

In order to integrate out the velocity degrees of freedom we define the relevant distribution functions

$$\begin{aligned}
\psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) &\equiv \int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t), \\
\psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) \bar{\mathbf{v}}_i(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) &\equiv \int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t) \mathbf{v}_i, \\
\psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) \bar{\boldsymbol{\omega}}_i(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) &\equiv \int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t) \boldsymbol{\omega}_i,
\end{aligned} \tag{S3}$$

such that the zeroth moment of Eq. (S2) reads

$$\partial_t \psi^{(N)} = - \sum_i \nabla_i \cdot (\psi^{(N)} \bar{\mathbf{v}}_i) - \sum_i \mathcal{R}_i \cdot (\psi^{(N)} \bar{\boldsymbol{\omega}}_i); \tag{S4}$$

its first moment in \mathbf{v}_i reads

$$\begin{aligned}
\partial_t (\psi^{(N)} \bar{\mathbf{v}}_i) &= - \sum_j \nabla_j \cdot \left(\int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)} \mathbf{v}_j \otimes \mathbf{v}_i \right) - \sum_j \mathcal{R}_j \cdot \left(\int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)} \boldsymbol{\omega}_j \otimes \mathbf{v}_i \right) \\
&\quad + \frac{\gamma_t}{m} \left(- \bar{\mathbf{v}}_i - \gamma_t^{-1} \nabla_i [V(\mathbf{r}_i, \hat{\mathbf{e}}_i) + U^{\text{eff}}(\mathbf{r}^N, \hat{\mathbf{e}}^N)] + v_0 \hat{\mathbf{e}}_i \right) \psi^{(N)};
\end{aligned} \tag{S5}$$

and its first moment in $\boldsymbol{\omega}_i$ reads

$$\begin{aligned}
\partial_t (\psi^{(N)} \bar{\boldsymbol{\omega}}_i) &= - \sum_j \nabla_j \cdot \left(\int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)} \mathbf{v}_j \otimes \boldsymbol{\omega}_i \right) - \sum_j \mathcal{R}_j \cdot \left(\int d\mathbf{v}^N d\boldsymbol{\omega}^N f^{(N)} \boldsymbol{\omega}_j \otimes \boldsymbol{\omega}_i \right) \\
&\quad + \frac{\gamma_r}{I} \left(- \bar{\boldsymbol{\omega}}_i - \gamma_r^{-1} \mathcal{R}_i [V(\mathbf{r}_i, \hat{\mathbf{e}}_i) + U^{\text{eff}}(\mathbf{r}^N, \hat{\mathbf{e}}^N)] \right) \psi^{(N)}.
\end{aligned} \tag{S6}$$

Here we used the notation $\mathbf{a} \otimes \mathbf{b}$ to denote the dyadic product of two vectors \mathbf{a} and \mathbf{b} . Next, we integrate out the remaining degrees of freedom of all but one particles. At this point we assume the effective interactions to be pairwise, i.e. $U^{\text{eff}}(\mathbf{r}^N, \hat{\mathbf{e}}^N) = \sum_{i < j} \phi_{\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j}^{\text{eff}}(\mathbf{r}_j - \mathbf{r}_i)$. This approximation is made purely for simplification purposes; the methods presented here can be extended to three- and higher-body interactions [3, 4]. We define the one-body distribution functions

$$\begin{aligned}
f(\mathbf{r}_1, \hat{\mathbf{e}}_1, \mathbf{v}_1, \boldsymbol{\omega}_1, t) &\equiv N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} d\mathbf{v}^{(2 \rightarrow N)} d\boldsymbol{\omega}^{(2 \rightarrow N)} f^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t), \\
\psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) &\equiv N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t), \\
\psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \bar{\mathbf{v}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) &\equiv N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) \bar{\mathbf{v}}_1(\mathbf{r}^N, \hat{\mathbf{e}}^N, t), \\
\psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \bar{\boldsymbol{\omega}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) &\equiv N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) \bar{\boldsymbol{\omega}}_1(\mathbf{r}^N, \hat{\mathbf{e}}^N, t),
\end{aligned} \tag{S7}$$

and the two-body distribution function

$$\psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) \equiv N(N-1) \int d\mathbf{r}^{(3 \rightarrow N)} d\hat{\mathbf{e}}^{(3 \rightarrow N)} \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t), \tag{S8}$$

where $\int d\mathbf{r}^{(n \rightarrow N)}$ denotes an integration over $\mathbf{r}_n, \mathbf{r}_{n+1}, \dots, \mathbf{r}_N$ (and similarly for $\int d\hat{\mathbf{e}}^{(n \rightarrow N)}$ etc). Integrating over all but one particles then yields for Eq. (S4)

$$\partial_t \psi = - \nabla_1 \cdot (\psi \bar{\mathbf{v}}) - \mathcal{R}_1 \cdot (\psi \bar{\boldsymbol{\omega}}); \tag{S9}$$

for Eq. (S5)

$$\begin{aligned} \partial_t(\psi\bar{\mathbf{v}}) = & -\nabla_1 \cdot \left(\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \mathbf{v}_1 \otimes \mathbf{v}_1 \right) - \mathcal{R}_1 \cdot \left(\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \boldsymbol{\omega}_1 \otimes \mathbf{v}_1 \right) \\ & + \frac{\gamma_t}{m} \left([-\bar{\mathbf{v}} - \gamma_t^{-1} \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) + v_0 \hat{\mathbf{e}}_1] \psi - \gamma_t^{-1} \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) \right); \end{aligned} \quad (\text{S10})$$

and for Eq. (S6)

$$\begin{aligned} \partial_t(\psi\bar{\boldsymbol{\omega}}) = & -\nabla_1 \cdot \left(\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \mathbf{v}_1 \otimes \boldsymbol{\omega}_1 \right) - \mathcal{R}_1 \cdot \left(\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 \right) \\ & + \frac{\gamma_r}{I} \left([-\bar{\boldsymbol{\omega}} - \gamma_r^{-1} \mathcal{R}_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1)] \psi - \gamma_r^{-1} \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \mathcal{R}_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) \right). \end{aligned} \quad (\text{S11})$$

To arrive at the evolution equation for the momentum density, we finally integrate over the orientations $\hat{\mathbf{e}}_1$. Upon defining

$$\begin{aligned} \rho(\mathbf{r}_1, t) & \equiv \int d\hat{\mathbf{e}}_1 \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t), \\ \mathbf{m}(\mathbf{r}_1, t) & \equiv \int d\hat{\mathbf{e}}_1 \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \hat{\mathbf{e}}_1, \\ \rho(\mathbf{r}_1, t) \bar{\mathbf{v}}(\mathbf{r}_1, t) & \equiv \int d\hat{\mathbf{e}}_1 \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \bar{\mathbf{v}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t), \end{aligned} \quad (\text{S12})$$

integrating over the orientations yields for Eq. (S9)

$$\partial_t \rho = -\nabla_1 \cdot (\rho \bar{\mathbf{v}}), \quad (\text{S13})$$

whereas it yields for Eq. (S10)

$$\begin{aligned} m \partial_t (\rho \bar{\mathbf{v}}) = & -\nabla_1 \cdot \left(m \int d\hat{\mathbf{e}}_1 d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \mathbf{v}_1 \otimes \mathbf{v}_1 \right) - \gamma_t \rho \bar{\mathbf{v}} \\ & - \int d\hat{\mathbf{e}}_1 \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) + \gamma_t v_0 \mathbf{m} + \int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t). \end{aligned} \quad (\text{S14})$$

Using

$$\int d\hat{\mathbf{e}}_1 d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \mathbf{v}_1 \otimes \mathbf{v}_1 = \int d\hat{\mathbf{e}}_1 d\mathbf{v}_1 d\boldsymbol{\omega}_1 f (\mathbf{v} - \bar{\mathbf{v}}) \otimes (\mathbf{v} - \bar{\mathbf{v}}) + \rho \bar{\mathbf{v}} \otimes \bar{\mathbf{v}}, \quad (\text{S15})$$

together with Eq. (S13), allows one to rewrite Eq. (S14) as

$$m \rho \frac{D\bar{\mathbf{v}}}{Dt} = -\nabla_1 \cdot \mathbf{P}(\mathbf{r}_1, t) - \gamma_t \rho \bar{\mathbf{v}} - \int d\hat{\mathbf{e}}_1 \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) + \gamma_t v_0 \mathbf{m}, \quad (\text{S16})$$

where we defined the material derivative $D/Dt \equiv \partial_t + \bar{\mathbf{v}} \cdot \nabla_1$, and where the pressure tensor

$$\begin{aligned} \mathbf{P}(\mathbf{r}_1, t) & \equiv m \int d\hat{\mathbf{e}}_1 d\mathbf{v}_1 d\boldsymbol{\omega}_1 f (\mathbf{v}_1 - \bar{\mathbf{v}}) \otimes (\mathbf{v}_1 - \bar{\mathbf{v}}) \\ & - \frac{1}{2} \int d\hat{\mathbf{e}}_1 d\mathbf{r}_{12} d\hat{\mathbf{e}}_2 \int_0^1 du \mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_{12}) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1 - u \mathbf{r}_{12}, \mathbf{r}_1 + (1-u) \mathbf{r}_{12}) \end{aligned} \quad (\text{S17})$$

is defined in terms of momentum flux (with respect to the mean velocity $\bar{\mathbf{v}}$), and of interaction forces, here in the Kirkwood-Irving form, as is standard for the definition of the local pressure tensor (or negative of the stress tensor) [4, 5].

Having properly defined the momentum balance (S16) for the underdamped dynamics, we are now ready to consider the overdamped limit. As explained below Eq. (S1), the separation of time scales implies that the velocities \mathbf{v}_i and $\boldsymbol{\omega}_i$ evolve much faster than the positions \mathbf{r}_i and orientations $\hat{\mathbf{e}}_i$, respectively. Motivated by this separation of time scales, we extend the approach of Enculescu and Stark [6] and make a local-equilibrium Maxwell-Boltzmann approximation² [3] for the N -body distribution function

$$f^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, \mathbf{v}^N, \boldsymbol{\omega}^N, t) \simeq \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) \left(\frac{mI\beta^2}{4\pi^2} \right)^{\frac{3N}{2}} \prod_i e^{-\frac{\beta m}{2}(\mathbf{v}_i - \bar{\mathbf{v}}_i(\mathbf{r}^N, \hat{\mathbf{e}}^N, t))^2 - \frac{\beta I}{2}(\boldsymbol{\omega}_i - \bar{\boldsymbol{\omega}}_i(\mathbf{r}^N, \hat{\mathbf{e}}^N, t))^2}, \quad (\text{S18})$$

consistent with Eq. (S3). We defined here $\beta \equiv (k_B T)^{-1}$.

