_{1/2} = \frac{3k_c^2\tau_L\mu_0}{\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right]}\left[1 + \frac{4k^2_c\mu_0\tau_L}{\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right]^2}\right]^{1/2} - \frac{1 + k_c\tau_L (1 + \mu_0 / \tau_L)}{2}
$$

(S33)

which is eqn (31a). For situations where $1 + k_c\mu_0 + k_e\tau_L \neq 0$, using eqn (S32) the equality $c^*_M(K^* = K^*_{1/2}) = c^*_M(K^* = 0) / 2$ leads after lengthy algebra and simplifications to the result

$$
K^*_{1/2} = \frac{3k_c^2\tau_L\mu_0}{\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right]}\left[1 + \frac{4k^2_c\mu_0\tau_L}{\left[1 + k_c\tau_L (1 + \mu_0 / \tau_L)\right]^2}\right]^{1/2} - \frac{1 + k_c\tau_L (1 + \mu_0 / \tau_L)}{2}
$$

(S34)

which identifies with eqn (31b).
II. 2. Physical interpretation of the term \( \frac{(1 + K^*)}{(k_c \tau_L)} \).

As indicated in the main text (§II.5), eqns (11)-(12) considered at \( t \to \infty \) provide
\[
\frac{\bar{\phi}_u^T(\infty)}{\bar{\phi}_u^M(\infty)} = K^* \quad \text{and} \quad \frac{\bar{\phi}_u^T(\infty)}{\bar{\phi}_u^T(\infty)} = (1 + K^*) \beta_a c_{M}^{*\infty} / \left( K_M + \beta_a c_{M}^{*\infty} \right).
\]

The latter expression can be rearranged in the form
\[
\bar{\phi}_u^T(\infty) = k_c \left[ V_{\text{soft}} + (\beta_a \varphi)^{-1} V_p \right] / K_H k_{\text{int}} S_a \quad \text{so that} \quad K_H k_{\text{int}} \quad \text{can be expressed as a function of} \quad \tau_L \quad \text{according to} \quad K_H k_{\text{int}} = \left[ V_{\text{soft}} + (\beta_a \varphi)^{-1} V_p \right] / (S_a \tau_L). \]

In turn, substituting the latter expression in the above equation defining \( \phi_u^T(\infty) \), we obtain
\[
\frac{1 + K^*}{k_c \tau_L} = \frac{S_a c_p \bar{\phi}_u^T(\infty)}{c_M^{*\infty}} \times \frac{1 + \beta_a c_{M}^{*\infty} / K_M}{1 + \bar{\varphi}} \quad \text{(S35)}
\]

where we further used the relationships \( c_p = \varphi / V_p \) and \( J_u^* = k_{\text{int}} K_H K_M \). The quantity \( S_a c_p \phi_u^T(\infty) \) in eqn (S35) corresponds to the amount of (free and complexed) metal forms accumulated at \( t \to \infty \) in the overall (intracellular) volume occupied by the microorganisms present at a cell number density \( c_p \). The term \( \left( 1 + \bar{\varphi} \right)^{-1} \approx 1 - \bar{\varphi} \) corrects \( c_p \) determined on the basis of the volume \( V_p \) of a microorganism for the presence of a soft peripheral corona where metals are not accumulated after internalisation. The term \( 1 + \beta_a c_{M}^{*\infty} / K_M \) accounts for the finite number of internalisation sites at the membrane surface with the limits \( \beta_a c_{M}^{*\infty} / K_M < 1 \) and \( \beta_a c_{M}^{*\infty} / K_M > 1 \) corresponding to the Henry and saturation adsorption regimes, respectively. In turn, the quantity \( \left( 1 + K^* \right) / (k_c \tau_L) \) is analogous to a dimensionless thermodynamic constant of an equilibrium reaction \( (t \to \infty) \) pertaining to the conversion of bulk free metal ions into internalized free and complexed metal forms in the overall microorganism suspension. In the (unrealistic) situation where \( \bar{\varphi} \to \infty \), which applies for microorganisms with very thick surface layer compared to the typical dimension of their cytoplasm, we have \( \left( 1 + K^* \right) / (k_c \tau_L) \to 0 \) meaning that the accumulation of metals is thermodynamically unfavorable, which is physically consistent. In addition, increasing \( K^* \) at fixed \( k_c \tau_L \) displaces the reaction toward the formation of ML complexes. Conversely, increasing \( k_c \tau_L \) at fixed \( K^* \) favors the excretion of internalized free metal forms and thus leads to a
decrease in $\phi_u^T(\infty)$. In the saturation regime $\beta_a c_{M}^{*,\infty} / K_M >> 1$, we have $\frac{1+\bar{K}^*}{k_c \tau_L} = \frac{\beta_a S_a c_{P}^T(\infty)}{K_M (1+\bar{V} \varphi)}$ whereas the result $\frac{1+\bar{K}^*}{k_c \tau_L} = \frac{S_a c_{P}^T(\infty)}{c_{M}^{*,\infty} (1+\bar{V} \varphi)}$ applies in the Henry regime $\beta_a c_{M}^{*,\infty} / K_M << 1$.

**II. 3. Formal demonstration of the relationship** $c_{M}^{*,\infty} (\bar{K}^* >> 1) / c_{M}^{*,\infty} (\bar{K}^* = 0) \to 0$.

After simplification of eqn (S2) taken in the limit $t \to \infty$ under conditions allowing for a neglect of the intracellular MLS complex dissociation step (condition satisfied for $\bar{K}^* >> 1$), we obtain

$$\frac{\bar{\lambda}}{1+\bar{\lambda}} R_T J_{u}^* \frac{c_{M}^{a,\infty}}{K_M + c_{M}^{a,\infty}} - \left( c_{M}^{a,\infty} - \beta_a^{-1} c_{M}^{a,\infty} \right) = 0 \quad (S36)$$

, where we used eqn (7). After some algebra, eqn (S36) can be transformed in the polynomial equation

$$\left( x^{a,\infty} \right)^2 + x^{a,\infty} \left( 1 - x^{*,\infty} + \frac{\bar{\lambda}}{1+\bar{\lambda}} Bn^{-1} \right) - x^{*,\infty} = 0 \quad (S37)$$

, where we defined the dimensionless $x^{a,\infty} = c_{M}^{a,\infty} / K_M$ and $x^{*,\infty} = \beta_a c_{M}^{*,\infty} / K_M$. The equilibrium eqn (S37) (valid for $\bar{K}^* >> 1$) must be satisfied irrespective of the value taken by $Bn^{-1}$ and $\bar{\lambda}$ and it must further be consistent with the thermodynamic Boltzmann relationship $x^{a,\infty} = x^{*,\infty}$ (or $c_{M}^{a,\infty} = \beta_a c_{M}^{*,\infty}$) that necessarily applies at equilibrium (see main text, §5). The only solution satisfying these conditions is $x^{a,\infty} = x^{*,\infty} = 0$, which demonstrates that bulk metal concentration at $t \to \infty$ and at $\bar{K}^* >> 1$ is 0, i.e. $c_{M}^{*,\infty} (\bar{K}^* >> 1) / c_{M}^{*,\infty} (\bar{K}^* = 0) \to 0$.

