Adhesive elastocapillary force on a cantilever beam: Supplementary material

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In this supplementary material, the beam model of regimes 2 and 3 is presented in detail, as well as a strategy for solving the corresponding equations numerically. Finally, both the alternative hypothesis of a constant moment \( M_d \) and the adhesion of a dry contact are discussed in the two last sections.

1 Main model

The beam deflection is modelled as described in the schematics of Fig. 1. Points D, W and C represent positions of the right end of the beam/substrate apparent contact zone, the beam/liquid/air contact line and the clamp. In regime 2, D coincides with the beam tip while in regime 3, both are separated by a distance \( d \). The \( s \)-axis is tangent to the beam in D, which corresponds to its origin \( s = 0 \). In regime 2, it makes an angle \( \alpha_s \) with the substrate. The beam deflection \( y(s) \) is measured perpendicularly to this \( s \)-axis. The local beam slope is \( \varphi(s) = \arctan(dy/ds) \). By definition, \( y(0) = \varphi(0) = 0 \). The clamp is at position \( (s_c, y_c) \) and the beam slope satisfies \( \varphi(s_c) = \alpha_c - \alpha_d \).

The forces that apply on the beam were described in the main text. They are recalled in the schematics of figure 1. The reaction force in D can be projected in the \( (s, y) \) coordinates: \( N_0 = N_d \cos \alpha_d - T_d \sin \alpha_d \) along \( y \) and \( T_0 = N_d \sin \alpha_d + T_d \cos \alpha_d \) along \( s \). Therefore, \( T_0 = N_0 \frac{\tan \alpha_d + \mu}{1 - \mu \tan \alpha_d} \).

As the beam is a slender body, it always prefers bending instead of stretching, so its deflection \( y(s) \) satisfies the Euler-Bernoulli’s equation, \( B \kappa(s) = M(s) \) where \( \kappa(s) \) is the local beam curvature and \( M(s) \) the moment at abscissa \( s \). In the wet part between D and W, \( \kappa(s) = 0 \), and \( M(s) = M_d + N_0 s - T_0 y - \int_0^s \frac{\sigma W}{R} \left[ (s-s') ds' + (y-y') dy' \right] \).

**Fig. 1** Two-dimensional schematics of the beam (green), the clamp (orange), the substrate (apricot) and the capillary bridge (blue). (a) Regime 2, represented with an inclination \( \alpha_s \), and (b) regime 3. Forces and moments applied to the beam are represented in red. The main variables of the model are indicated.

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while in the dry part between W and C,
\[ M(s) = M_d + N_0 s - T_0 y - \int_0^s \frac{\sigma W}{R} [(s - s') ds' + (y - y') dy'] \]
- \( \sigma W (s - s_w) \sin(\theta_b + \varphi_w) \)
+ \( \sigma W (y - y_w) \cos(\theta_b + \varphi_w) \)  
(3)
where \( s_w, y_w \) and \( \varphi_w \) are the position, deflection and inclination at point W.

We consider small beam deflections, i.e. \( y^2 \ll s^2 \) and \( \tan \phi = dy/ds \ll 1 \), so the bending curvature can be approximated by \( d^2 y / ds^2 \) and the Euler-Bernoulli equation simplifies into:
\[ \frac{d^2 y}{ds^2} + \frac{T_0}{B} y = \frac{M_d}{B} + \frac{N_0}{B} s - \frac{s^2}{2R} \]
(4)
for the wet part, and
\[ \frac{d^2 y}{ds^2} + \frac{T_0}{B} y = \frac{M_d}{B} + \frac{N_0}{B} s - \frac{s^2}{2R} \frac{\ell^2}{s^2} \frac{w}{s^2} \frac{\cos(\theta_b + \varphi_w)}{2R} \]
(5)
for the dry part.