The overdamped version of the Fokker-Planck equation (S2) is obtained by using Eq. (S18) in Eq. (S10) and (S11), and combining the result with Eq. (S9). Upon noting that

$$\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \mathbf{v}_1 \otimes \mathbf{v}_1 \stackrel{(\text{S18})}{=} \frac{k_B T}{m} \psi \mathbb{1} + N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1, \quad (\text{S19})$$

and

$$\int d\mathbf{v}_1 d\boldsymbol{\omega}_1 f \boldsymbol{\omega}_1 \otimes \mathbf{v}_1 \stackrel{(\text{S18})}{=} N \int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\boldsymbol{\omega}}_1 \otimes \bar{\mathbf{v}}_1, \quad (\text{S20})$$

Eq. (S10) becomes

$$\begin{aligned} m \left[\partial_t (\psi \bar{\mathbf{v}}) + N \nabla_1 \cdot \left(\int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1 \right) + N \mathcal{R}_1 \cdot \left(\int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\boldsymbol{\omega}}_1 \otimes \bar{\mathbf{v}}_1 \right) \right] \\ = [-\gamma_t \bar{\mathbf{v}} - \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) + \gamma_t v_0 \hat{\mathbf{e}}_1 - k_B T \nabla_1] \psi - \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t). \end{aligned} \quad (\text{S21})$$

Similarly, Eq. (S11) becomes

$$\begin{aligned} I \left[\partial_t (\psi \bar{\boldsymbol{\omega}}) + N \nabla_1 \cdot \left(\int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\mathbf{v}}_1 \otimes \bar{\boldsymbol{\omega}}_1 \right) + N \mathcal{R}_1 \cdot \left(\int d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}^{(2 \rightarrow N)} \psi^{(N)} \bar{\boldsymbol{\omega}}_1 \otimes \bar{\boldsymbol{\omega}}_1 \right) \right] \\ \stackrel{(\text{S18})}{=} [-\gamma_r \bar{\boldsymbol{\omega}} - \mathcal{R}_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) - k_B T \mathcal{R}_1] \psi - \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \mathcal{R}_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t). \end{aligned} \quad (\text{S22})$$

In the overdamped limit ($m, I \rightarrow 0$), the left-hand sides of Eq. (S21) and (S22) disappear, yielding

$$\begin{aligned} \psi \bar{\mathbf{v}} &= [-\gamma_t^{-1} \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) + v_0 \hat{\mathbf{e}}_1 - \gamma_t^{-1} k_B T \nabla_1] \psi - \gamma_t^{-1} \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t), \\ \psi \bar{\boldsymbol{\omega}} &= [-\gamma_r^{-1} \mathcal{R}_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) - \gamma_r^{-1} k_B T \mathcal{R}_1] \psi - \gamma_r^{-1} \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \mathcal{R}_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t). \end{aligned} \quad (\text{S23})$$

Together with Eq. (S9), Eq. (S23) forms the Smoluchowski equation for interacting overdamped ABPs. For the non-interacting case, upon defining $\mathbf{j}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \equiv \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \bar{\mathbf{v}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t)$ and $\mathbf{j}_{\hat{\mathbf{e}}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \equiv \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) \bar{\boldsymbol{\omega}}(\mathbf{r}_1, \hat{\mathbf{e}}_1, t)$, this Smoluchowski equation reduces to the non-interacting Smoluchowski equation (1) of the main text.

Finally, we find the overdamped version of the momentum balance (S16). In order to find the expression for the pressure tensor (S17) under the approximation (S18), we note that

$$m \int d\hat{\mathbf{e}}_1 d\mathbf{v}_1 d\boldsymbol{\omega}_1 f (\mathbf{v}_1 - \bar{\mathbf{v}}) \otimes (\mathbf{v}_1 - \bar{\mathbf{v}}) = \rho k_B T \mathbb{1} + mN \int d\hat{\mathbf{e}}_1 d\mathbf{r}^{(2 \rightarrow N)} d\hat{\mathbf{e}}_2^{2 \rightarrow N} \psi^{(N)}(\mathbf{r}^N, \hat{\mathbf{e}}^N, t) (\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}) \otimes (\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}),$$

² Alternatively, the separation of time scales can be exploited to explicitly solve Eq. (S2) for $f^{(N)}$ by means of a multiple time scale theory, see [7].

such that the the momentum balance (S16) in the overdamped limit ($m, I \rightarrow 0$) reads

$$0 = -\nabla_1 \cdot \mathbf{P}(\mathbf{r}_1, t) - \gamma_t \rho \bar{\mathbf{v}} - \int d\hat{\mathbf{e}}_1 \nabla_1 V(\mathbf{r}_1, \hat{\mathbf{e}}_1) \psi(\mathbf{r}_1, \hat{\mathbf{e}}_1, t) + \gamma_t v_0 \mathbf{m}, \quad (\text{S24})$$

with the overdamped pressure tensor given as

$$\mathbf{P}(\mathbf{r}_1, t) = \rho(\mathbf{r}_1, t) k_B T \mathbb{1} - \frac{1}{2} \int d\hat{\mathbf{e}}_1 d\mathbf{r}_{12} d\hat{\mathbf{e}}_2 \int_0^1 du \mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_{12}) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1 - u \mathbf{r}_{12}, \mathbf{r}_1 + (1-u) \mathbf{r}_{12}). \quad (\text{S25})$$

In the non-interacting case, and in the steady state of the main text, Eq. (S25) reduces to Eq. (3) of the main text, upon defining the particle current $\mathbf{J}(\mathbf{r}) \equiv \rho(\mathbf{r}) \bar{\mathbf{v}}(\mathbf{r})$, and the local pressure $P(\mathbf{r}) \equiv \frac{1}{3} \text{Tr}[\mathbf{P}(\mathbf{r})] = \rho(\mathbf{r}) k_B T$. This justifies the interpretation of Eq. (3) as a force balance.

2. DERIVATION OF THE SOLVENT FORCE BALANCE

In this section we derive that Eq. (5) of the main text is the force balance governing the solvent flow on a scale where the colloids can be regarded as a continuum. To this end, we start from the hydrodynamic problem that governs the solvent flow around a single swimmer, and coarse-grain this problem to the desired larger scale. As done throughout this Electronic Supplementary Information, Latin indices i, j, k shall label particles, whereas Greek indices α, β, γ shall refer to the Cartesian components x, y, z . We apply the Einstein summation convention *only* to the latter Greek indices, and only in this section. Furthermore, we use the notation $A_{(\alpha\beta)} \equiv \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$ to denote the symmetrization of a tensor \mathbf{A} with respect to its Greek indices *only*.

To describe the solvent flow around a single swimmer, we consider a model in which the swimming is generated by a nonzero slip-velocity at the surface of the single particle. This models for example biological swimmers - so-called ‘squirmers’ - that move by the beating motion of small flagella at their body surface, or by small body deformations [8], but also the swimming of active colloidal particles [9, 10]. The hydrodynamic problem is as follows. The swimmer/particle i , occupying a volume V_i enclosed by the surface S_i , is assumed to have a fixed overall shape, such that it can only undergo rigid body motion, with center-of-mass velocity \mathbf{v}_i and angular velocity $\boldsymbol{\omega}_i$ around its center-of-mass position \mathbf{r}_i . It swims in an ambient flow $\mathbf{u}^\infty(\mathbf{r})$ that is assumed to solve the Stokes equation for all \mathbf{r} in the absence of any particles. In the fluid region V_f , bounded by S_i and by a spherical surface S_∞ with radius R that we plan to take towards ∞ , the fluid velocity $\mathbf{u}^{\text{out}}(\mathbf{r})$ and pressure $p^{\text{out}}(\mathbf{r})$ satisfy

$$\left. \begin{aligned} \nabla \cdot \mathbf{u}^{\text{out}} &= 0 \\ -\nabla p^{\text{out}} + \eta \nabla^2 \mathbf{u}^{\text{out}} &= \mathbf{0} \end{aligned} \right\} \text{ with b.c.'s } \left\{ \begin{aligned} \mathbf{u}^{\text{out}}(\mathbf{r}) &= \mathbf{u}_i^{\text{RBM}}(\mathbf{r}) + \mathbf{u}_i^s(\mathbf{r}), & \text{for } \mathbf{r} \in S_i, \\ \mathbf{u}^{\text{out}}(\mathbf{r}) &= \mathbf{u}^\infty(\mathbf{r}), & \text{for } \mathbf{r} \in S_\infty, \end{aligned} \right. \quad (\text{S26})$$

where η is the dynamic solvent viscosity, where $\mathbf{u}_i^{\text{RBM}}(\mathbf{r}) \equiv \mathbf{v}_i + \boldsymbol{\omega}_i \times (\mathbf{r} - \mathbf{r}_i)$ is the surface velocity of particle i due to its rigid body motion, and where $\mathbf{u}_i^s(\mathbf{x})$ is the additional slip velocity satisfying $\mathbf{u}_i^s(\mathbf{x}) \cdot \hat{\mathbf{n}}_i = 0$, $\hat{\mathbf{n}}_i$ being the normal vector pointing from particle i into V_f . The stress tensor $\boldsymbol{\sigma}^{\text{out}}$ of the solvent is given as $\sigma_{\alpha\beta}^{\text{out}} = -p^{\text{out}} \delta_{\alpha\beta} + 2\eta \partial_{(\alpha} u_{\beta)}^{\text{out}}$; the second equation of (S26), known as the Stokes equation, can thus also be written as $\nabla \cdot \boldsymbol{\sigma}^{\text{out}} = \mathbf{0}$.

