The effect of connectivity on information in neural networks

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Supporting information: clustering analysis of cell networks

Starting from graphs generated on squared grids of sides of 200 pixel having different number of nodes (25, 50, 100, 150, 200) and connection probability (from 0.1 to 1) combination, we imported the nodes coordinates matrix and the connectivity information (contained in the adjacency matrix), to quantify some network parameters including the clustering coefficient, the characteristic path length and then the ‘smallworldness’ coefficient. These parameters give an indication of the connectivity properties of the nodes in a graph and allow to distinguish between graphs of different types (regular, random or small world).

In graph theory, the clustering coefficient ($C_c$) is a measure of the degree to which nodes in a graph tend to cluster together. $C_c$ ranges from 0 (none of the possible connections among the nodes are realized) to 1 (all possible connections are realized and nodes group together to form a single aggregate).

The clustering coefficient is defined as

$$C_i = \frac{2E_i}{k(k-1)}$$

(1)
where \( k \) is the number of neighbors of a generic node \( i \), \( E^i \) is the number of existing connections between those, \( k(k - 1)/2 \) being the maximum number of connections, or combinations, that can exist among \( k \) nodes. Notice that the clustering coefficient \( C_i \) is defined locally, and a global value, \( C_c \), is derived upon averaging \( C_i \) over all the nodes that compose the graph.

\[
C_i = \frac{2 \cdot 0}{3 \cdot (2)} = 0 ; \quad C_i = \frac{2 \cdot 1}{3 \cdot (2)} = \frac{1}{3} ; \quad C_i = \frac{2 \cdot 3}{3 \cdot (2)} = 1.
\]

The characteristic path length (\( C_{pl} \)) is defined as the average number of steps along the shortest paths for all possible pairs of network nodes. The shortest path length (\( S_{pl} \)) between two nodes is the path that connects the two nodes with the shortest number of edges and it is the minimum distance between a generic couple of nodes.
Fig 2. In this example, between nodes 1 and 4, there are two shortest path of length 2: \{1,2,4\} and \{1,5,4\}. Longer paths between the two nodes are \{1, 2, 5, 4\}, \{1, 5, 2, 4\}, \{1, 2, 3, 5, 4\}, \{1, 5, 3, 2, 4\}. Shorter paths are desirable to enhance speed of communication.

We show now how to calculate the \( S_p^{l} \) for a couple of nodes \( n_l \) and \( n_m \) premising that, in the imported adjacency matrix \( A = a_{ij} \), the indices \( i \) and \( j \) run through the number of nodes \( n \) in the graph and \( a_{ij} = 1 \), if there exists a connection between \( i \) and \( j \), \( a_{ij} = 0 \) otherwise. In the analysis, reciprocity between nodes is assumed, and thus if information can flow from \( i \) to \( j \), it can reversely flow from \( j \) to \( i \). Notice that this property translates into symmetry of \( A \) being \( a_{ij} = a_{ji} \). Moreover, \( a_{ij} = 0 \).

In \( A \), \( a_{li} \) and \( a_{lm} \) account for all the pairs of nodes which are connected to \( n_l \) and \( n_m \) respectively. The sum of \( a_{li} \) and \( a_{lm} \) over all the nodes in \( A \), is stored in a new matrix \( A_2 = \sum a_{li}a_{lm} \) for all the \( l \) and \( m \) and \( A_2 \) has the same dimension of \( A \). Now multiplicate \( A_2 \) and \( A \) repeatedly \( A_2 = A_2A \), until all the terms of \( A_2 \) are non-zero and those terms in position \( ij \) will be the \( S_p^{l} \) between node \( i \) and node \( j \).

Finally, the characteristic path length \( C_p^{l} \) is calculated like the average of \( S_p^{l} \) over \( A_2 \).

Once obtained the \( C_c \) and \( C_p^{l} \) values, we defined a precise measure of ‘smallworldness’, the ‘smallworldness’ coefficient (SW), based on the trade off between high local clustering and short path length.

A network \( G \) with \( n \) nodes and \( m \) edges is a small-world network if it has a similar path length but greater clustering of nodes than an equivalent Erdos-Rényi (E–R) random graph with the same \( m \) and \( n \) (an E–R graph is constructed by uniquely assigning each edge to a node pair with uniform probability). Let \( C_p^{l}_u \) and \( C_c^u \) be the mean shortest path length and the mean clustering coefficient for the E–R random graphs, obtained meaning the \( C_p^{l} \) and the \( C_c \) of 20 uniform distributions, and \( C_p^{l}_{graph} \) and \( C_c_{graph} \) the corresponding quantities for the graphs derived using the methods described above. We can calculate:

\[
\gamma = \frac{C_c_{graph}}{C_c^u} \quad (4)
\]

\[
\lambda = \frac{C_p^{l}_{graph}}{C_p^{l}_u} \quad (5)
\]

Thus, the 'smallworldness' coefficient is
\[ SW = \frac{\gamma}{\lambda} \] (6)

The categorical definition of small-world network above implies \( \lambda \geq 1 \gamma >> 1 \), which, in turn, gives \( SW > 1 \).