Besides boundary conditions, the beam is subjected to three geometrical constraints. Firstly, as it cannot stretch, its length \( L \) should satisfy
\[ L = d + \int_0^x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, ds \approx d + s_c + \frac{1}{2} \int_0^x \frac{1}{2} \tan^2 \phi \, ds \]
(6)
Secondly, the circular liquid-air interface of the capillary bridge should connect to both the substrate and the beam with contact angles \( \theta_b \) and \( \theta_s \), respectively. This is satisfied if
\[ z_w = s_w \sin \alpha_d + y_w \cos \alpha_d = R [\cos \theta_s + \cos(\theta_b + \alpha_d + \varphi_w)] \]
(7)
And thirdly, the volume of liquid per unit width \( V/W = \Omega L^2 \), given by
\[ \Omega L^2 \approx \int_0^{s_w} z \, dx + \frac{R^2}{2} [\theta_s + \theta_b + \alpha_d + \varphi_w - \pi] \]
+ \( \frac{R^2}{2} \sin(\theta_b + \alpha_d + \varphi_w) [2 \cos \theta_s + \cos(\theta_b + \alpha_d + \varphi_w)] \]
- \( \frac{R^2}{2} \cos \theta_s \sin \theta_s \)
(8)
should remain constant, where \( x = s \cos \alpha_d - y \sin \alpha_d \) and
\[ \int_0^{s_w} z \, dx = \frac{s_w^2 - y_w^2}{2} \sin \alpha_d \cos \alpha_d \]
+ \( \int_0^{s_w} [y \cos^2 \alpha_d - s \frac{dy}{ds} \sin^2 \alpha_d] \, ds \).
(9)

The reaction forces and moment at the clamp are given by
\[ \frac{N_c}{B} = \frac{N_d}{B} \frac{s_w}{s_w^2} - \frac{1}{2} \frac{\sin(\theta_b + \alpha_d + \varphi_w)}{\pi} \]
\[ \frac{T_c}{B} = \frac{T_d}{B} + \frac{z_w}{R} \frac{\ell^2}{s_w^2} - \frac{1}{2} \frac{\cos(\theta_b + \alpha_d + \varphi_w)}{\pi} \]
\[ M_c = \frac{d^2 y}{dx^2} \frac{s_w}{s_w^2} \]
(10)