It is not Eq. (S26) that we shall coarse-grain, but an integral representation of these differential equations. In e.g. the book of Kim and Karilla [11], this integral representation is derived for a rigid, non-swimming particle, i.e. for $\mathbf{u}_i^s(\mathbf{r}) = \mathbf{0}$, a result that we extend to a finite swimming velocity $\mathbf{u}_i^s(\mathbf{r})$. It is important to realize that Eq. (S26) is an equation for the solvent velocity *outside* the particle, that we have called $\mathbf{u}^{\text{out}}(\mathbf{x})$ to emphasize this. Of course, no solvent is present inside the particle, yet it shall be convenient to formally consider an ‘extended’ solvent velocity profile defined on both V_f and V_i as

$$\mathbf{u}(\mathbf{r}) = \begin{cases} \mathbf{u}^{\text{out}}(\mathbf{r}), & \text{if } \mathbf{r} \in V_f, \\ \mathbf{u}_i^{\text{in}}(\mathbf{r}), & \text{if } \mathbf{r} \in V_i, \end{cases} \quad (\text{S27})$$

where $\mathbf{u}_i^{\text{in}}(\mathbf{r})$ is defined as the velocity field solving the Stokes equation *inside* particle i (i.e. in V_i) subject to the boundary condition

$$\mathbf{u}_i^{\text{in}}(\mathbf{r}) = \mathbf{u}_i^{\text{RBM}}(\mathbf{r}) \text{ for } \mathbf{r} \in S_i. \quad (\text{S28})$$

This velocity profile inside the particle is easily solved as $\mathbf{u}_i^{\text{in}}(\mathbf{r}) = \mathbf{v}_i + \boldsymbol{\omega}_i \times (\mathbf{r} - \mathbf{r}_i)$, meaning that $\mathbf{u}_i^{\text{in}}(\mathbf{r})$ has the same functional form as $\mathbf{u}_i^{\text{RBM}}(\mathbf{r})$, yet is defined on V_i rather than only on S_i . It is therefore clear that this $\mathbf{u}_i^{\text{in}}(\mathbf{r})$ indeed satisfies the boundary condition (S28); that this $\mathbf{u}_i^{\text{in}}(\mathbf{r})$ also solves the Stokes equations follows from $\nabla \cdot \mathbf{u}_i^{\text{in}}(\mathbf{r}) = 0$ and $\eta \partial_{(\alpha} u_{i,\beta)}^{\text{in}} = 0$, which implies that the corresponding stress tensor reads $\boldsymbol{\sigma}_i^{\text{in}}(\mathbf{r}) = -p_i^0 \mathbf{1}$ with a spatially constant pressure p_i^0 . In order to find an integral representation for the solvent velocity profile $\mathbf{u}(\mathbf{r})$, as defined by (S27), we follow the exact same procedure as done in [11], but for a nonzero $\mathbf{u}_i^{\text{s}}(\mathbf{r})$. The resulting integral representation for $\mathbf{u}(\mathbf{r})$ makes use of the Green's functions that correspond to equations (S26). These Green's functions, $\mathcal{G}_{\alpha\beta}$, \mathcal{P}_α , and $\Sigma_{\alpha\beta\gamma}$ are defined by

$$\begin{cases} \partial_\alpha \mathcal{G}_{\alpha\beta}(\mathbf{r}) = 0, \\ 8\pi\eta \partial_\gamma \Sigma_{\alpha\beta\gamma}(\mathbf{r}) = -\partial_\alpha \mathcal{P}_\beta(\mathbf{r}) + \eta \nabla^2 \mathcal{G}_{\alpha\beta}(\mathbf{r}) = -8\pi\eta \delta_{\alpha\beta} \delta(\mathbf{r}). \end{cases} \quad (\text{S29})$$

The integral formulation for the velocity field $\mathbf{u}(\mathbf{r})$ then reads

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^\infty(\mathbf{r}) - \frac{1}{8\pi\eta} \oint_{S_i} dS(\boldsymbol{\xi}) (\boldsymbol{\sigma}^{\text{out}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i) \cdot \mathcal{G}(\mathbf{r} - \boldsymbol{\xi}) - \oint_{S_i} dS(\boldsymbol{\xi}) \mathbf{u}_i^{\text{s}}(\boldsymbol{\xi}) \cdot \Sigma(\mathbf{r} - \boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i. \quad (\text{S30})$$

For $\mathbf{u}_i^{\text{s}}(\mathbf{r}) = \mathbf{0}$, Eq. (S30) shows that the solvent flow can be understood as resulting from a collection of force monopoles $-\boldsymbol{\sigma}^{\text{out}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i$ distributed over the surface of the particle. For nonzero $\mathbf{u}_i^{\text{s}}(\mathbf{r})$, the effect of the last term in (S30) is that an additional surface distribution of force *dipoles* comes into play. To see this, we use $\Sigma_{\alpha\beta\gamma} = (8\pi\eta)^{-1} (-\mathcal{P}_\beta \delta_{\alpha\gamma} + 2\eta \partial_{(\gamma} \mathcal{G}_{\alpha)\beta})$ to rewrite this term as

$$\begin{aligned} - \oint_{S_i} dS(\boldsymbol{\xi}) u_{i\alpha}^{\text{s}}(\boldsymbol{\xi}) \Sigma_{\alpha\beta\gamma}(\mathbf{r} - \boldsymbol{\xi}) \hat{n}_{i\gamma} &= \frac{1}{8\pi\eta} \oint_{S_i} dS(\boldsymbol{\xi}) \mathcal{P}_\beta(\mathbf{r} - \boldsymbol{\xi}) \mathbf{u}_i^{\text{s}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i - \frac{1}{4\pi} \oint_{S_i} dS(\boldsymbol{\xi}) \partial_{(\gamma} \mathcal{G}_{\alpha)\beta}(\mathbf{r} - \boldsymbol{\xi}) u_{i\alpha}^{\text{s}}(\boldsymbol{\xi}) \hat{n}_{i\gamma} \\ &= -\frac{1}{4\pi} \oint_{S_i} dS(\boldsymbol{\xi}) \lim_{\epsilon \downarrow 0} \left\{ [\mathcal{G}_{\alpha\beta}(\mathbf{r} - (\boldsymbol{\xi} - \epsilon \hat{\boldsymbol{\gamma}})) - \mathcal{G}_{\alpha\beta}(\mathbf{r} - \boldsymbol{\xi})] \frac{u_{i(\alpha}^{\text{s}}(\boldsymbol{\xi}) \hat{n}_{i\gamma)}(\boldsymbol{\xi})}{\epsilon} \right\}, \end{aligned} \quad (\text{S31})$$

where we used $\mathbf{u}_i^{\text{s}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i(\boldsymbol{\xi}) = 0$, wrote out the definition of the derivative ∂_γ , with $\hat{\boldsymbol{\gamma}}$ denoting the unit vector in the γ -direction, and used $A_{(\alpha\beta)} B_{\alpha\beta} = A_{(\alpha\beta)} B_{(\alpha\beta)} = A_{\alpha\beta} B_{\alpha\beta}$ for any tensors \mathbf{A} and \mathbf{B} . A summation over γ is implied in the last line of (S31), and will be in similar terms arising from this term. The combination of Eq. (S30) and (S31) shows that the solvent velocity profile $\mathbf{u}(\mathbf{r})$, as defined by Eq. (S27), satisfies the problem

$$\begin{aligned} \nabla \cdot \mathbf{u}(\mathbf{r}) &= 0, \\ -\partial_\alpha p(\mathbf{r}) + \eta \nabla^2 u_\alpha(\mathbf{r}) &= \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \sigma_{\alpha\beta}^{\text{out}}(\boldsymbol{\xi}) \hat{n}_{i\beta}(\boldsymbol{\xi}) \delta^3(\mathbf{r} - \boldsymbol{\xi}) \\ &\quad + 2\eta \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \lim_{\epsilon \downarrow 0} \left\{ \frac{u_{i(\alpha}^{\text{s}}(\boldsymbol{\xi}) \hat{n}_{i\gamma)}(\boldsymbol{\xi})}{\epsilon} [\delta^3(\mathbf{r} - (\boldsymbol{\xi} - \epsilon \hat{\boldsymbol{\gamma}})) - \delta^3(\mathbf{r} - \boldsymbol{\xi})] \right\}, \end{aligned} \quad (\text{S32})$$

where we now account for many possible particles i present, and where $\mathbf{u}(\mathbf{r})$ is subject to the boundary condition $\mathbf{u}(\mathbf{r}) = \mathbf{u}^\infty(\mathbf{r})$ for $\mathbf{r} \in S_\infty$ (with $R \rightarrow \infty$). Eq. (S32) indeed shows that the velocity profile $\mathbf{u}(\mathbf{r})$ can be thought of as resulting from a distribution of force monopoles, *and*, for $\mathbf{u}_i^{\text{s}}(\mathbf{r}) \neq \mathbf{0}$, force dipoles distributed over the surfaces of the particles.

It is equation (S32) that we shall coarse-grain. In order to do so, we define a window $w(\mathbf{r})$ around \mathbf{r} , that satisfies $\int d\mathbf{r}w(\mathbf{r}) = 1$, and whose ‘width’ determines the coarse-graining scale L . For definiteness, we shall take

$$w(\mathbf{r}) = \frac{1}{L^3} \prod_{\alpha=x,y,z} \Theta\left(\frac{L}{2} - |r_\alpha|\right), \quad (\text{S33})$$

such that $w(\mathbf{r})$ is only nonzero (and equal to L^{-3}) inside a cube with ribbons of length L centered at \mathbf{r} , that we shall refer to as $\mathcal{C}(\mathbf{r}, L)$. We assume the window to ‘contain’ many colloids; for our cubical window (S33) we thus assume $L \gg a, b$, where a is the particle radius and b the typical particle separation. We define the coarse-grained version of any solvent property $f(\mathbf{r})$ as

$$\langle f \rangle(\mathbf{r}) = \int_{V_f^+} d\mathbf{r}' w(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') = \frac{1}{L^3} \int_{\mathcal{C}(\mathbf{r}, L) \cap V_f^+} d\mathbf{r}' f(\mathbf{r}'), \quad (\text{S34})$$

where V_f^+ denotes the fluid volume V_f , plus, for every particle i , a thin shell of width δ enclosing the particle surface S_i ³. Note that the integration is *not* over the volume *inside* the particles, while we *do* divide by the entire window volume L^3 . Consequently, the coarse-grained solvent velocity $\langle \mathbf{u} \rangle(\mathbf{r})$ is the *physical* velocity $\mathbf{u}^{\text{out}}(\mathbf{r})$ *volume-averaged* over $\mathcal{C}(\mathbf{r}, L)$. It is related to the average velocity *per solvent particle* $\mathbf{u}^{\text{av}}(\mathbf{r})$ as $\langle \mathbf{u} \rangle(\mathbf{r}) = (1 - \phi(\mathbf{r}))\mathbf{u}^{\text{av}}(\mathbf{r})$, where $\phi(\mathbf{r})$ is the local volume fraction of colloids.