**II. 4. Detailed asymptotic behavior of eqn (29) for $\bar{K}^* << 1$ and $\bar{K}^* >> 1$, and details on the behavior of eqn (31) at $k_c \tau_L << 1$ and $k_c \tau_L >> 1$ (Table S1).**

The Taylor series expansions of eqns (S31)-(S32) with respect to $\bar{K}^*$ and $1/\bar{K}^*$ are useful for apprehending the behavior of $c_{M}^{*,\infty} (\bar{K}^*) / c_{M}^{*,\infty} (\bar{K}^* = 0)$ at $\bar{K}^* << 1$ and $\bar{K}^* >> 1$, respectively. After calculations with help of Mathcad software (version 15, PTC), we obtain the results collected in Table S1 and written in reduced forms. The expressions of $c_{M}^{*,\infty} (\bar{K}^*) / c_{M}^{*,0}$ (defined by eqn (29)) in the limits $\bar{K}^* << 1$ and $\bar{K}^* >> 1$ are simply obtained from the results given in Table S1 using the relation $c_{M}^{*,\infty} (\bar{K}^*) / c_{M}^{*,0} = c_{M}^{*,\infty} (\bar{K}^*) / c_{M}^{*,\infty} (\bar{K}^* = 0) \times c_{M}^{*,\infty} (\bar{K}^* = 0) / c_{M}^{*,0}$ where $c_{M}^{*,\infty} (\bar{K}^* = 0) / c_{M}^{*,0}$ is provided by
\[
\frac{c_M^*}{c_M^*} (\overline{K}^* = 0) = \frac{1}{2 \chi_0} \left[ - \left( 1 + \frac{\mu_0}{\tau_L} + \frac{1}{k_e \tau_L} \right) + \frac{1}{1 + 2 k_e \tau_L \left( 1 + \frac{\mu_0}{\tau_L} \right)} \left[ k_e \tau_L \left( 1 - \frac{\mu_0}{\tau_L} \right) \right]^2 \right]^{1/2}
\]  

(S38)

, which is derived from eqn (29) taken at \( \overline{K}^* = 0 \).

Similarly, the limiting behavior of \( K_{1/2}^* \) at \( k_e \tau_L \ll 1 \) and \( k_e \tau_L \gg 1 \) is described by the Taylor series expansions provided in Table S1 where we discriminate between cases \( k_e \mu_0 > -1, k_e \mu_0 = -1 \) and \( k_e \mu_0 < -1 \). The limiting expressions given in Table S1 corresponds to the curves denoted as (a) and (b) in Figures 2A-2B.

\[
\begin{align*}
\overline{K}^* &\ll 1: \\
&\begin{cases}
\text{for } 1 + k_e \mu_0 + k_e \tau_L = 0: & \frac{c_M^*}{c_M^*} (\overline{K}^* = 0) \approx \frac{1}{2 k_e \mu_0 (1 + k_e \mu_0)} \\
\text{for } 1 + k_e \mu_0 + k_e \tau_L = 0: & \frac{c_M^*}{c_M^*} (\overline{K}^* = 0) \approx \frac{\mu_0 (1 + k_e \mu_0)}{k_e \tau_L (1 + \mu_0 / \tau_L)} \left[ 1 + \frac{4 k_e^2 \mu_0 \tau_L}{(1 + k_e \tau_L (1 + \mu_0 / \tau_L))^2} \right]^{1/2}
\end{cases}
\end{align*}
\]

\[
\overline{K}^* \gg 1: \\
\begin{cases}
\text{for } 1 + k_e \mu_0 + k_e \tau_L \neq 0: & \frac{c_M^*}{c_M^*} (\overline{K}^* = 0) = \frac{2 k_e \mu_0 \tau_L}{(1 + k_e \tau_L (1 + \mu_0 / \tau_L))^2} \left[ 1 + \frac{4 k_e^2 \mu_0 \tau_L}{(1 + k_e \tau_L (1 + \mu_0 / \tau_L))^2} \right]^{-1/2}
\end{cases}
\]

Table S1. Taylor series expansions for the ratio \( \frac{c_M^*}{c_M^*} (\overline{K}^*) / c_M^* (\overline{K}^* = 0) \) at \( \overline{K}^* \ll 1 \) and \( \overline{K}^* \gg 1 \) (indicated) and for \( K_{1/2}^* \) in the limits \( k_e \tau_L \ll 1 \) and \( k_e \tau_L \gg 1 \) (indicated). Results are valid up to first order terms in \( \overline{K}^* \) or \( 1/\overline{K}^* \), \( k_e \tau_L \) or \( 1/(k_e \tau_L) \).

II. 5. Evolution of the ratio \( \frac{c_M^*}{c_M^*} (\overline{K}^*) / c_M^* (\overline{K}^* = 0) \) with \( \overline{K}^* \) at different \( k_e \tau_L \) (Figure S1).
**Figure S1.** Dependence of the ratio $c_{M}^{\ast \infty} / c_{M}^{\ast \infty} (K^{\ast} = 0)$ on $K^{\ast} / K_{1/2}^{\ast}$ at various values of $k_{e} \tau_{L}$ (indicated). Simulations are given for $k_{e} \mu_{0} = -5/2$.

**III. 1. Derivation of eqn (32) valid for strong ML₅ complexes ($K^{\ast} \gg 1$).**

For strong intracellular ML₅ complexes, eqn (27) becomes after rearrangements

$$\frac{dc_{M}^{\ast}}{dt} = -k_{e} (1 + \lambda) c_{M}^{\ast} (t) +$$

$$+ \beta_{a}^{-1} \left\{ \frac{dc_{M}^{a}}{dt} \left[ 1 + \frac{Bn^{-1}}{(1 + c_{M}^{a} (t) / K_{M})^{2}} \right] + k_{e} (1 + \lambda) c_{M}^{a} (t) \left[ 1 + \frac{\lambda}{1 + \lambda \frac{Bn^{-1}}{1 + c_{M}^{a} (t) / K_{M}}} \right] \right\}$$

(S39)

, where we used $\Sigma_{t} = c_{M}^{a} (t) / (K_{M} + c_{M}^{a} (t))$. The general solution of eqn (S39) can be written in the form

$$c_{M}^{\ast} (t) = C_{1} e^{-k_{e} (1 + \lambda)t} + C_{2} (t) e^{-k_{e} (1 + \lambda)t}$$

(S40)

, with $C_{1}$ a scalar independent of time and $C_{2} (t)$ is a function of time defined by

$$\frac{dC_{2} (t)}{dt} = K_{M} \beta_{a}^{-1} e^{k_{e} (1 + \lambda)t} \left\{ \frac{dy(t)}{dt} \left[ 1 + \frac{Bn^{-1}}{(1 + y(t))^{2}} \right] + k_{e} (1 + \lambda) y(t) \left[ 1 + \frac{\lambda}{1 + \lambda \frac{Bn^{-1}}{1 + y(t)}} \right] \right\}$$