1.1 Wet part
1.1.1 Regime 2:
In this regime, \( M_d = 0 \) and, in the limit of low friction where \( T_0 \ll B \), the wet part obeys
\[ \frac{d^2 y}{ds^2} = \frac{N_0}{B} s - \frac{s^2}{2R} \]
(11)
The solution to this equation that satisfies \( y = dy/ds = 0 \) in \( s = 0 \) is
\[ \frac{dy}{ds} = -\frac{N_0}{2B} s^3 \]
(12)
\[ y = -\frac{N_0}{6B} s^4 \]
(13)
Since \( dy/ds = \tan \varphi_w \) in \( s = s_w \), equations 12 and 13 can be rewritten in \( s_w \) as:
\[ \frac{s_w}{6B} \frac{\tan \varphi_w}{s_w} = \frac{N_0}{6B} \frac{\tan \varphi_w}{s_w} \]
(14)
\[ y_w = \frac{N_0}{2B} \frac{s_w}{s_w} \frac{\tan \varphi_w}{s_w} \]
(15)
Combining them with the geometrical condition on the liquid-air interface (eq. 7) yields:
\[ \left( \frac{N_0}{2B} \right)^2 + 2 \left( \frac{\tan \varphi_w}{s_w} + 6 \frac{\tan \alpha_d}{s_w} \right) \frac{N_0}{2B} s_w - 3 \frac{\tan \varphi_w}{s_w} + 4 \frac{\tan \alpha_d}{s_w} = 0 \]
(16)
This quadratic equation can be solved for \( N_0 s_w / (2B) \). The discriminant \( \rho \) satisfies
\[ \rho^2 = \frac{4}{s_w^2} (\tan \varphi_w + 3 \tan \alpha_d)^2 + 2 \frac{\cos \theta_b + \cos(\theta_b + \alpha_d + \varphi_w)}{\ell^2 \cos \alpha_d} \]
(16)
Only the solution \( N_0 > 0 \) is kept since the beam touches the substrate in regime 2, namely
\[ \frac{N_0 s_w}{2B} = \frac{\rho - \frac{\tan \varphi_w + 6 \tan \alpha_d}{s_w}} \]
(17)
Then,
\[ \frac{s^2_w}{6B^2} = \frac{\rho - 2 \frac{\tan \varphi_w + 6 \tan \alpha_d}{s_w}} \]
(18)
\[ y_w = \frac{s_w^2}{12} \left[ \rho + 2 \tan \phi_w - 6 \tan \alpha_d \right] \]  
\[ \text{The liquid volume is evaluated by considering that} \]  
\[ \int_0^{s_w} \frac{dyds}{ds} = \frac{s_w^2}{60} \left( 2\rho + \frac{\tan \phi_w - 12 \tan \alpha_d}{s_w} \right) \]  
\[ \int_0^{s_w} \frac{dyds}{ds} = \frac{s_w^2}{20} \left( \rho + \frac{3 \tan \phi_w - 6 \tan \alpha_d}{s_w} \right) \]  
\[ \text{The wet beam length is} \]  
\[ L_w = s_w + \frac{s_w^2}{504t^4R^2} \frac{N_0d^6}{72B^2R^2} + \frac{N_0^2s_w^6}{40B^2} \]  
\[ \text{1.1.2 Regime 3:} \]  
In this regime, \( \alpha_d = 0 \) which implies \( N_0 = N_d \), and \( T_0 = 0 \). The wet part obeys  
\[ \frac{d^2y}{dx^2} = (s + md) \frac{N_d}{B} - \frac{1}{2R^2} \left[ s^2 + d^2(1 - 2m) \right] \]  
\[ \text{The solution to this equation that satisfies} \]  
\[ \frac{dy}{ds} = \frac{N_d}{2B} \left( s^2 + 2mds \right) - \frac{s^3 + 3d^2(1 - 2m)s^2}{6L^2R} \]  
\[ y = \frac{N_d}{6B} \left( s^2 + 3md^2s^2 \right) - \frac{s^4 + 6d^2(1 - 2m)s^2}{24L^2R} \]  
\[ \text{For the sake of simplifying notations, we define} \]  
\[ a = \frac{md}{s_w}, \quad b = \left( 1 - 2m \right) \frac{d^2}{s_w^2} \]  
\[ \text{Since} dy/ds = \tan \phi_w \text{ in } s = s_w, \text{equations 23 and 24 can be rewritten in } s_w \text{ as:} \]  