We now coarse-grain Eq. (S32), i.e. we calculate $\langle (S32) \rangle(\mathbf{r})$. First, we note that any distribution $f(\mathbf{r})$ satisfies

$$\begin{aligned} \langle \nabla f \rangle(\mathbf{r}) &= \int_{V_f^+} d\mathbf{r}' w(\mathbf{r} - \mathbf{r}') \nabla' f(\mathbf{r}') = - \int_{V_f^+} d\mathbf{r}' \nabla' w(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') = \nabla \int_{V_f^+} d\mathbf{r}' w(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') \\ &= \nabla \langle f \rangle(\mathbf{r}). \end{aligned} \quad (\text{S35})$$

Using this property, the left-hand side of the coarse-grained version of Eq. (S32) becomes

$$-\partial_\alpha \langle p \rangle(\mathbf{r}) + \eta \nabla^2 \langle u_\alpha \rangle(\mathbf{r}). \quad (\text{S36})$$

The coarse-grained version of the first term on the right-hand side of Eq. (S32) is

$$\begin{aligned} \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) w(\mathbf{r} - \boldsymbol{\xi}) \sigma_{\alpha\beta}^{\text{out}}(\boldsymbol{\xi}) \hat{n}_{i\beta}(\boldsymbol{\xi}) &\approx \frac{1}{L^3} \sum_{i \in \mathcal{C}(\mathbf{r}, L)} \oint_{S_i} dS(\boldsymbol{\xi}) \sigma_{\alpha\beta}^{\text{out}}(\boldsymbol{\xi}) \hat{n}_{i\beta}(\boldsymbol{\xi}) \\ &= \frac{1}{L^3} \sum_{i \in \mathcal{C}(\mathbf{r}, L)} F_{i,\alpha}^H, \end{aligned} \quad (\text{S37})$$

where $\mathbf{F}_i^H = \oint_{S_i} dS \boldsymbol{\sigma}^{\text{out}} \cdot \hat{\mathbf{n}}$ is the hydrodynamic force exerted on particle i , and where we neglected any contributions from particles contained only partially in $\mathcal{C}(\mathbf{r}, L)$, which is justified by virtue of the assumption

³ Formally we take $\delta \rightarrow 0$, while ensuring that $\delta > \epsilon$ at all times. All the monopoles *and* dipoles appearing in Eq. (S32) are thus entirely contained in V_f^+ .

$L \gg a, b$. The coarse-grained version of the second term on the right-hand-side of Eq. (S32) is

$$\begin{aligned}
& 2\eta \int_{V_f^+} d\mathbf{r}' w(\mathbf{r} - \mathbf{r}') \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \lim_{\epsilon \downarrow 0} \left\{ \frac{u_{i(\alpha)}^s(\boldsymbol{\xi}) \hat{n}_{i\gamma}(\boldsymbol{\xi})}{\epsilon} [\delta^3(\mathbf{r}' - (\boldsymbol{\xi} - \epsilon \hat{\gamma})) - \delta^3(\mathbf{r}' - \boldsymbol{\xi})] \right\} \\
&= 2\eta \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \lim_{\epsilon \downarrow 0} \left\{ \frac{w(\mathbf{r} - (\boldsymbol{\xi} - \epsilon \hat{\gamma})) - w(\mathbf{r} - \boldsymbol{\xi})}{\epsilon} \right\} u_{i(\alpha)}^s(\boldsymbol{\xi}) \hat{n}_{i\gamma}(\boldsymbol{\xi}) \\
&= 2\eta \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \frac{\partial w(\mathbf{r} - \boldsymbol{\xi})}{\partial r_\gamma} u_{i(\alpha)}^s(\boldsymbol{\xi}) \hat{n}_{i\gamma}(\boldsymbol{\xi}) \\
&\stackrel{(S33)}{=} -\frac{2\eta}{L^3} \sum_i \oint_{S_i} dS(\boldsymbol{\xi}) \left(\prod_{\beta \neq \gamma} \Theta\left(\frac{L}{2} - |r_\beta - \xi_\beta|\right) \right) \left\{ \delta\left(r_\gamma - \left(\xi_\gamma + \frac{L}{2}\right)\right) - \delta\left(r_\gamma - \left(\xi_\gamma - \frac{L}{2}\right)\right) \right\} u_{i(\alpha)}^s(\boldsymbol{\xi}) \hat{n}_{i\gamma}(\boldsymbol{\xi}) \\
&= -\frac{2\eta}{L^3} \sum_i \left(\oint_{\delta\mathcal{C}_\gamma^+(\mathbf{r}, L) \cap S_i} dl(\boldsymbol{\xi}) - \oint_{\delta\mathcal{C}_\gamma^-(\mathbf{r}, L) \cap S_i} dl(\boldsymbol{\xi}) \right) u_{i(\alpha)}^s(\boldsymbol{\xi}) \hat{n}_{i\gamma}(\boldsymbol{\xi}), \tag{S38}
\end{aligned}$$

where in the last line $\delta\mathcal{C}_\gamma^\pm(\mathbf{r}, L)$ denotes the face of the $\mathcal{C}(\mathbf{r}, L)$ cube with outward normal $\pm\hat{\gamma}$. The integrations in the last line thus run over the intersection of the $\delta\mathcal{C}_\gamma^\pm(\mathbf{r}, L)$ -face with the surface S_i of any particle i that intersects it. For any particle i , the integration domain is thus an intersection between two surfaces, which forms a line. The magnitude of the contributions (S37) and (S38) can now be estimated. Denoting the magnitude of F_i^H by F , and the colloid density by ρ , such that the number of colloids inside $\mathcal{C}(\mathbf{r}, L)$ approximately equals ρL^3 , the magnitude of the contribution (S37) is estimated as $L^{-3}(\rho L^3)F = \rho F \approx F\phi a^{-3}$, where ϕ denotes the packing fraction of the colloids, and where a denotes the particle size. To estimate the contribution of either integral in (S38), we note that i) the number of particles intersecting $\delta\mathcal{C}_\gamma^\pm(\mathbf{r}, L)$ has as approximate upper bound $(\rho L^3)^{\frac{2}{3}}$ (in fact, the number of intersecting particles is much less for a dilute suspension), ii) for any particle intersecting $\delta\mathcal{C}_\gamma^\pm(\mathbf{r}, L)$, the length of the intersection line is of the order a , and iii) $\eta u_i^s \approx a\sigma^{\text{out}} \approx F/a$. Therefore, the contribution of either integral in (S38) is approximated as $L^{-3}(\rho L^3)^{\frac{2}{3}} a(F/a) = F\rho^{2/3} L^{-1} \approx F\phi^{\frac{2}{3}} a^{-2} L^{-1}$. As $L^{-1} \ll a^{-1}$, the contribution of (S38) is negligible as compared to the contribution of (S37). Therefore, the coarse-grained version of Eq. (S32) reads

$$-\partial_\alpha \langle p \rangle(\mathbf{r}) + \eta \nabla^2 \langle u_\alpha \rangle(\mathbf{r}) = \frac{1}{L^3} \sum_{i \in \mathcal{C}(\mathbf{r}, L)} F_i^H. \tag{S39}$$

The hydrodynamic force \mathbf{F}_i^H experienced by a spherical particle i can be decomposed as $\mathbf{F}_i^H = -\gamma_t \mathbf{v}_i + \gamma_t v_0 \hat{\mathbf{e}}_i$, where \mathbf{v}_i is the velocity of particle i and $\hat{\mathbf{e}}_i$ its orientation [12]. The evolution of $\mathbf{v}_i(t)$ and $\hat{\mathbf{e}}_i(t)$ are governed by the Langevin dynamics of the particles ⁴. Consequently, even though it was left implicit so far, the solvent pressure $\langle p \rangle(\mathbf{r}, t)$ and velocity $\langle \mathbf{u} \rangle(\mathbf{r}, t)$ actually depend on time, via Eq. (S39). To relate the right-hand side of Eq. (S39) to the probability distribution function $\psi(\mathbf{r}, \mathbf{e}, t)$ of the particles - whose time evolution is governed by the Smoluchowski equation - we assume the dynamics of the particles not to change significantly throughout a window, meaning that e.g. the external potential $V^{\text{ext}}(\mathbf{r})$ must not vary significantly under $r_\alpha \rightarrow r_\alpha + L$ ⁵. This implies that ψ is approximately constant within any window, i.e. $\psi(\mathbf{r}', \mathbf{e}, t) \approx \psi(\mathbf{r}, \mathbf{e}, t)$ for $\mathbf{r}' \in \mathcal{C}(\mathbf{r}, L)$, for any \mathbf{r} . In this case, the sum over all particles in $\mathcal{C}(\mathbf{r}, L)$ (which are many) coincides with a sum over different realizations of the noise appearing in the Langevin equation, such

⁴ This is under the assumption that the effect of the slip velocity \mathbf{u}_i^s is to displace particle i only translationally; if it also rotates the particle an additional ‘self-torque’ has to be added to the Langevin equations.