(S41)
, where we introduced the normalized surface metal concentration \( y(t) = \frac{c_M^a(t)}{K_M} \). Integration of eqn (S41) provides after calculation

\[
C_2(t) = K_M \beta_a^{-1} e^{k_c (1+\lambda) t} \left\{ y(t) - Bn^{-1} \left[ \frac{1}{1+y(t)} - k_c (1+\lambda) e^{k_c (1+\lambda) t} - \int_0^t e^{k_c (1+\lambda) \nu} \left( \frac{1+\lambda}{1+\lambda y(\nu)} \right) d\nu \right] \right\} 
\]

(S42)

The constant \( C_1 \) is obtained from the initial boundary condition \( C_1 = c_M^{*,0} - C_2(0) \) with \( C_2(0) \) defined from eqn (S42) according to \( C_2(0) = K_M \beta_a^{-1} \left\{ y(0) - Bn^{-1} \left[ \frac{1}{1+y(0)} \right] \right\} \). The steady state transport condition given by eqn (10) and considered at \( t = 0 \) further provides

\[
c_M^{*,0} - K_M \beta_a^{-1} y(0) = J_u^* R_T \frac{y(0)}{1+y(0)} - k_c R_T \phi_u^{M,0} 
\]

(S43)

, where we used eqns (7)-(8) at \( t = 0 \). In turn, it comes \( C_1 = K_M \beta_a^{-1} Bn^{-1} - k_c R_T \phi_u^{M,0} \) after realizing that the product \( J_u^* R_T \) can be rewritten \( K_M \beta_a^{-1} Bn^{-1} \). After rearrangements, \( C_1 \) reduces to

\[
c_1 / c_M^{*,0} = x_o^{-1} Bn^{-1} \left( 1 - \phi_u^{M,0} \right) \text{ where we recall that } x_o = \left[ K_M / \left( \beta_a c_M^{*,0} \right) \right]^{-1}. 
\]

Finally, we obtain

\[
c_M^*(t) / c_M^{*,0} = x_o^{-1} Bn^{-1} \left( 1 - \phi_u^{M,0} \right) e^{-k_c (1+\lambda) t} + \]

\[
x_o^{-1} \left\{ \frac{c_M^a(t)}{K_M} - Bn^{-1} \left[ \frac{1}{1+c_M^a(t) / K_M} - k_c (1+\lambda) e^{-k_c (1+\lambda) t} - \int_0^t e^{k_c (1+\lambda) \nu} \left( \frac{1+\lambda}{1+\lambda c_M^a(\nu) / K_M} \right) d\nu \right] \right\} 
\]

(S44)

, which is eqn (32) in the main text.

### III. 2. Derivation of the expressions for the surface and bulk metal concentrations for \( K^* \gg 1 \) in the regime of strong and weak affinity of \( M \) for the internalisation sites (eqns (33)-(34) and eqns (35)-(37), respectively). Simplifications of eqns (37)-(38) for situations where \( k_o = k_+ = k_- \) and \( k_- = k_c (1+\lambda) \).

**Regime of strong M affinity for the internalisation sites in the limit \( K^* \gg 1 \) (eqns (33)-(34)).**

In this regime, we have \( K_M << c_M^a(t) \) at any time \( t \) so that \( J_u(t) = J_u^* \). Simplifying eqn (S6) accordingly and discarding the MLs dissociation terms not relevant in the \( K^* \gg 1 \) limit, it comes

\[
\frac{d c_M^a(t)}{dt} + k_c (1+\lambda) c_M^a(t) = -\frac{K_M}{\tau_L} \left[ 1 + k_c \lambda t + k_c (\tau_o - \lambda \Delta \tau) \right] 
\]

(S45)
After simple algebra, the integration of eqn (S45) leads to

\[ c^a_M(t) / K_M = \left[ \frac{c^{a,0}_M}{K_M} + \frac{1 + k_e (1 + \lambda) (\tau_o - \lambda \Delta \tau)}{k_e \tau_L (1 + \lambda)^2} \right] e^{-k_e (1+\lambda)t} - \frac{1 + k_e (1 + \lambda) (\tau_o + \lambda (t - \Delta \tau))}{k_e \tau_L (1 + \lambda)^2} \]  

(S46)

which is eqn (33) in the main text. To derive \( c^*_M(t) \), we now use eqn (S44) (or eqn (32)) in the limit \( J_u(t) = J^*_u \) or equivalently \( y(t) / (1 + y(t)) \approx 1 \) with \( y(t) = c^*_M(t) / K_M \). It comes after simplification

\[ c^*_M(t) = \left( K_M \beta_a^{-1} Bn^{-1} - k_c R_T \phi^M_u \right) e^{-k_e (1+\lambda)t} + \left[ \beta_a^{-1} c^*_M(t) + \frac{\lambda}{1 + \lambda} K_M \beta_a^{-1} Bn^{-1} (1 - e^{-k_e (1+\lambda)t}) \right] \]  

(S47)

In addition, from eqns (8) and (28) we easily derive that the initial M transport flux \( J_M(0) \) is given by

\[ J_M(0) / J^*_u = x_o Bn \left[ 1 + \frac{1 + Bn^{-1} (1 - \phi^M_u)}{x_o} \right] \left[ 1 - \frac{4x_o^{-1} Bn^{-1} \left[ 1 - \phi^M_u \right] (1 - x_o^{-1})}{1 + x_o^{-1} \left[ 1 + Bn^{-1} (1 - \phi^M_u) \right]^2} \right]^{-1/2} / 2 . \]  

(S48)

This expression identifies with that derived in Ref [1] in the limit \( \phi^M_u = 0 \) where the dimensionless metal surface affinity parameter and bioconversion capacity of the microorganism noted \( A \) and \( B \) in Ref [1], respectively, are defined here by \( A = x_o^{-1} \) and \( B = x_o^{-1} Bn^{-1} \). At \( t = 0 \), the condition \( J_u(0) = J^*_u \) applies and eqn (10) then simplifies into \( J_M(0) / J^*_u = 1 - \phi^M_u \). The latter expression is consistent with eqn (S48) provided that \( x_o^{-1} \ll 1 \) and \( x_o^{-1} Bn^{-1} (1 - \phi^M_u) \ll 1 \). Simplifying eqn (28) in these limits, we obtain \( c^a_M / (\beta_a c^*_M) = 1 \) or, equivalently, \( K_M \beta_a^{-1} Bn^{-1} - k_c R_T \phi^M_u = R_T J_M(0) = 0 \). In turn, eqn (S47) becomes

\[ \beta_a c^*_M(t) / K_M = \left[ c^a_M(t) / K_M + \frac{\lambda}{1 + \lambda} Bn^{-1} \left( 1 - e^{-k_e (1+\lambda)t} \right) \right] \]  

(S49)

which is eqn (34) in the main text.