\[ (1 + 3b) \frac{s_w^2}{6L^2R} = (1 + 2a) \frac{N_ds_w}{2B} - \frac{\tan \phi_w}{s_w} \]  
\[ 4(1 + 3b) \frac{\tan \phi_w}{s_w} = (1 + 6a - 6b) \frac{N_ds_w}{6B} \]  
\[ + (1 + 6b) \frac{\tan \phi_w}{s_w} \]  
\[ \text{Combining them with the geometrical condition on the liquid-air interface (eq. 7) yields:} \]  
\[ (1 + 6a - 6b)(1 + 2a) \left( \frac{N_ds_w}{2B} \right)^2 \]  
\[ + 2(1 + 12b + 18ab) \frac{\tan \phi_w}{s_w} N_ds_w \]  
\[ - 3 \frac{\tan^2 \phi_w}{s_w}(1 + 6b) \]  
\[ - 2(1 + 3b) \frac{\cos \theta_b + \cos(\theta_b + \phi_w)}{\ell^2} \]  
\[ = 0 \]  
This quadratic equation can be solved for \( N_d s_w/(2B) \). The discriminant is then  
\[ \rho^2 = 4(1 + 3a)^2(1 + 3b)^2 \frac{\tan^2 \phi_w}{s_w} + 2(1 + 3b)(1 + 2a)(1 + 6a - 6b) \frac{\cos \theta_b + \cos(\theta_b + \phi_w)}{\ell^2} \]  
and only the solution \( N_d > 0 \) is kept.  
\[ \text{The liquid volume is evaluated by considering} \]  
\[ \int_0^{s_w} \frac{dyds}{ds} = \frac{N_d s_w^2}{4B} (1 + 4a) - \frac{s_w^5}{120 \ell^2 R} (1 + 10b) \]  
\[ \text{The wet beam length is} \]  
\[ L_w = s_w + \frac{s_w^2}{504t^4R^2} \frac{N_d^6d^6}{72B^2R^2} \]  
\[ + \left( \frac{N_d^2}{4B^2} \frac{M_d}{3B^2R^2} \right) s_w \]  
\[ + \frac{N_d M_d s_w^6}{8B^2} + \frac{M_d^2 s_w^6}{6B^2} \]  
where we recall that  
\[ \frac{M_d}{B} = \frac{N_d}{B} \]  
\[ - \frac{d^2(1 - 2m)}{2R^2} = a \]  
\[ - \frac{b s_w^2}{2R^2} \]  
\[ 1.2 \text{ Dry part} \]  
We define  
\[ K = \frac{T_0}{B} - \frac{\cos(\theta_b + \phi_w)}{\ell^2} \]  
If \( K > 0 \), then \( k = \sqrt{K} \) and Euler-Bernoulli's differential equation becomes  
\[ \frac{d^2y}{dx^2} + k^2y = Gs + H \]  
where  
\[ G = \frac{N_0}{B} - \frac{s_w}{\ell^2} \frac{\sin(\theta_b + \phi_w)}{\ell^2} \]  
\[ H = \frac{M_d}{B} + \frac{s_w}{2\ell^2} \]  
\[ + \frac{s_w \sin(\theta_b + \phi_w)}{\ell^2} - \frac{y_w \cos(\theta_b + \phi_w)}{\ell^2} \]  
The solution that satisfies boundary conditions at the contact line is:  
\[ y(x) - y_w = \frac{t}{k} \tan \phi_w = \frac{C_1}{K} \]  
\[ \tan \phi(x) - \tan \phi_w = \frac{C_1 k}{K} \]  
\[ t - \sin t \]  
\[ \tan \phi(x) - \tan \phi_w = \frac{C_1 k}{K} \sin t + \frac{C_2}{K} (1 - \cos t) \]  
where \( t = k(s - s_w) \) and  
\[ C_1 = \frac{M_d}{B} + \frac{N_0}{B} s_w \]  
\[ - \frac{T_0}{B} y_w - \frac{s_w^2}{2\ell^2 R} \]  
\[ C_2 = \frac{N_0}{B} \]  
\[ - \frac{T_0}{B} \tan \phi_w - \frac{s_w}{\ell^2} \frac{\sin \theta_b}{\ell^2} \]  
\[ \cos \phi_w \]
The dry length $L_d$ between points W and C is given by
\begin{equation}
L_d = \left(1 + \frac{1}{2} \tan^2 \varphi_w \right) \frac{L_c}{K} + \frac{C_1}{K} \left(1 - \cos t_c \right) \right)^2 \\
+ \frac{C_1^2}{8K^2} (2t_c - \sin 2t_c) + \frac{C_2^2}{8K^2} (6t_c - 8 \sin t_c + \sin 2t_c) \\
+ \frac{C_1 \tan \varphi_w}{K} (1 - \cos t_c) + \frac{C_2 \tan \varphi_w}{K} (t_c - \sin t_c) \\
(38)
\end{equation}