⁵ In the main text we *do* consider a membrane potential that changes on the scale $a \ll L$. However, this is in a planar geometry; if one employs a window that is thin in the direction perpendicular to the membrane, and elongated in the parallel direction(s), it can still contain many colloids, yet have an approximately constant $V(\mathbf{r})$ inside.

that the coarse-grained Stokes equation (S39) becomes

$$\begin{aligned}
-\nabla\langle p\rangle(\mathbf{r},t) + \eta\nabla^2\langle\mathbf{u}\rangle(\mathbf{r},t) &= \int d\hat{\mathbf{e}}\psi(\mathbf{r},\hat{\mathbf{e}},t) [-\gamma_t\bar{\mathbf{v}}(\mathbf{r},\hat{\mathbf{e}},t) + \gamma_tv_0\hat{\mathbf{e}}] \\
&\stackrel{(S12)}{=} -\gamma_t\rho(\mathbf{r},t)\bar{\mathbf{v}}(\mathbf{r},t) + \gamma_tv_0\mathbf{m}(\mathbf{r},t) \\
&\equiv \mathbf{f}^f(\mathbf{r},t) + \mathbf{f}^p(\mathbf{r},t),
\end{aligned} \tag{S40}$$

where we defined the frictional body force $\mathbf{f}^f(\mathbf{r},t) \equiv -\gamma_t\rho(\mathbf{r},t)\bar{\mathbf{v}}(\mathbf{r},t)$ and the propulsion body force $\mathbf{f}^p(\mathbf{r},t) \equiv \gamma_tv_0\mathbf{m}(\mathbf{r},t)$, which are the internal body forces appearing in the colloidal force balance (3) of the main text. Note that in the main text we simply denoted $\langle\mathbf{u}(\mathbf{r},t)\rangle$ by $\mathbf{u}(\mathbf{r},t)$, and $\langle p(\mathbf{r},t)\rangle$ by $P_s(\mathbf{r},t)$.

Both Eq. (S39) and (S40) show (upon bringing all the terms to the left-hand side) that on the coarse-grained scale the solvent flow is simply governed by the Stokes equation, equipped with body forces equal to the opposite of the hydrodynamic forces experienced by the particles. We remark here that in the overdamped limit, the Langevin equation (S1) reduces to $\mathbf{0} = \mathbf{F}_i^H - \nabla_i(V + U^{\text{eff}}) + \sqrt{2\gamma_t k_B T}\boldsymbol{\eta}_i^t$, showing that the hydrodynamic, external, interaction and Brownian forces acting on any particle exactly balance. It is therefore only when $-\nabla_i(V + U^{\text{eff}}) + \sqrt{2\gamma_t k_B T}\boldsymbol{\eta}_i^t = \mathbf{0}$, that the hydrodynamic force $\mathbf{F}_i^H = \mathbf{0}$, and that the motion is usually referred to as ‘force-free’ [12]. However, when $-\nabla_i(V + U^{\text{eff}}) + \sqrt{2\gamma_t k_B T}\boldsymbol{\eta}_i^t \neq \mathbf{0}$, the hydrodynamic force $\mathbf{F}_i^H \neq \mathbf{0}$ contributes to Eq. (S39). As the stochastic force $\boldsymbol{\eta}_i^t$ time-averages to zero, the essential factor distinguishing these two cases is whether $-\nabla_i(V + U^{\text{eff}})$ is nonzero. In the setting of the main text, where U^{eff} was neglected for the dilute suspension, the fact that the hydrodynamic force is nonzero (i.e. the fact that $\mathbf{f}^f + \mathbf{f}^p \neq \mathbf{0}$) near the membrane, is in this sense a consequence of the external force $-\nabla_i V$ exerted on the colloids by the membrane.

3. OSMOTIC PRESSURE WITH INTERACTIONS

This section shows the conclusion of the main paper - that the activity-induced increase in osmotic pressure can be attributed to an increase in the chemical potential of the solvent - to hold true also in the presence of interactions.

We start by writing the force balance of the overdamped colloids (S24) as

$$\mathbf{0} = \mathbf{f}^e(\mathbf{r},t) + \mathbf{f}^f(\mathbf{r},t) + \mathbf{f}^p(\mathbf{r},t) - \nabla \cdot \mathbf{P}(\mathbf{r},t), \tag{S41}$$

where we defined the *external* body force $\mathbf{f}^e(\mathbf{r},t) \equiv -\int d\hat{\mathbf{e}}\nabla V(\mathbf{r},\hat{\mathbf{e}})\psi(\mathbf{r},\hat{\mathbf{e}},t)$, and where we recall the definitions of the frictional body force $\mathbf{f}^f(\mathbf{r},t) = -\gamma_t\rho(\mathbf{r},t)\bar{\mathbf{v}}(\mathbf{r},t)$ and the propulsion body force $\mathbf{f}^p(\mathbf{r},t) = \gamma_tv_0\mathbf{m}(\mathbf{r},t)$, both of which are *internal*. Furthermore, we recall from section 1 that

$$-\nabla_1 \cdot \mathbf{P}(\mathbf{r}_1,t) = -\nabla_1\rho(\mathbf{r}_1,t)k_B T - \int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1)\psi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t). \tag{S42}$$

The second term in (S42) represents an effective force exerted on colloids at \mathbf{r}_1 by other colloids. These effective interactions are due to both direct colloid-colloid interactions and solvent induced interactions, which can be made explicit as

$$\begin{aligned}
-\int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}^{\text{eff}}(\mathbf{r}_2 - \mathbf{r}_1)\psi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) &= -\int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}(\mathbf{r}_2 - \mathbf{r}_1)\psi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) \\
&\quad - \int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 s}(\mathbf{r}_2 - \mathbf{r}_1)\psi_{\hat{\mathbf{e}}_1 s}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t).
\end{aligned} \tag{S43}$$

Here $\phi_{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}(\mathbf{r}_2 - \mathbf{r}_1)$ and $\phi_{\hat{\mathbf{e}}_1 s}(\mathbf{r}_2 - \mathbf{r}_1)$ denote the bare colloid-colloid and the bare colloid-solvent interaction pair potential, respectively. Furthermore, $\psi_{\hat{\mathbf{e}}_1 s}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t)$ denotes the two-body colloid-solvent distribution function.

Next, we consider the momentum balance for the solvent, i.e. the analogue of Eq. (S16) for the solvent. As we consider a regime in which the Reynolds number of the solvent $Re \ll 1$, the inertial terms in the solvent momentum balance are negligible. Therefore, the various body forces acting on the solvent, to wit, a possible external body force $\mathbf{f}_s^e(\mathbf{r}, t)$, and the negative of the internal forces $\mathbf{f}^f(\mathbf{r}, t)$ and $\mathbf{f}^p(\mathbf{r}, t)$ that act on the colloids, have to balance the divergence of the solvent momentum flux tensor $\mathcal{J}_s^{\text{mom}}(\mathbf{r}, t)$, and the solvent-colloid and solvent-solvent interaction forces:

$$\mathbf{0} = \mathbf{f}_s^e(\mathbf{r}_1, t) - \mathbf{f}^f(\mathbf{r}_1, t) - \mathbf{f}^p(\mathbf{r}_1, t) - \nabla_1 \cdot \mathcal{J}_s^{\text{mom}}(\mathbf{r}_1, t) - \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{s\hat{\mathbf{e}}_2}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{s\hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}_2 \nabla_1 \phi_{ss}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{ss}^{(2)}(\mathbf{r}_1, \mathbf{r}_2). \quad (\text{S44})$$

The momentum flux tensor $\mathcal{J}_s^{\text{mom}}(\mathbf{r}_1, t)$ of the solvent is defined, analogous to the colloidal momentum flux (first term on the right-hand side of Eq. (S17)), as

$$\mathcal{J}_s^{\text{mom}}(\mathbf{r}_1, t) = m \int d\mathbf{v}_1 f_s(\mathbf{r}_1, \mathbf{v}_1, t) (\mathbf{v}_1 - \mathbf{u}) \otimes (\mathbf{v}_1 - \mathbf{u}). \quad (\text{S45})$$

Here the probability density $f_s(\mathbf{r}_1, \mathbf{v}_1, t)$ to find a solvent particle at position \mathbf{r}_1 with velocity \mathbf{v}_1 at time t defines the solvent number density $\rho_s(\mathbf{r}_1, t)$ and mean solvent velocity $\mathbf{u}(\mathbf{r}_1, t)$ as

$$\rho_s(\mathbf{r}_1, t) \equiv \int d\mathbf{v}_1 f_s(\mathbf{r}_1, \mathbf{v}_1, t),$$

$$\rho_s(\mathbf{r}_1, t) \mathbf{u}(\mathbf{r}_1, t) \equiv \int d\mathbf{v}_1 f_s(\mathbf{r}_1, \mathbf{v}_1, t) \mathbf{v}_1. \quad (\text{S46})$$

In equilibrium (where $\mathbf{u} = \mathbf{0}$), the equipartition theorem implies $\mathcal{J}_s^{\text{mom}}(\mathbf{r}, t) = \rho_s(\mathbf{r}) k_B T$. The divergence of this momentum flux then combines with the interaction terms in Eq. (S44), characterized by the equilibrium two-body distribution functions $\psi_{s\hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \psi_{s\hat{\mathbf{e}}_2}^{(2),\text{eq}}(\mathbf{r}_1, \mathbf{r}_2)$ and $\psi_{ss}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \psi_{ss}^{(2),\text{eq}}(\mathbf{r}_1, \mathbf{r}_2)$, to give [13]

$$-\nabla_1 \rho_s(\mathbf{r}_1) k_B T - \int d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{s\hat{\mathbf{e}}_2}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{s\hat{\mathbf{e}}_2}^{(2),\text{eq}}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}_2 \nabla_1 \phi_{ss}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{ss}^{(2),\text{eq}}(\mathbf{r}_1, \mathbf{r}_2) = -\rho_s(\mathbf{r}_1) \nabla_1 \mu_s^{\text{int}}(\mathbf{r}_1), \quad (\text{S47})$$

where the *intrinsic chemical potential of the solvent* is defined as $\mu_s^{\text{int}}(\mathbf{r}_1) \equiv \delta F[\psi, \rho_s] / \delta \rho_s(\mathbf{r}_1)$, $F[\psi, \rho_s]$ being the free energy functional of the colloid-solvent mixture. To proceed out of equilibrium we follow Archer [3]. Out of equilibrium, the momentum flux tensor generally receives an additional contribution, whose divergence can, under suitable approximations, be written as $-\eta^{(K)} \nabla^2 \mathbf{u}$, such that

$$-\nabla \cdot \mathcal{J}_s^{\text{mom}}(\mathbf{r}, t) = -\nabla \rho_s(\mathbf{r}, t) k_B T + \eta^{(K)} \nabla^2 \mathbf{u}(\mathbf{r}, t). \quad (\text{S48})$$