**Regime of weak M affinity for the internalisation sites in the limit \( K^* \gg 1 \) (eqns (35)-(37)).**

In this regime where the condition \( K_M \gg c^a_M(t) \) applies, the uptake flux (eqn (7)) can be linearized according to \( J_u(t) / J^*_u = c^a_M(t) / K_M = y(t) \). Equation (S6) taken in the extreme \( K^* \gg 1 \) then reduces to

\[ \frac{dy(t)}{dt} + \frac{1 + k_e \tau_L (1 + \lambda) - k_e \lambda \Delta \tau}{\tau_E} y(t) + \frac{k_e \tau_o}{\tau_E} + \frac{k_e \lambda}{\tau_E} \int_0^t y(v) dv = 0 . \]  

(S50)
Equation (S50) leads to the second-order differential equation in
\[ \Sigma_t + \frac{1 + k_e \tau_L}{\tau_E} \Sigma_t + \frac{k_e}{\tau_E} \Sigma(t) = -\frac{k_e}{\tau_E} \cdot \frac{\lambda}{\tau_E}. \]  
(S51)

The characteristic polynomial equation associated with eqn (S51) admits the discriminant \( \Delta \) that reads

\[ \Delta \tau_E^2 = (1 + k_e \tau_L)^2 + 2k_e \tau_E (k_e \tau_L - 1) \lambda + \lambda^2 (k_e \tau_E)^2 \]  
(S52)

The quantity \( \Delta \tau_E^2 \) is always \( \geq 0 \) irrespective of the value of \( \lambda \), which can be shown after evaluation of the discriminant of the second order polynomial eqn (S52) in \( \lambda \). Let us first examined cases where \( \Delta \tau_E^2 > 0 \). Then, the general solution of eqn (S51) can be written in the form

\[ \Sigma(t) = \alpha_+ e^{-k_+ t} + \alpha_- e^{-k_- t} - \tau_0 / \lambda \]  
(S53)

, where \( \alpha_\pm \) are independent of time and \( k_\pm (\geq 0) \) (with \( k_- \neq k_+ \)) are the two kinetic constants defined by

\[ k_\pm \tau_E = -\left[1 + k_e (\tau_L + \lambda \tau_E)\right] \left\{-1 \pm \sqrt{1 - \frac{4 \lambda k_e \tau_E}{[1 + k_e (\tau_L + \lambda \tau_E)]^2}}\right\} / 2. \]  
(S54)

\( \alpha_\pm \) are determined from the boundaries \( \Sigma(0) = 0 \) and \( \Sigma_t(0) = y(0) \), which provides after evaluation

\[ \alpha_\pm = \mp \left[ \frac{c_M^{a,0} / K_M + k_\pm \tau_0 / \lambda}{k_+ - k_-} \right]. \]

Using the relationship \( y(t) = \Sigma_t \), we finally obtain

\[ c_M^a(t) / K_M = \sum_{j=+,--} A_j e^{-k_j t} \]  
(S55)

, with \( A_\pm = -\alpha_\pm k_\pm = \mp k_\pm \left[ \frac{c_M^{a,0} / K_M + k_\pm \tau_0 / \lambda}{k_+ - k_-} \right] \), which corresponds to eqn (35) in the main text. The expression for \( c_M^a(t) \) in the weak M affinity regime is obtained from eqn (32) after substitution therein of the linearized expressions \( (1 + y(t))^{-1} \approx 1 - y(t) \) and \( y(t) \left(1 + \frac{\lambda}{1 + \lambda} y(t)\right)^{-1} \approx 1 - y(t)/(1 + \lambda) \) with \( y(t) = c_M^a(t) / K_M \ll 1 \) defined by eqn (S55). After lengthy calculation and algebraic arrangements, we obtain the reduced expression

\[ \text{S14} \]
\[
\frac{\beta_a c^*_M(t)}{K_M} = \frac{Bn^{-1}}{1 + \lambda} \left[ \sum_{j=+, -, -} \left( \frac{A_j}{k_j} - \frac{(1 + \lambda)\phi_{M,0}^j}{k_c(1 + \lambda)} \right) e^{-k_c(1 + \lambda)t} + \sum_{j=+, -, -} \left[ \frac{\lambda Bn^{-1}}{1 + \lambda} \left( \frac{1}{k_j} - \frac{k_j}{k_c(1 + \lambda)} \right) \right] A_j e^{-k_j t} \right]
\]

(S56)

, where we used the relationship \( J_u^* R_T = K_M \beta_a^{-1} Bn^{-1} \). Equation (S56) is identical to eqn (37) given in the main text. Finally, the value of \( c_{M,0}^{a,0} \) is obtained from application of eqn (10) at \( t = 0 \) with \( J_u(0)/J_u^* = c_{M,0}^{a,0}/K_M \), *i.e.* \( R_T^{-1} \left( c_{M,0}^{a,0} - \beta_a^{-1} c_{M,0}^{a,0} \right) = J_u^* c_{M,0}^{a,0}/K_M - k_c \phi_{M,0}^{a,0} \) which leads to

\[
c_{M,0}^{a,0}/K_M = \left(1 + Bn^{-1}\right)^{-1} \left(x_o + \phi_{M,0}^{a,0} Bn^{-1}\right)
\]

as indicated in the main text.

*Analysis of the case* \( k_+ = k_- = k_o \).

Let us now address the degeneracy case where \( \Delta \tau_E^2 = 0 \). This situation is achieved for \( k_c \tau_L = 0 \) and \( \lambda k_c \tau_E = 1 \) with the result \( k_+ = k_- = k_o = \tau_E^{-1} \), which is inferred from eqn (36) in agreement with Figure 4. Then, the pendant of eqn (35) (or eqn (S55)) is \( c_M^a(t)/K_M = (A_o + B_o t)e^{-k_o t} \) with \( A_o = c_{M,0}^{a,0}/K_M \) and \( B_o = -k_o \left(c_{M,0}^{a,0}/K_M + k_o \tau_o/\lambda\right) \). In addition, it is can be shown after some algebra that the equivalent of eqn (37) in the limit \( k_+ = k_- = k_o = \tau_E^{-1} \) is provided by

\[
\frac{\beta_a c^*_M(t)}{K_M} = \frac{Bn^{-1}}{1 + \lambda} \left[ \frac{A_o}{k_c(1 + \lambda) - k_o} - \frac{B_o}{k_o} - \frac{(1 + \lambda)\phi_{M,0}^j}{k_c(1 + \lambda)} e^{-k_c(1 + \lambda)t} \right] + \left[ \frac{\lambda Bn^{-1}}{1 + \lambda} \left( \frac{1}{k_j} - \frac{k_j}{k_c(1 + \lambda)} \right) \right] A_o + B_o \left(1 + Bn^{-1}\right)t + \frac{k_c Bn^{-1}}{\left(k_c(1 + \lambda) - k_o\right)^2} \left(1 - \left[k_c(1 + \lambda) - k_o\right] t \right) e^{-k_o t} \quad \text{(S57)}
\]