with $t_c = k(s_c - s_w)$. The clamp position $s_c$ is determined through the non-stretching condition $L = d + L_w + L_d$. Finally, the clamping condition $\varphi(s_c) = \alpha_c - \alpha_d$ needs to be imposed.

If $K < 0$, then $k = \sqrt{-K}$ and
\begin{equation}
\frac{d^2 y}{dx^2} - K^2 y = G s + H \\
(39)
\end{equation}

The solution that satisfies boundary conditions at the contact line is:
\begin{equation}
y(s) - y_w - \frac{t}{k} \tan \varphi_w = \frac{C_1}{K} (1 - \cosh t) + \frac{C_2}{K} (t - \sinh t) \\
\tan \varphi(s) - \tan \varphi_w = - \frac{C_1 k}{K} \sinh t + \frac{C_2}{K} (1 - \cosh t) \quad (40)
\end{equation}

where again $t = k(s - s_w)$.

The dry length is then given by
\begin{equation}
L_d = \left(1 + \frac{1}{2} \tan^2 \varphi_w \right) \frac{L_c}{K} + \frac{C_1}{K} \left(1 - \cosh t_c \right)^2 \\
+ \frac{C_1^2}{8K^2} (2t_c - \sinh 2t_c) + \frac{C_2^2}{8K^2} (6t_c - 8 \sinh t_c + \sinh 2t_c) \\
+ \frac{C_1 \tan \varphi_w}{K} (1 - \cosh t_c) + \frac{C_2 \tan \varphi_w}{K} (t_c - \sinh t_c) \\
(41)
\end{equation}

1.3 Solving the system of equations

In the previous section, the model of beam deflection has been progressively reduced from a system of differential equations with boundary conditions to a system of non-linear algebraic equations. This latter has been solved iteratively in Matlab according to the following procedure:

1. For a given value of $\varphi_w$ (here chosen within $[-5^\circ, 12^\circ]$),
   (a) Consider a range of values for $s_w$ (here chosen between $10^{-4}L$ and $L$),
   (b) Calculate $\rho(s_w)$, $N_0(s_w)$, $R(s_w)$, $y_w(s_w)$ and $V(s_w)$ according to section 1.1. In regime 3, both solutions $\pm \rho$ should be considered, and only the one yielding $V > 0$ and $y_w > 0$ is kept.
   (c) Find $s_w$ that yields the desired liquid volume $V$ [the function $V(s_w)$ is monotonic].
   (d) Find the clamp position $s_c$ that yields the desired beam length $L = d + L_w + L_d$ [the function $L(s_c)$ is monotonic]. Deduce $y_c$ and $\varphi_c$.
2. Loop on $\varphi_w$ (i.e., repeat step 1) until the boundary condition $\varphi_c = \alpha_c - \alpha_d$ at the clamp is satisfied. A bisection method was adopted, as there may be several solutions and no guarantee of convergence. Only the less deformed solution (i.e. the solution of smallest $|\varphi_c|$) was kept.

3. Calculate the reaction forces at the clamp $N_c$, $T_c$ and $M_c$.

This process involves one main loop ($\varphi_w$) and two independent secondary loops ($s_w$ and $s_c$). It is therefore much simpler and less time consuming to solve than the initial model based on the Euler-Bernoulli differential equation with several boundary conditions and geometrical constraints.

2 Alternative hypothesis of constant $M_d$

The model of regime 3 is based on the hypothesis of a distributed reaction force through the parameter $m$. In the previous models of elastocapillary adhesion, the reaction force was assumed to be localized in D and constant (possibly equal to zero). Following a similar approach to section 1.1 with this new hypothesis on $M_d$, we find
\begin{equation}
\rho^2 = \left( \frac{M_d}{B} + \frac{2\tan \varphi_w}{s_w} \right)^2 + 2 \cos \theta_a + \cos(\theta_b + \varphi_w) \frac{L_c}{B} + \frac{2\tan \varphi_w}{s_w} \\
\frac{N_0 s_w}{2B} = \rho - \frac{2M_d}{B} - \frac{\tan \varphi_w}{s_w} \\
\frac{N_0^2}{6B^2} = \frac{M_d}{B} - \frac{2\tan \varphi_w}{s_w} \\
(42)
\end{equation}

and a deflection
\begin{equation}
y = \frac{M_d}{2B} s^2 + \frac{N_0}{6B} s^3 - \frac{s^4}{24B^2} \\
(43)
\end{equation}
in the wet zone.