Also the interaction terms in Eq. (S44) give additional contributions as compared to equilibrium, due to deviations of the correlation functions from their equilibrium values. These extra contributions can, under similar approximations, be shown to be $\eta^{(V)} \nabla^2 \mathbf{u}$. Together, these two out-of-equilibrium corrections add up to $\eta \nabla^2 \mathbf{u}$, where the viscosity $\eta = \eta^{(K)} + \eta^{(V)}$ comprises both kinetic contributions and contributions from interactions, respectively. For details see Archer [3] and Kreuzer [14]. Putting it all together, the solvent force balance (S44) thus reads

$$\mathbf{0} = \mathbf{f}_s^e(\mathbf{r}_1, t) - \mathbf{f}^f(\mathbf{r}_1, t) - \mathbf{f}^p(\mathbf{r}_1, t) - \rho_s(\mathbf{r}_1, t) \nabla_1 \mu_s^{\text{int}}(\mathbf{r}_1, t) + \eta \nabla_1^2 \mathbf{u}(\mathbf{r}_1, t). \quad (\text{S49})$$

As an aside, for the case of no colloid-solvent interactions, i.e. as in the main text (where we neglected any effective colloid-colloid interactions, and hence, according to Eq. (S43), also colloid-solvent interactions), the term $-\rho_s \nabla_1 \mu_s^{\text{int}}$ can be written as

$$-\rho_s(\mathbf{r}_1, t) \nabla_1 \mu_s^{\text{int}}(\mathbf{r}_1, t) \stackrel{\phi_{s\hat{\mathbf{e}}_2} = 0}{=} -\nabla_1 \rho_s(\mathbf{r}_1, t) k_B T - \int d\mathbf{r}_2 \nabla_1 \phi_{ss}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{ss}^{(2),\text{eq}}(\mathbf{r}_1, \mathbf{r}_2) = -\nabla \cdot \mathbf{P}_s(\mathbf{r}_1, t), \quad (\text{S50})$$

i.e. as minus the divergence of the solvent pressure tensor ⁶

$$\mathbf{P}_s(\mathbf{r}_1, t) = \rho_s(\mathbf{r}_1, t)k_B T \mathbb{1} - \frac{1}{2} \int d\mathbf{r}_{12} \int_0^1 du \mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{ss}(\mathbf{r}_{12}) \psi_{ss}^{(2), \text{eq}}(\mathbf{r}_1 - u\mathbf{r}_{12}, \mathbf{r}_1 + (1-u)\mathbf{r}_{12}), \quad (\text{S51})$$

that contains the solvent-solvent interactions in Kirkwood-Irving form [4]. Upon assuming the pressure tensor (S51) to be isotropic, the solvent force balance (S49) then reduces to the solvent force balance (5) of the main text (and Eq. (S40) of this Electronic Supplementary Information). In the presence of arbitrary solvent-colloid interactions however, an identification akin to (S51) is not generally possible, and we work with Eq. (S49) as the solvent force balance.

The force balance of the complete suspension is obtained by adding the colloid force balance (S41) (using Eq. (S42) and (S43)) and the solvent force balance in the form of (S44), as

$$\begin{aligned} 0 &= \mathbf{f}^e(\mathbf{r}_1, t) + \mathbf{f}_s^e(\mathbf{r}_1, t) - \nabla_1 \cdot \left(\rho(\mathbf{r}_1, t)k_B T \mathbb{1} + \mathcal{J}_s^{\text{mom}}(\mathbf{r}_1, t) \right) - \int d\hat{\mathbf{e}}_1 d\mathbf{r}_2 d\hat{\mathbf{e}}_2 \nabla_1 \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \\ &\quad - \int d\hat{\mathbf{e}} d\mathbf{r}_2 \left(\nabla_1 \phi_{\hat{\mathbf{e}} s}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{\hat{\mathbf{e}} s}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) + \hat{\mathbf{e}} \leftrightarrow s \right) \\ &\quad - \int d\mathbf{r}_2 \nabla_1 \phi_{ss}(\mathbf{r}_2 - \mathbf{r}_1) \psi_{ss}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \\ &= \mathbf{f}^e(\mathbf{r}_1, t) + \mathbf{f}_s^e(\mathbf{r}_1, t) - \nabla_1 \cdot \mathbf{P}_{\text{tot}}(\mathbf{r}_1, t), \end{aligned} \quad (\text{S52})$$

where the pressure tensor of the total suspension $\mathbf{P}_{\text{tot}}(\mathbf{r}_1, t)$ includes the momentum flux of the colloids and of the solvent, and all colloid-colloid, colloid-solvent and solvent-solvent interactions in Kirkwood-Irving form [4], as

$$\begin{aligned} \mathbf{P}_{\text{tot}}(\mathbf{r}_1, t) &= \left(\rho(\mathbf{r}_1, t)k_B T \mathbb{1} + \mathcal{J}_s^{\text{mom}}(\mathbf{r}_1, t) \right) \mathbb{1} \\ &\quad - \frac{1}{2} \int d\hat{\mathbf{e}}_1 d\mathbf{r}_{12} d\hat{\mathbf{e}}_2 \int_0^1 du \mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}(\mathbf{r}_{12}) \psi_{\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2}^{(2)}(\mathbf{r}_1 - u\mathbf{r}_{12}, \mathbf{r}_1 + (1-u)\mathbf{r}_{12}) \\ &\quad - \frac{1}{2} \int d\hat{\mathbf{e}} d\mathbf{r}_{12} \int_0^1 du \left(\mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{\hat{\mathbf{e}} s}(\mathbf{r}_{12}) \psi_{\hat{\mathbf{e}} s}^{(2)}(\mathbf{r}_1 - u\mathbf{r}_{12}, \mathbf{r}_1 + (1-u)\mathbf{r}_{12}) + \hat{\mathbf{e}} \leftrightarrow s \right) \\ &\quad - \frac{1}{2} \int d\mathbf{r}_{12} \int_0^1 du \mathbf{r}_{12} \otimes \frac{\partial}{\partial \mathbf{r}_{12}} \phi_{ss}(\mathbf{r}_{12}) \psi_{ss}^{(2)}(\mathbf{r}_1 - u\mathbf{r}_{12}, \mathbf{r}_1 + (1-u)\mathbf{r}_{12}). \end{aligned} \quad (\text{S53})$$

As in the main text, we now consider a flux-free steady state, where the solvent velocity $\mathbf{u}(\mathbf{r}, t) = \mathbf{0}$ (on the scale coarse-grained over the colloids). Adding the colloid and solvent force balance in the form of Eq. (S41) and (S49), respectively, and comparing with the total force balance (S52) then reveals

$$\nabla \cdot \mathbf{P}_{\text{tot}}(\mathbf{r}) = \nabla \cdot \mathbf{P}(\mathbf{r}) + \rho_s(\mathbf{r}) \nabla \mu_s^{\text{int}}(\mathbf{r}). \quad (\text{S54})$$

In an isotropic bulk, the pressure tensor of the effective colloids-only system reduces to a scalar as $\mathbf{P} = P \mathbb{1}$. Similarly, the total pressure tensor $\mathbf{P}_{\text{tot}} = P_{\text{tot}} \mathbb{1}$. Furthermore, in a bulk characterized by colloid density ρ , solvent density ρ_s , and colloid propulsion speed v_0 , all the two-body correlation functions are uniquely characterized by (ρ, ρ_s, v_0) (at fixed temperature). Therefore, Eq. (S25) and (S53) reveal the pressures

⁶ There is some ambiguity in what quantity to call the solvent pressure tensor here. We could also define the solvent pressure tensor

$$\tilde{\mathbf{P}}_s(\mathbf{r}, t) \equiv \mathcal{J}_s^{\text{mom}}(\mathbf{r}, t) - \frac{1}{2} \int d\mathbf{r}_{12} \int_0^1 du \mathbf{r}_{12} \otimes \frac{d}{d\mathbf{r}_{12}} \phi_{ss}(\mathbf{r}_{12}) \psi_{ss}^{(2)}(\mathbf{r}_1 - u\mathbf{r}_{12}, \mathbf{r}_1 + (1-u)\mathbf{r}_{12}),$$

which is more analogous to the pressure tensor of the total suspension $\mathbf{P}_{\text{tot}}(\mathbf{r}, t)$ defined in Eq. (S53), and in terms of which the solvent force balance without colloid-solvent interactions would read

$$\mathbf{0} = \mathbf{f}_s^e(\mathbf{r}, t) - \mathbf{f}^p(\mathbf{r}, t) - \nabla \cdot \tilde{\mathbf{P}}_s(\mathbf{r}, t).$$

The two solvent pressure tensors are related as $-\nabla \cdot \tilde{\mathbf{P}}_s(\mathbf{r}, t) = -\nabla \cdot \mathbf{P}_s(\mathbf{r}, t) + \eta \nabla^2 \mathbf{u}(\mathbf{r}, t)$.

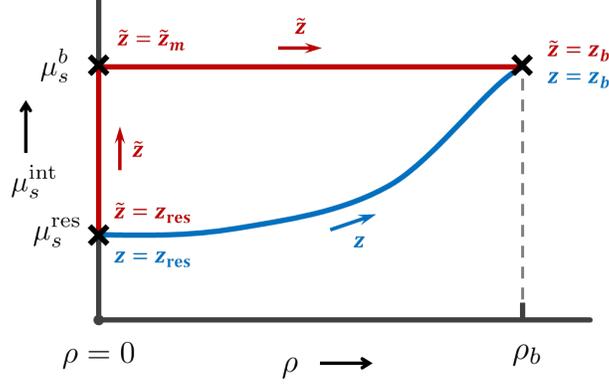


FIG. S1. Paths in $(\rho, \mu_s^{\text{int}})$ space. The physically realized path (blue) is parameterized by z and the path proposed to calculate the integral (S57) (red) by \tilde{z} . The marked points $(\rho = 0, \mu_s^{\text{res}})$ and (ρ_b, μ_s^b) correspond to the reservoir at $z = \tilde{z} = z_{\text{res}}$ and the bulk suspension at $z = \tilde{z} = z_b$ respectively (in the text the bulk density ρ_b is simply called ρ). The marked point $(\rho = 0, \mu_s^b)$ at $\tilde{z} = z_m$ corresponds to a region large enough for a local density approximation to apply.