In turn the expression of the transport flux \( J_M(t) \) for \( k_+ = k_- = k_o = \tau_E^{-1} \) is obtained from eqns (8) and (S57) together with \( c_M^a(t)/K_M = (A_o + B_o t)e^{-k_o t} \), which yields
\[ J_M(t) / J_u^* = \frac{1}{1 + \lambda} \left[ \frac{A_o - \frac{B_o}{k_e (1 + \lambda) - k_o}}{1 - \frac{k_o}{k_e (1 + \lambda)}} - (1 + \lambda) \bar{\phi}_u^{M,0} \right] e^{-k_e (1 + \lambda) t} + \right. \]
\[ \left. \left\{ \frac{\lambda A_o}{1 + \lambda} \left( \frac{1 - k_o}{k_e \lambda} \right) + \frac{B_o}{1 - \frac{k_o}{k_e (1 + \lambda)}} \right\} t e^{-k_e t} \right) \] (S58)

**Analysis of the case** \( k_- = k_e (1 + \lambda) \).

We now examine cases where \( k_- = k_e (1 + \lambda) \), and the conditions leading to such a situation are detailed below (in particular it is shown that \( k_+ \) can not take the value \( k_e (1 + \lambda) \)). Then, realizing that \( A_\lambda \left( e^{k_e (1 + \lambda) k_+} - 1 \right) / [k_e (1 + \lambda) - k_-] \approx A_\lambda t \), substitution of eqn (35) into eqn (32) now provides

\[ \frac{\beta_{eM}^* (t)}{K_M} = Bn^{-1} \left[ \frac{A_+}{1 - \frac{k_o}{k_e (1 + \lambda)}} - (1 + \lambda) \bar{\phi}_u^{M,0} + \left( \frac{1 + Bn^{-1}}{Bn^{-1} - k_e (1 + \lambda)} \right) \right] e^{-k_e (1 + \lambda) t} + \left[ \frac{\lambda Bn^{-1}}{1 + \lambda} \left( \frac{1 - k_+}{k_e (1 + \lambda)} \right) \right] A_\lambda e^{-k_e t} \] (S59)

, and the metal transport flux expression now reads as

\[ J_M(t) / J_u^* = \frac{1}{1 + \lambda} \left[ \left( \frac{A_+}{1 - \frac{k_o}{k_e (1 + \lambda)}} - (1 + \lambda) \bar{\phi}_u^{M,0} \right) e^{-k_e (1 + \lambda) t} + \frac{1 - k_+}{k_e (1 + \lambda)} \right] \lambda A_\lambda e^{-k_e t} \] (S60)

**Evaluation of the conditions leading to** \( k_- = k_e (1 + \lambda) \).

Let us first show that the condition \( k_+ = k_e (1 + \lambda) \) is never met. Finding the critical value \( \Lambda_c \) taken by \( \Lambda = \lambda k_e \tau_E \) such that \( k_+ = k_e (1 + \lambda) \) comes to solve the equation \( k_+ \tau_E = k_e \tau_E + \Lambda \) in \( \Lambda \) where \( k_+ \tau_E \) is defined by eqn (36). After calculation, we obtain \( \Lambda_c = k_e \tau_E (k_e \tau_E - k_e \tau_L)^{-1} (1 + k_e \tau_L - k_e \tau_E) \). We exclude here the case \( \tau_E = \tau_L \) (i.e. \( Bn^{-1} = 0 \)) for which the kinetic constant \( k_e (1 + \lambda) \) is not operational in the practical limit \( \bar{\phi}_u^{M,0} = 0 \) (ESI, part III.6). As \( k_+ \tau_E \leq 1 \) over the whole range of \( \Lambda \) (see Figure 3A), \( \Lambda_c \) must satisfy the condition \( k_e \tau_E + \Lambda_c \leq 1 \). In addition, the condition \( \Lambda_c > 0 \) further imposes that
\((k_e \tau_L <) k_e \tau_E < 1 + k_e \tau_L\). Substituting the above expression defining \(\Lambda_c\) into the inequality \(k_e \tau_E + \Lambda_c \leq 1\), we get \(1 \leq 1 - (k_e \tau_E)^{-1}\). This inequality is never verified because \(k_e \tau_E \geq 0\), so that there are no conditions leading to \(k_+ = k_e (1 + \lambda)\). We can now attempt finding the value of \(\Lambda_c\) such that \(k_- = k_e (1 + \lambda)\). Adopting the same procedure as above, we again obtain the expression \(\Lambda_c = k_e \tau_E (k_e \tau_E - k_e \tau_L)^{-1}(1 + k_e \tau_L - k_e \tau_E)\). However, \(\Lambda_c\) must now satisfy the condition \(k_e \tau_E + \Lambda_c \geq 1 + k_e \tau_L\) as \(k_- \tau_E \geq 1 + k_e \tau_L\) over the whole range of \(\Lambda = \lambda k_e \tau_E\) (see Figure 3B). Substituting this latter inequality into the expression of \(\Lambda_c\), we obtain \((k_e \tau_L <) k_e \tau_E < 1 + k_e \tau_L\), which is also consistent with the necessity to have positive values of \(\Lambda_c\). In turn, the relationship \((k_e \tau_L <) k_e \tau_E < 1 + k_e \tau_L\) is verified at the value \(\lambda = \lambda_c\) (specified below) provided that the \((k_e \tau_L, k_e \tau_E)\) couple is in line with the condition \((k_e \tau_L <) k_e \tau_E < 1 + k_e \tau_L\) : \(k_- \tau_E\) then reduces to \((1 - \tau_L / \tau_E)^{-1}\) and the critical value of \(\lambda_c = \Lambda_c / (k_e \tau_E)\) where \(k_- = k_e (1 + \lambda_c)\) is given by \(\lambda_c = \left[\frac{k_e (\tau_E - \tau_L)}{k_e (1 + \lambda_c)}\right]^{-1} - 1\).

**III. 3. Limits of eqns (35), (37) at \(\lambda \to 0\).**

It can be shown that \(A_s\) involved in eqn (35) and (37) satisfies the limit \(A_s|_{\lambda \to 0} \approx -k_o / (1 + k_e \tau_L)\) while \(A_s|_{\lambda \to 0} \approx c^a_{0,0}/K_M + k_o / (1 + k_e \tau_L)\), where we used the limits \(k_+|_{\lambda \to 0} \approx 0\), \(k_-|_{\lambda \to 0} \approx (1 + k_e \tau_L)/\tau_E\). Replacing these limits into eqn (35) we obtain

\[
\frac{c^a_M(t)}{K_M} = \left(\frac{c^a_M(0)}{K_M} + \frac{k_o \tau_o}{1 + k_e \tau_L}\right) e^{-t/\tau_d} - \frac{k_e \tau_o}{1 + k_e \tau_L}
\]