3 Dry adhesion in regime 3

In the absence of a liquid bridge, the aforementioned equations are greatly simplified. If we further assume that there is still no friction, the Euler-Bernoulli equation becomes
\begin{equation}
B \frac{d^2 y}{dx^2} = N_d s + M_d. \quad (44)
\end{equation}

The beam deflection is then
\begin{equation}
y(s) = (-2y_c + \alpha_c s_c) \left( \frac{s}{s_c} \right)^3 + (3y_c - \alpha_c s_c) \left( \frac{s}{s_c} \right)^2, \quad (45)
\end{equation}

where $s_c = L - d$. It satisfies $y(0) = y'(0) = 0$, $y(s_c) = y_c$ and $y'(s_c) = \alpha_c$. The curvature is
\begin{equation}
y'' = \left( -12 \frac{y_c}{s_c^2} + 6 \frac{\alpha_c}{s_c} \right) \frac{s}{s_c} + \left( 6 \frac{y_c}{s_c} - 2 \frac{\alpha_c}{s_c} \right). \quad (46)
\end{equation}

from which we infer
\begin{equation}
N_d = \frac{B}{s_c} \left( -12 \frac{y_c}{s_c^2} + 6 \frac{\alpha_c}{s_c} \right) \quad \text{and} \quad M_d = B \left( 6 \frac{y_c}{s_c} - 2 \frac{\alpha_c}{s_c} \right). \quad (47)
\end{equation}

As the beam shall not penetrate the underlying substrate, $y''(0) \geq 0$, which yields
\begin{equation}
s_c \leq \frac{3y_c}{\alpha_c}. \quad (48)
\end{equation}
We may here attempt to determine \( s_c \) (and so \( d \)) through energy arguments. The internal bending energy of the beam is

\[
U = \int_0^s \frac{B}{2} (y'')^2 \, dx = \frac{2B}{s_c} \left[ \frac{3y_c^2}{s_c^2} - \frac{3y_c \alpha_c}{s_c} + \alpha_c^2 \right].
\]  

(49)

The external work of the loads is

\[
W = -\alpha_c M_c + y_c N_c.
\]  

(50)

If the beam is free to slide along the substrate, there is no external force in the \( s \) direction so \( W \) is independent of \( s_c \). We may consider an additional adhesive energy \( E_a = -\xi (L - s_c) \).

The total potential energy is therefore

\[
\Pi(y_c, \alpha_c, s_c) = U + W + E_a
\]  

(51)

\[
= \frac{2B}{s_c} \left[ \frac{3y_c^2}{s_c^2} - \frac{3y_c \alpha_c}{s_c} + \alpha_c^2 \right] - \alpha_c M_c + y_c N_c - \xi (L - s_c).
\]

It should be minimum regarding the three possible displacements \( y_c, \alpha_c \) and \( s_c \):

\[
\frac{\partial \Pi}{\partial y_c} = 0 \Rightarrow \frac{N_c}{B} = -2 \frac{y_c}{s_c^2} + 6 \frac{\alpha_c}{s_c}
\]  

(52)

\[
\frac{\partial \Pi}{\partial \alpha_c} = 0 \Rightarrow \frac{M_c}{B} = -6 \frac{y_c}{s_c^2} + 4 \frac{\alpha_c}{s_c}
\]  

(53)

\[
\frac{\partial \Pi}{\partial s_c} = 0 \Rightarrow \sqrt{\frac{\xi B}{2}} \frac{y_c}{s_c} + \alpha_c s_c - 3y_c = 0.
\]  

(54)

This latter equation can only be satisfied in the adhesive regime \( (\xi > 0) \). If the solid-solid interaction is not energetically favourable \( (\xi < 0) \), then \( \frac{\partial \Pi}{\partial s_c} < 0 \) and the minimum is found when \( s_c = 1 \), i.e. when the contact area between the beam and the substrate is reduced to 0.

For \( \xi > 0 \), an equilibrium in \( s_c < 1 \) satisfying equation (54) is necessarily stable since

\[
\frac{\partial^2 \Pi}{\partial s_c^2} = 6B \left[ (3y_c - \alpha_c s_c)^2 + 3y_c^2 \right] > 0.
\]  

(55)

The solution \( s_c \) to Eq. (54) is in the range \( [0, L] \) when

\[
y_c < \frac{\alpha_c L}{3} + \frac{L^2}{3} \sqrt{\frac{\xi}{2B}}.
\]  

(56)

We note that \( M_d = \sqrt{2\xi B} \) is constant.