$P(\rho, \rho_s, v_0)$ and $P_{\text{tot}}(\rho, \rho_s, v_0)$ to be state functions in the bulk. Also the solvent chemical potential μ_s^{int} is in bulk a function of the colloid density ρ and solvent density ρ_s , owing to its definition in terms of the free energy functional. This state function $\mu_s^{\text{int}}(\rho, \rho_s)$ can be inverted to $\rho_s(\mu_s^{\text{int}}, \rho)$, such that the bulk solvent can be characterized by μ_s^{int} instead of ρ_s . Consequently, the bulk pressures can be expressed as $P(\rho, \mu_s^{\text{int}}, v_0)$ and $P_{\text{tot}}(\rho, \mu_s^{\text{int}}, v_0)$.

To calculate the osmotic pressure, we again specialize to the planar geometry of Fig. 1 of the main text, where $\mathbf{f}^e(\mathbf{r}) = f_z^e(z)\hat{\mathbf{z}}$, and where we now also allow for a non-perfect membrane that can exert a force $\mathbf{f}_s^e(\mathbf{r}) = f_{sz}^e(z)\hat{\mathbf{z}}$ on the solvent. The osmotic pressure then follows as

$$\Pi = \int_{z_{\text{res}}}^{z_b} dz (f_z^e(z) + f_{sz}^e(z)) \stackrel{\text{(S52)}}{=} P_{\text{tot}}(\rho, \mu_s^b, v_0) - P_{\text{tot}}(\rho = 0, \mu_s^{\text{res}}, v_0), \quad (\text{S55})$$

i.e. as the difference in total pressure on the opposing sides of the membrane. This difference can be obtained by integrating Eq. (S54) from the reservoir to the bulk at the opposing side of the membrane, to be

$$P_{\text{tot}}(\rho, \mu_s^b, v_0) - P_{\text{tot}}(\rho = 0, \mu_s^{\text{res}}, v_0) = P(\rho, \mu_s^b, v_0) + \int_{z_{\text{res}}}^{z_b} dz \rho_s(z) \partial_z \mu_s^{\text{int}}(z). \quad (\text{S56})$$

To calculate the remaining integral in (S56), we note that the z -coordinate parameterizes a path $(\rho(z), \mu_s^{\text{int}}(z))$ in $(\rho, \mu_s^{\text{int}})$ -space, as illustrated in blue in Fig. S1. This path corresponds to the physically realized profiles $\rho(z)$ and $\mu_s^{\text{int}}(z)$. The crucial observation is that the three pressure terms in Eq. (S56) are functions of only (ρ, μ_s^b) and $(\rho = 0, \mu_s^{\text{res}})$ (at fixed v_0), i.e. only of the *endpoints* of the path. Therefore, the same holds true for the integral. This implies that the integral yields the same value when evaluated for a *different* path with the *same* endpoints. The path traced out in $(\rho, \mu_s^{\text{int}})$ -space can be altered by applying a nonzero external potential $V_s(z)$ (and thus an external force $\mathbf{f}_s^e(z) = -\rho_s(z)\partial_z V_s(z)\hat{\mathbf{z}}$) to the solvent, for according to Eq. (S49) this alters the profiles $\rho(z)$ and $\mu_s^{\text{int}}(z)$. In particular, we could apply an external potential $V_s(z)$ that vanishes in both the reservoir and the bulk, but is nonzero in between in such a way that the intrinsic solvent chemical potential $\mu_s^{\text{int}}(z)$ increases from μ_s^{res} at $z = z_{\text{res}}$ to μ_s^b already at the reservoir side of the membrane (where $\rho(z) = 0$), and such that it remains $\mu_s^{\text{int}}(z) = \mu_s^b$ from there to $z = z_b$ in the bulk suspension on the other side. This situation corresponds to the red path in Fig. S1, parameterized by \tilde{z} . The marked point $(\rho = 0, \mu_s^b)$ along this path corresponds to the region just on the reservoir side of the membrane, where $\rho(\tilde{z}) = 0$ and $\mu_s^{\text{int}}(\tilde{z}) = \mu_s^b$. We can choose the external solvent potential $V_s(z)$ in such a way that this region is large enough to be considered as a bulk. Upon naming one \tilde{z} -coordinate in this intermediate bulk \tilde{z}_m (see

Fig. S1), the red path can be utilized to calculate the remaining integral in Eq. (S56) as

$$\begin{aligned}
\int_{z_{\text{res}}}^{z_b} dz \rho_s(z) \partial_z \mu_s^{\text{int}}(z) &= \int_{z_{\text{res}}}^{z_b} d\tilde{z} \rho_s(\tilde{z}) \partial_z \mu_s^{\text{int}}(\tilde{z}) = \int_{z_{\text{res}}}^{\tilde{z}_m} d\tilde{z} \rho_s(\tilde{z}) \partial_z \mu_s^{\text{int}}(\tilde{z}) \\
&\stackrel{(S54)}{=} \int_{z_{\text{res}}}^{\tilde{z}_m} d\tilde{z} \partial_{\tilde{z}} P_{\text{tot}}(\tilde{z}) \\
&= P_{\text{tot}}(\rho = 0, \mu_s^b, v_0) - P_{\text{tot}}(\rho = 0, \mu_s^{\text{res}}, v_0) \\
&\equiv \Delta P_s,
\end{aligned} \tag{S57}$$

where in the second step we used that $\partial_{\tilde{z}} \mu_s^{\text{int}}(\tilde{z}) = 0$ for $\tilde{z}_m \leq \tilde{z} \leq z_b$, and in the third step that $P(\tilde{z}) = 0$ for $z_{\text{res}} \leq \tilde{z} \leq \tilde{z}_m$, as there are no colloid on the reservoir side of the membrane. As the quantity ΔP_s defined by Eq. (S57) is a difference in total pressures at colloid density $\rho = 0$, it is natural to think of it as a difference in solvent pressures. Indeed, in the limit of a dilute suspension this definition reduces to the difference ΔP_s used in the main text. To see this, note that the total pressure tensor of Eq. (S53) at colloid density $\rho = 0$ coincides with the solvent pressure tensor of Eq. (S51) (where $\psi_{ss}^{(2),\text{eq}}$ can be replaced by $\psi_{ss}^{(2)}$, as we are considering a vanishing solvent flow, such that $\eta^{(V)} \nabla^2 \mathbf{u} = \mathbf{0}$).

Combining Eq. (S55), (S56) and (S57), shows the osmotic pressure to be

$$\Pi = P(\rho, \mu_s^b, v_0) + \Delta P_s, \tag{S58}$$

which is the extension of the main text's result (8) to a system with interactions. Eq. (S58) shows that the osmotic pressure of a passive system $\Pi = P(\rho, \mu_s^b = \mu_s^{\text{res}}, v_0)$ increases with activity in two ways. Firstly, the effective colloids-only pressure $P(\rho, \mu_s^b, v_0)$ of Eq. (S25) changes, as the two-body colloid-colloid correlation function changes with activity [15], and as the increasing solvent chemical potential may modify the effective colloid-colloid interaction potential and the colloid-colloid correlation function. Secondly, the increase in solvent chemical potential induces an increase ΔP_s , that can be interpreted as an increase in solvent pressure.

4. DERIVATION OF THE SOLVENT PRESSURE DIFFERENCE ΔP_s (WITH INTERACTIONS)

The solvent pressure difference ΔP_s can be expressed as

$$\Delta P_s \stackrel{(S57)}{=} \int_{z_{\text{res}}}^{z_b} dz \rho_s(z) \partial_z \mu_s^{\text{int}}(z) \stackrel{(S49)}{=} - \int_{z_{\text{res}}}^{z_b} dz f_z^p(z) = -\gamma_t v_0 \int_{z_{\text{res}}}^{z_b} dz m_z(z), \tag{S59}$$

where we used $\mathbf{f}^p(z) = f_z^p(z) \hat{\mathbf{z}} = \gamma_t v_0 m_z(z) \hat{\mathbf{z}}$, and where we assumed the membrane to exert no force on the solvent, i.e. $\mathbf{f}_s^e(\mathbf{r}) = \mathbf{0}$. In the dilute limit, the same expression follows alternatively from finding $\Delta P_s = P_s(z_b) - P_s(z_{\text{res}})$ from Eq. (5) of the main text (with $\mathbf{f}_s^e(\mathbf{r}) = \mathbf{f}^f(\mathbf{r}) = \eta \nabla^2 \mathbf{u}(\mathbf{r}) = \mathbf{0}$). An expression for the polarization $m_z(z)$ is found from the Smoluchowski equation, formed by inserting Eq. (S23) into Eq. (S9). Indeed, the first moment (in the variable $\hat{\mathbf{e}}$) of this Smoluchowski equation yields, in steady state, and in the same geometry as in the main text,

$$\begin{aligned}
2 \frac{k_B T}{\gamma_r} m_z(z) &= -\partial_z \left\{ v_0 \left(\frac{1}{3} \rho(z) + \mathcal{S}_{zz}(z) \right) - \gamma_t^{-1} \int d\hat{\mathbf{e}} \psi(z, \theta) e_z \partial_z V(z, \theta) - \gamma_t^{-1} k_B T \partial_z m_z(z) \right. \\
&\quad \left. - \gamma_t^{-1} \int d\hat{\mathbf{e}} d\mathbf{r}' d\hat{\mathbf{e}}' e_z \partial_z \phi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{\text{eff}}(\mathbf{r}' - \mathbf{r}) \psi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{(2)}(\mathbf{r}, \mathbf{r}') \right\} \\
&\quad + \gamma_r^{-1} \int d\hat{\mathbf{e}} \sin(\theta) \partial_\theta V(z, \theta) \psi(z, \theta) + \gamma_r^{-1} \int d\hat{\mathbf{e}} d\mathbf{r}' d\hat{\mathbf{e}}' \sin(\theta) \partial_\theta \phi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{\text{eff}}(\mathbf{r}' - \mathbf{r}) \psi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{(2)}(\mathbf{r}, \mathbf{r}'),
\end{aligned} \tag{S60}$$

where $e_z = \cos \theta$ and where $\mathcal{S}(z) \equiv \int d\hat{\mathbf{e}} \psi(z, \theta) (\hat{\mathbf{e}}\hat{\mathbf{e}} - \frac{1}{3}\mathbb{1})$ is the traceless alignment tensor in three dimensions. According to Eq. (S59), the difference in solvent pressure ΔP_s essentially follows from integrating