(S61)

where \(\tau_d = \tau_E / (1 + k_e \tau_L)\). Equation (S61) correctly corresponds to the result derived in Ref [2] for microorganisms in the absence of intracellular MLs formation and in the weak affinity limit. The following limits can be further shown

\[
\lambda \to 0 \quad \frac{1 + k_e \tau_L}{1 + k_e (\tau_E - \tau_L)} \approx \frac{1 + k_e \tau_L}{1 + k_e (\tau_L - \tau_E)}
\]

\[
\sum_{j=+,-} \left(\frac{A_j}{k_j} \frac{k_o}{1 + k_e (1 + \lambda)}\right) \approx -\frac{\tau_o + \tau_E c^a_M(0) / K_M}{1 + k_e (\tau_L - \tau_E)}
\]

so that substitution into eqn (37) leads after some arrangements to

\[
c^*_M(t) = -k_e e^{-k_e t} \left\{\frac{K_M}{\beta_a} \left[\frac{\tau_o + \tau_E c^a_M(0) / K_M}{1 + k_e (\tau_L - \tau_E)} + R_T \phi^M_{0,0}\right]\right\}
\]
\[
\frac{K_M}{\beta_a} e^{-t/\tau_d} \left( c_M^0 / K_M + \frac{k_e \tau_o}{1+k_e \tau_L} \right) \left[ 1 + \frac{Bn^{-1} (1+k_e \tau_L)}{1+k_e (\tau_L - \tau_E)} \right] \frac{K_M}{\beta_a} \frac{k_e \tau_o}{1+k_e \tau_L}.
\]

(S62)

This expression correctly compares with the result obtained in Ref [2] for cases where intracellular ML formation is not operational.

**III. 4. Evolution of** \( k_+ \tau_E \) **with** \( \lambda k_c \tau_E \) **at different** \( k_c \tau_L \) (Figure S2).

*Figure S2.* Dependence of the (dimensionless) kinetic constants \( k_+ \tau_E \) on \( \lambda k_c \tau_E = \Lambda \) at various values of \( k_c \tau_L \) (indicated). Dotted lines represent the evolution of \( k_+ \tau_E \) at low and large \( \lambda k_c \tau_E = \Lambda \) and the corresponding analytical Taylor-series expressions are indicated in Figure 3A.

**III. 5. Physical interpretation of the timescale** \( \tau_E = \left( k_a \rho_s^{V,1} \tau_E + k_c \tau_L \right) \), and comments on Table 1.

Using eqns (21)-(22), \( \tau_E \) can be rewritten in the form

\[
\tau_E = R_S \left\{ 4\pi \int_a^c \xi^2 \beta_\xi d\xi / S_a - J_u^* \Omega_1 Bn^{-1} \right\}.
\]

(S63)

In the following we define the Warburg-like element \( Z_W \) with value

\[
Z_W = 4\pi \int_a^c \xi^2 \beta_\xi d\xi / S_a - J_u^* \Omega_1 Bn^{-1}.
\]

(S64)
After algebraic rearrangements, \( Z_W \) can be expressed according to

\[
Z_W = (\tau_E / \tau_L) (1 + \bar{V} \varphi) V_p / (S_a \varphi)
\]

(S65)

, where we used eqns (21)-(22). Equation (S65) corresponds to that provided in the main text (see caption Table 1). In addition, the dimensionless product \( k_a \tau_L = 4\pi k_c R_S \int \frac{r_e}{a} \bar{\xi}^2 \beta \bar{\xi} d\bar{\xi} / S_a \) (eqn (21)) can be viewed as the ratio between the M membrane transfer resistance \( R_S \) and the excretion resistance \( R_{c,\varphi} = S_a \left( 4\pi k_c \int \frac{r_e}{a} \bar{\xi}^2 \beta \bar{\xi} d\bar{\xi} \right)^{-1} \). Under the electrostatic conditions specified in the main text (i.e. \( 1/\kappa << \delta \)), \( R_{c,\varphi} \) can be rewritten as \( R_{c,\varphi} = R_{c,\varphi} / (1 + \bar{V} \varphi) \) with \( R_c = S_a / (k_e V_p) \) and \( \bar{V} = \beta_a V_{\text{soft}} / V_p \). The timescale \( \tau_d = \tau_E / (k_a^* \rho_S^V \tau_E + k_c \tau_L) \) corresponds to \( (k_-)^{-1} \) in the limit \( k_a^* \rho_S^V >> (1 + k_c \tau_L) / \tau_E \) (see main text). It can be written in the following form

\[
\tilde{\tau}_d = R_S Z_W / \left( k_a^* \rho_S^V \tau_E + \frac{R_S}{R_{c,\varphi}} \right)
\]

(S66)

, or, equivalently,

\[
\tilde{\tau}_d = \frac{R_S}{k_a^* \rho_S^V \tau_E} \frac{R_{c,\varphi}}{R_{c,\varphi} + \frac{R_S}{k_a^* \rho_S^V \tau_E}} Z_W .
\]

(S67)

Equation (S67) is analogous to that defining the characteristic time for discharging a (Warburg-like) element with value \( Z_W \) across an interface with charge transfer resistance \( R_{ct} = \hat{R}_S R_{c,\varphi} / (R_{c,\varphi} + \hat{R}_S) \) with \( \hat{R}_S = R_S / k_a^* \rho_S^V \tau_E \). \( R_{ct} \) is nothing else than the equivalent resistance for the resistances \( \hat{R}_S \) and \( R_{c,\varphi} \) set in parallel, in agreement with the electrochemical circuit schemed in Table 1 (case \( (k_-)^{-1} \) for the limit \( k_a^* \rho_S^V >> (1 + k_c \tau_L) / \tau_E \)). Considering the inequality \( k_a^* \rho_S^V \tau_E >> (1 + k_c \tau_L) (> 1) \) defining the range of applicability of the limit \( (k_-)^{-1} \approx \tilde{\tau}_d \), it is easy to verify that \( \hat{R}_S / R_S << 1 \) and \( R_{c,\varphi} >> \hat{R}_S \) (see Table 1), the latter inequality stemming from \( k_a^* \rho_S^V \tau_E >> (1 + R_S / R_{c,\varphi}) > R_S / R_{c,\varphi} \). Stated differently, fast MLs formation kinetics leads to a decrease of the membrane transfer resistance \( R_S \) and microorganisms display a stronger propensity to accumulate M than excrete internalized metal ions. In
turn, fast ML$_S$ formation facilitates the intracellular sequestration of metal ions in the form of ML$_S$ complexes.