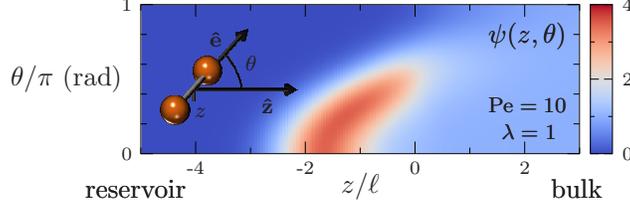


FIG. S2. Normalized probability $\psi(z, \theta)$ to find a dumbbell at (scaled) position z with orientation θ , obtained as a numerical solution to Eq. (1) of the main text for a soft potential $V(z)$ at $z < 0$ acting on both halves of the dumbbell (see text), with potential strength λ and activity Pe as indicated. Note that $\psi(z < -4, \theta) \approx 0$, indicating that (almost) no particles penetrate into the wall beyond $z = -4$, and that $\psi(z > 3, \theta) \approx 1$, indicating a homogeneous and isotropic bulk for $z > 3$. The probability peak corresponds to dumbbells persistently propelling into the membrane.

Eq. (S60) from the reservoir to the bulk suspension. The first two lines of the right-hand side of Eq. (S60) are easily integrated, as they form a derivative with respect to z . Note that all the terms acted upon by the derivative vanish in the reservoir, whereas only the density term and the interaction term are nonzero in the bulk suspension. The resulting solvent pressure difference is

$$\begin{aligned} \Delta P_s &= \frac{\gamma_t \gamma_r v_0^2}{6k_B T} \rho - \frac{\gamma_r v_0}{2k_B T} \int d\hat{\mathbf{e}} d\mathbf{r}' d\hat{\mathbf{e}}' e_z \partial_z \phi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{\text{eff}}(\mathbf{r}' - \mathbf{r}_b) \psi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{(2)}(\mathbf{r}_b, \mathbf{r}') \\ &\quad - \frac{\gamma_t v_0}{2k_B T} \int_{z_{\text{res}}}^{z_b} dz \int d\hat{\mathbf{e}} \sin(\theta) \partial_\theta V(z, \theta) \psi(z, \theta) \\ &\quad - \frac{\gamma_t v_0}{2k_B T} \int_{z_{\text{res}}}^{z_b} dz \int d\hat{\mathbf{e}} d\mathbf{r}' d\hat{\mathbf{e}}' \sin(\theta) \partial_\theta \phi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{\text{eff}}(\mathbf{r}' - \mathbf{r}) \psi_{\hat{\mathbf{e}}\hat{\mathbf{e}}'}^{(2)}(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (\text{S61})$$

where \mathbf{r}_b is a point in the bulk suspension. In the dilute limit (where the interaction terms are negligible), this solvent pressure difference reduces to Eq. (6), as was claimed in the main text.

5. NUMERICAL SOLUTION DETAILS

The plots in figures 3 and 5 are based upon a numerical solution $\psi(z, \theta)$ to the Smoluchowski equation (1), in steady state in the planar geometry of the main text. From this solution the other quantities of interest (e.g. polarization and pressure profiles) follow. The solutions $\psi(z, \theta)$ were obtained, using COMSOL Multiphysics[®], for z -values $-20\ell \leq z \leq 20\ell$, where $\ell \equiv \sqrt{\gamma_r/\gamma_t}$, and θ -values $0 \leq \theta \leq \pi$. As Fig. 3 corresponds to spherical particles, the considered colloid-membrane interaction potential, $V(z) = \lambda k_B T (z/\ell)^2$ with $\lambda = 1$ for $z < 0$ and $V(z) = 0$ for $z \geq 0$, is independent of θ . Fig. 5 corresponds to dumbbells, consisting of two point particles with separation ℓ , both of which are subject to the same potential $V(z)$, where λ is varied. The potential experienced by one dumbbell, with center of mass position z and orientation characterized by the polar angle θ , follows as $V(z, \theta) = V(z - \ell \cos \theta/2) + V(z + \ell \cos \theta/2)$. The applied boundary conditions are in both cases (i) a uniform distribution in the bulk, i.e. $\psi(20\ell, \theta) = c$ with arbitrary normalization c , (ii) either no flux $j_z(z = -20\ell, \theta) = 0$ or no particles $\psi(z = -20\ell, \theta) = 0$ far into the quadratic potential (both conditions yield equivalent solutions), and (iii) $\partial_\theta \psi(z, 0) = 0$ and $\partial_\theta \psi(z, \pi) = 0$, which follow from the symmetries $\psi(z, \theta) = \psi(z, -\theta)$ and $\psi(z, \pi + \theta) = \psi(z, \pi - \theta)$, respectively. A typical solution $\psi(z, \theta)$ for dumbbells, shown in Fig. S2, displays the same physics as encountered for spheres in the main text, namely an accumulation of dumbbells at the membrane caused by dumbbells persistently propelling into the membrane.

6. SOLVENT FLOW IN A PIPE

This section illustrates that the net polarization of colloids near a semipermeable membrane can also lead to solvent flow. Consider a dilute, active suspension in a cylindrical pipe, of radius R and length L , confined on one side by a semipermeable membrane, as illustrated in Fig. 2 of the main text. The difference with the setup of Fig. 1 of the main text is in the boundary conditions: whereas Fig. 1 corresponds to no-flux boundary conditions, the boundary conditions instead imposed here are equal solvent pressures on either end. We adopt a cylindrical coordinate system, where the z -axis coincides with the symmetry axis of the cylinder, $z = \pm L/2$ corresponding to either end of cylinder, where r is the radial distance from the z -axis, $\hat{\mathbf{r}}$ being the corresponding unit vector, and where ϕ is the azimuthal angle. We shall assume the solvent velocity $|\mathbf{u}(\mathbf{r})| \ll v_0$, such that the effect of advection is negligible. The dynamics of the colloids is then governed by the Smoluchowski equation (1) of the main text. We again consider a steady state where the colloid flux vanishes (such that $\mathbf{f}^f(\mathbf{r}) = \mathbf{0}$). We then expect the colloids to accumulate at, and form a net polarization towards, both the membrane and the outer walls of the pipe. We assume the radius R to be large, i.e. $R \gg \sqrt{\gamma_r/\gamma_t}$ and $R \gg \gamma_r v_0/k_B T$, such that in the centre of the pipe the effect of the pipe's outer wall is not felt. In this centre region, that we denote as $r < R^* < R$, the results for the planar geometry of the main text then carry over. More precisely, we expect the propulsion force $\mathbf{f}^p(r, z) = f_z^p(r, z)\hat{\mathbf{z}} + f_r^p(r, z)\hat{\mathbf{r}}$ to be given by $\mathbf{f}^p(r, z) = f_z^p(z)\hat{\mathbf{z}}$ for $r < R^* < R$, where $f_z^p(z)$ describes the polarization profile next to a membrane in a planar geometry. In particular, $-\int_{-L/2}^{L/2} dz f_z^p(z) = \Delta P_s$, as calculated in Eq. (6) of the main text. Note that the interpretation of ΔP_s as a solvent pressure difference is *not* valid in this setting, for this interpretation relies on the solvent flow $\mathbf{u}(\mathbf{r}) = \mathbf{0}$. Instead, at this point ΔP_s should purely be regarded as the right-hand side of Eq. (6). Finally, we assume the colloid polarization to be such that $\nabla \times \mathbf{f}^p(r, z) = \mathbf{0}$, also for $R^* \leq r \leq R$. The steady state solvent velocity profile $\mathbf{u}(r, z)$ is then governed by the Stokes equation (5)

$$-\mathbf{f}^p - \nabla P_s + \eta \nabla^2 \mathbf{u} = 0, \quad (\text{S62})$$

which again features the opposite propulsion force $-\mathbf{f}^p$ as a body force (note that we used $\mathbf{f}_s^c = \mathbf{f}^f = \mathbf{0}$). Eq. (S62) is to be solved, together with the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, subject to the boundary conditions of equal solvent pressure $P_s(r, z = -L/2) = P_s(r, z = L/2) = P_0$ for $r < R^*$, and the no-slip condition $\mathbf{u}(r = R, z) = \mathbf{0}$. Upon using the ansatz $\hat{\mathbf{r}} \cdot \mathbf{u} = 0$, the incompressibility condition reveals that $\mathbf{u}(r, z) = u_z(r)\hat{\mathbf{z}}$. Inserting this into (S62) yields

$$-f_r^p - \partial_r P_s = 0, \quad (\text{S63a})$$

$$-f_z^p - \partial_z P_s + \eta \frac{1}{r} \partial_r (r \partial_r u_z) = 0. \quad (\text{S63b})$$

Deriving Eq. (S63b) with respect to r , and using $\nabla \times \mathbf{f}^p = \mathbf{0}$ together with Eq. (S63a), shows that $\eta \frac{1}{r} \partial_r (r \partial_r u_z(r)) = c$. The constant c follows by integrating (S63b) from $z = -L/2$ to $z = L/2$ (for $r < R^*$), resulting in $c = -\Delta P_s/L$. Solving for $u_z(r)$, and using the no-slip boundary condition, yields the velocity profile

$$u_z(r) = \frac{\Delta P_s}{4\eta L} (R^2 - r^2)\hat{\mathbf{z}}. \quad (\text{S64})$$

Eq. (S64) shows that the solvent flow - driven by the body force $-\mathbf{f}_p$ - coincides with a Poiseuille-Hagen flow driven by a solvent pressure difference ΔP_s applied between the ends of the pipe.

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