Following the above methodology, it is possible to evaluate expressions of charge transfer resistances associated to $1/k_+$. In the limit $k_a^*\rho_S^V << (1/2 + k_c\tau_L) / \tau_E$ (see Table 1), Figure 3A shows that $(k_+)^{-1} \approx \tau_E (1 + k_c\tau_L) / \Lambda$ with $\Lambda = \lambda k_c\tau_E$, or, equivalently, $(k_+)^{-1} \approx (k_a^*\rho_S^V)^{-1} (1 + k_c\tau_L)$. As the inequality $k_a^*\rho_S^V << (1/2 + k_c\tau_L) / \tau_E (< (1 + k_c\tau_L) / \tau_E)$ applies, we infer $(k_a^*\rho_S^V)^{-1} >> \tau_E (1 + k_c\tau_L)^{-1}$, i.e. $(k_+)^{-1} >> \tau_E$ (see Table 1). In addition, $(k_+)^{-1}$ can be written in the form

$$(k_+)^{-1} = R_S Z_W \frac{1 + k_c\tau_L}{k_a^*\rho_S^V \tau_E}$$

(S68)

, which corresponds to the time for discharging the Warburg-like element $Z_W$ across a charge transfer resistance defined here by $R_{ct} = R_S (1 + k_c\tau_L) / (k_a^*\rho_S^V \tau_E) >> R_S$ (see Table 1). This means that a sluggish ML$_S$ formation ($k_a^*\rho_S^V << (1/2 + k_c\tau_L) / \tau_E$) effectively leads to increasing the M membrane transfer resistance (blocking effect). In the extreme of fast intracellular ML$_S$ formation ($k_a^*\rho_S^V >> (1/2 + k_c\tau_L) / \tau_E (> (1 + k_c\tau_L) / \tau_E)$), we have $1/k_+ \approx \tau_E (1 - k_c\tau_L / \Lambda)^{-1}$ (see Figure 3A) or

$$(1/k_+) \approx R_S Z_W \left[ 1 - \left( k_a^*\rho_S^V \tau_E \right)^{-1} R_S / R_{e,\phi} \right]^{-1}$$

Using $k_a^*\rho_S^V \tau_E >> 1 + k_c\tau_L > k_c\tau_L$ (= $R_S / R_{e,\phi}$) that holds in the limit examined, it comes $1/k_+ \approx R_S Z_W \left[ 1 + \left( k_a^*\rho_S^V \tau_E \right)^{-1} k_c\tau_L \right]$, which corresponds to the time for discharging the Warburg-like element into a charge transfer resistance that identifies with

$R_S \left[ 1 + \left( k_a^*\rho_S^V \tau_E \right)^{-1} k_c\tau_L \right] \approx R_S$, in line with the result reported in Table 1. The absence of any excretion contribution in this charge transfer resistance agrees with the limit $c_M^{*,\infty} / c_M^{*,\infty} (\bar{K}^* = 0) \rightarrow 0$ discussed in §II.5 for $\bar{K}^* >> 1$ (Table 1 applies for such strong ML$_S$ complexes), recalling that excretion is required to obtain a finite non-zero M bulk concentration at equilibrium.$^2$
As stated in the main text, it is possible to evaluate from Table 1 which of the timescale \((k_+)^{-1}\) or \((k_-)^{-1}\) is operational in the regime of slow and fast intracellular \(ML_S\) formation (left and right column of Table 1, respectively). In the regime of slow \(ML_S\) formation, the timescale \((k_+)^{-1}\) is much larger than the free M transfer time \(\tau_E\) between bulk and intracellular volume (see Table 1 and Figure 4). Accordingly, it is the only \((k_-)^{-1}\) contribution that will be operational in defining kinetics of M bulk depletion and overall uptake. On the opposite, the membrane transfer resistance associated with \((k_-)^{-1}\) in the limit of fast \(ML_S\) formation significantly facilitates M uptake-depletion compared to that relevant for the \((k_+)^{-1}\) case. In line with this, the processes subsumed in the \((k_+)^{-1}\) component thus now play a key role in governing/limiting M accumulation and bulk depletion kinetics.

III. 6. Simplification of eqns (37)-(38) for \(\phi_{M,0} = 0\).

Obviously, in the limit of fast intracellular \(ML_S\) formation reached at \(\lambda \to \infty\) we have \(e^{-k_+(1+\lambda)t} \to 0\) in eqn (37) and the time constant \((1+\lambda)^{-1}/k_c\) is then not operational in determining M uptake/depletion kinetics. For \(\lambda \to 0\), we showed in §III.3 that eqn (37) reduces to

\[
\frac{\phi_{M,0}}{\tau_0} = -c_M^0/(1 + c_M^0/k_M) \left[ R_T \phi_{M,0} + \frac{K_M}{\beta_a} \frac{1}{(1 + k_c)} \right] + \frac{c_M^0(0)/K_M}{1 + k_c(\tau_L - \tau_E)} \left[ 1 + \frac{Bn^{-1}(1 + k_c)}{1 + k_c(\tau_L - \tau_E)} \right] \frac{K_M}{\beta_a} \frac{k_c}{k_c + 1}. \tag{S69}
\]

In addition, under the condition \(K_M >> c_M^0(t)\) marking the validity of eqn (S69), the time constant \(\tau_0\) defined by eqn (20) may be written for \(\lambda \to 0\) in the form \(\tau_0 = -c_M^0/\tau_E / K_M - \phi_{M,0}^0 \left(1/J_a^0 + k_e \Omega_1 R_T \right)\).

This equation is derived after combining the expression \(\tau_E = -K_M \beta^{-1}_a \left[ c_M^0(0)/K_M \right] \left[ \Omega_2 \left(1 + Bn^{-1}\right) + 2\Omega_2 \right] \left[ \Omega_1 \right] \) given in the Supporting Information of our previous work\(^1\) with the relation \(c_M^0(0)/K_M = (1 + Bn^{-1})^{-1} \left(x_0 + \phi_{M,0}^0 Bn^{-1}\right)\) valid for \(K_M >> c_M^0(t)\) (see ESI, part III.2). In turn, for \(\phi_{M,0}^0 = 0\), we obtain \(\tau_0 = -c_M^0/\tau_E / K_M\) and \(\frac{K_M}{\beta_a} Bn^{-1} \left[ \frac{\tau_0 + \tau_E c_M^0(0)/K_M}{1 + k_c(\tau_L - \tau_E)} \right] + R_T \phi_{M,0}^0 = 0\). The prefactor of the \(e^{-k_+t}\) term in eqn (S69) becomes zero so that the time constant \(1/k_c\) is not operational, as stated in the main text.
IV. 1. Time-dependence of M transport flux at the membrane surface under conditions of Figure 5 (Figure S3) and time-dependent ratio $c_M^* / c_M^{*0}$ under conditions of Figure 7 (Figure S4).

**Figure S3.** Evolution of the dimensionless ratio $J_M(t)/(J_u(t)+J_M(t))$ (A) and $J_{\text{kin}}(t)/(J_{\text{kin}}(t)+J_u(t))$ (B) with time $t$ normalized by the M transfer timescale $\tau_E$, at various values of $\lambda$ (indicated). $J_M$ and $J_u$ are defined in the main text and $J_{\text{kin}}(t) = \frac{d\phi_u^M(t)}{dt} = k_u^s \rho_{3u} \phi_u^M(t) - k_u^s \phi_u^C(t)$ corresponds to the net kinetic flux pertaining to intracellular MLs complex formation. Model parameters: as in Figure 5 of the main text. The dotted lines in (A) and (B) represent the limits $J_u = J_M$ and $J_{\text{kin}} = J_u$, respectively.
**Figure S4.** Evolution of the ratio $c_M^* / c_M^{0*}$ versus $t / \tau_E$ for various values of the dimensionless M reciprocal affinity $1 / x_0 = K_M / \left( \beta_0 c_M^{0*} \right)$ (indicated). Model parameters: as in Figure 7 of the main text.

**IV. 2. Expressions of $c_M^{\infty*} / c_M^{0*}$ and $k_+ / k_-$ as a function of $\varphi / \varphi^*$ and details on the corresponding asymptotic behavior at $\varphi / \varphi^* << 1$ and $\varphi / \varphi^* >> 1$ (Table S2). Derivation of eqns (39)-(41).

Equation (29) defines the ratio $c_M^{\infty*} / c_M^{0*}$ according to

$$
c_M^{\infty*} / c_M^{0*} = \frac{1}{2x_0} \left[ - \left( 1 + \frac{\mu_o}{\tau_L} + \frac{1 + \bar{K}^*}{k_c \tau_L} \right) + \frac{1 + \bar{K}^*}{k_c \tau_L} \left[ 1 + 2k_c \tau_L \left( 1 + \frac{\mu_o}{\tau_L} \right) + \left[ \frac{k_c \tau_L}{1 + \bar{K}^*} \left( 1 - \frac{\mu_o}{\tau_L} \right) \right]^2 \right] \right]^{1/2} \tag{S70}
$$

, with $k_c \mu_o = k_c \tau_L \left[ (1 - \tau_E / \tau_L) \Delta \bar{c}_o x_o Bn - c_M^{0*} / K_M \right] - \tilde{\phi}_u^{T,0}$ (eqn (30)) which we rewrite in the form $k_c \mu_o = -k_c \tau_L x_o \left[ 1 - \Delta \bar{c}_o \left[ 1 - \frac{\zeta}{(1 + \bar{V} \varphi)} \right] \right] - \tilde{\phi}_u^{T,0}$ where we used eqn (24) valid for $\varphi << 1$. In turn we obtain $\mu_o / \tau_L = -x_o \left[ 1 - \Delta \bar{c}_o \left[ 1 - \frac{\zeta}{(1 + \bar{V} \varphi)} \right] \right] - \tilde{\phi}_u^{T,0} / (k_c \tau_L)$. Further using eqn (23) that defines $k_c \tau_L$ as a function of $\varphi$, we finally derive $\mu_o / \tau_L = -x_o \left[ 1 - \Delta \bar{c}_o \left[ 1 - \frac{\zeta}{(1 + \bar{V} \varphi)} \right] \right] - \tilde{\phi}_u^{T,0} / \left[ \varphi^* \left( \varphi^{-1} + \bar{V} \right) \right]$.

Substitution of that latter expression and of $k_c \tau_L = \varphi^* \left( \varphi^{-1} + \bar{V} \right)$ (eqn (23)) into eqn (S70) provides the general relationship that determines the evolution of $c_M^{\infty*} / c_M^{0*}$ as a function of the microorganism volume.
fraction $\varphi$ in the suspension. Then, after solving the equation $c^*_{M,d}(\varphi_{1/2})/c^*_{M,0} = 1/2$, we obtain eqn (39) that defines the cell volume fraction $\varphi_{1/2}$ needed to achieve a two-fold reduction of the bulk M concentration at $t \to \infty$ under the condition $\Delta \varepsilon_0 = 0$. Using Mathcad software (version 15, PTC) and after simplifications, we further obtain the Taylor series expansions reported in Table S2 for $c^*_{M,d}/c^*_{M,0}$ at $\varphi/\varphi^* << 1$ and $\varphi/\varphi^* >> 1$. For the sake of simplicity, we derived these expressions in the limit where the term $F\varphi$ is so small compared to unity (which is generally the case in practice) that it can be discarded in the mathematical developments.

As detailed in the main text, the kinetic constants $k_\pm (\geq 0)$ given by eqn (36) are fully defined upon upon the only specification of $k_e \tau_L$ and $\lambda k_e \tau_E \left( = k^*_{a,E} \rho_S \tau_E \right)$ and the same holds for the ratio $k_+ / k_-$. The expressions of $k_\pm$ and $k_+ / k_-$ as a function of cell volume fraction $\varphi$ are simply obtained after substituting into eqn (36) the relationships defining the dependence of $k_e \tau_L$ and $\lambda k_e \tau_E$ on $\varphi$, i.e. $k_e \tau_L = \varphi^* \left( \varphi^{-1} + F \right)$ (which is eqn (23)) and $\lambda k_e \tau_E = \lambda \varphi^* \left( \varphi^{-1} + F \right) \left[ 1 + \zeta B n^{-1} / \left( 1 + F \varphi \right) \right]$. Using Mathcad software (version 15, PTC), the first derivative of the ratio $k_+ / k_-$ with respect to $\varphi$ can be easily computed. Then, the position and the value of the maximum in $k_+ / k_-$ can be determined from the value in $\varphi$ where this derivative is zero. After lengthy simplifications and neglect of the $\varphi$-dependent component of $\zeta$ (see justification in ESI, Part I.2), the expression defining $\left( \varphi/\varphi^* \right)_{\text{max}}$ where $k_+ / k_-$ is maximum can be written in the form given by eqn (40) and the value $\left( k_+ / k_- \right)_{\text{max}}$ can be arranged according to eqn (41). Finally, we derived the Taylor-series expansion of the ratio $k_+ / k_-$ at $\varphi/\varphi^* << 1$ and $\varphi/\varphi^* >> 1$ and after arrangements we obtain the results in reduced form collected in Table S2.
Table S2. Taylor series expansions for the ratio \( \frac{c_{\infty}^*}{c_{M}^*} \) at \( \varphi / \varphi^* \ll 1 \) and \( \varphi / \varphi^* \gg 1 \) (indicated, valid with discarding term in \( T_0^* \)) and for \( k_+ / k_- \) in the limits \( \varphi / \varphi^* \ll 1 \) and \( \varphi / \varphi^* \gg 1 \) (indicated). Results are valid up to first order terms in \( \varphi / \varphi^* \) or \( \varphi^* / \varphi \).

IV. 3. Dependence of \( \frac{c_{\infty}^*}{c_{M}^*} \) on \( \varphi / \varphi^* \) and \( \overline{K}^* \) at various values of \( \Delta T_0^* \) and \( 1/x_0 \) (Figure S5).
Figure S5. Evolution of the ratio $c_{M}^{*,\infty} / c_{M}^{*,0}$ with microorganism volume fraction (normalized by $\varphi^{*} = R_{S} / R_{c}$ with $R_{c} = S_{a} / (k_{e} V_p)$ and $R_{S} = 1 / (k_{int} K_H \beta_a)$) and with intracellular dimensionless ML$_S$ complex stability constant ($\bar{K}^*$). Model parameters: $\phi^* = 10^{-3}$, $\bar{V} = \bar{\tau}_{u}^{T,0} = 0$, $a / r_o = \epsilon = \beta_a = 1$ with (A): $\Delta \tau_o = 0$, $x_o = 1$, (B): $\Delta \tau_o = 0$, $x_o = 5 \times 10^2$, (C): $\Delta \tau_o = 0$, $x_o = 5 \times 10^3$, (D): $\Delta \tau_o = 1$, $x_o = 5 \times 10^2$.

References.