Vibrational and coherence dynamics in molecules: Supplementary Information

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I. DERIVATION OF QUANTUM MASTER EQUATION

To derive the operator master equation in Eq.(2) in main text, we first need to diagonalize the system Hamiltonian by Bogoliubov transform: $H_s = E_1\eta_1^\dagger\eta_1 + E_2\eta_2^\dagger\eta_2$ where

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

and the mixture angle

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\Delta}{\varepsilon_1 - \varepsilon_2}\right), \quad \frac{\pi}{2} < \theta < \frac{3\pi}{4} \tag{2}$$

thus the interaction to T_1-bath is

$$\begin{aligned} H_{int}^{(1)} &= (-i)\sum_{i=1}^{2}\sum_{\nu=1}^{2}\left(U_{i\nu}\eta_\nu - U_{\nu i}^\dagger\eta_\nu^\dagger\right)\otimes B = \sum_{i=1}^{2}\sum_{\nu=1}^{2}\sum_{\omega_\nu} A_{i\nu}(\omega_\nu)\otimes B \\ B &= i\sum_{\mathbf{k},\sigma} g_{\mathbf{k}\sigma}\left(b_{\mathbf{k}\sigma}^{(1)} e^{i\mathbf{k}\cdot\mathbf{r}} - b_{\mathbf{k}\sigma}^{(1),\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}}\right) \end{aligned} \tag{3}$$

where the operator $A_{i\nu}(\omega_\nu)$ takes the form of

$$\begin{aligned} A_{i\nu}(\omega_\nu > 0) &= -iU_{i\nu}\eta_\nu, \quad A_{i\nu}^\dagger(\omega_\nu > 0) = iU_{\nu i}^\dagger\eta_\nu^\dagger \\ A_{i\nu}(\omega_\nu < 0) &= iU_{\nu i}^\dagger\eta_\nu^\dagger, \quad A_{i\nu}^\dagger(\omega_\nu < 0) = -iU_{i\nu}\eta_\nu \end{aligned} \tag{4}$$

Switching into interaction picture, the interaction reads

$$\begin{aligned} \tilde{H}_{int}^{(1)}(t) &= \sum_{i=1}^{2}\sum_{\nu=1}^{2}\sum_{\omega_\nu} A_{i\nu}(\omega_\nu)e^{-i\omega_\nu t}\otimes B(t) \\ B(t) &= i\sum_{\mathbf{k},\sigma} g_{\mathbf{k}\sigma}\left[b_{\mathbf{k}\sigma}^{(1)} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}\sigma}t)} - b_{\mathbf{k}\sigma}^{(1),\dagger} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}\sigma}t)}\right] \end{aligned} \tag{5}$$

and subsequently the correlation function of the environment can be obtained

$$\langle B(t)B(t-s)\rangle = \sum_{\mathbf{k},\sigma} g_{\mathbf{k}\sigma}^2\left[n_{\mathbf{k}\sigma}e^{i\omega_{\mathbf{k}\sigma}s} + (n_{\mathbf{k}\sigma}+1)e^{-i\omega_{\mathbf{k}\sigma}s}\right] \tag{6}$$

As is known, the general form of the dynamical equation of density matrix up to the 2nd order of system-bath interaction is

$$\frac{d\rho_s}{dt} = \frac{i}{\hbar}[\rho_s, H_s] - \frac{1}{\hbar^2}e^{-iH_st/\hbar}\mathrm{Tr}_R\int_0^t ds\ [\tilde{H}_{int}(s), [\tilde{H}_{int}(t), \tilde{\rho}_s(t)\otimes\rho_R(0)]]e^{iH_st/\hbar} \tag{7}$$

which, in terms of Eq.(4) gives the dissipation term contributed by the T_1-bath

$$\begin{aligned}
D^{(1)}(\rho_s) &= \frac{1}{\hbar^2}\mathrm{Tr}_R \int_0^t \left(\tilde{H}_{int}^{(1)}(t-s)\tilde{\rho}_s(t)\rho_R \tilde{H}_{int}^{(1)}(t) - \tilde{H}_{int}^{(1)}(t)\tilde{H}_{int}^{(1)}(t-s)\tilde{\rho}_s(t)\rho_R\right) ds + \text{h.c.} \\
&= \sum_{i,j=1}^{2}\sum_{\nu,\mu=1}^{2}\sum_{\omega_\nu,\omega_\mu} e^{i(\omega_\mu-\omega_\nu)t}\left[A_{i\nu}(\omega_\nu)\rho_s(t)A_{j\mu}^\dagger(\omega_\mu) - A_{j\mu}^\dagger(\omega_\mu)A_{i\nu}(\omega_\nu)\rho_s(t)\right] \times \int_0^t e^{i\omega_\nu s}\langle B(t)B(t-s)\rangle ds + \text{h.c.} \\
&= \frac{1}{\hbar^2}\sum_{i,j=1}^{2}\sum_{\nu,\mu=1}^{2}\sum_{\omega_\nu}\sum_{\omega_\mu} e^{i(\omega_\mu-\omega_\nu)t}\Gamma^1(\omega_\nu)\left[A_{i\nu}(\omega_\nu)\rho_s(t)A_{j\mu}^\dagger(\omega_\mu) - A_{j\mu}^\dagger(\omega_\mu)A_{i\nu}(\omega_\nu)\rho_s(t)\right] + \text{h.c.}
\end{aligned} \tag{8}$$

where

$$\Gamma^1(\omega_\nu) = \int_0^t e^{i\omega_\nu s}\langle B(t)B(t-s)\rangle ds = \begin{cases} \pi\sum_{\mathbf{k},\sigma} g_{\mathbf{k}\sigma}^2 n_{\mathbf{k}\sigma}^{T_1}\delta(\omega_\nu+\omega_{\mathbf{k}\sigma}) = \hbar^2\gamma n_\nu^{T_1}, \quad \text{when} \quad \omega_\nu<0 \\ \pi\sum_{\mathbf{k},\sigma} g_{\mathbf{k}\sigma}^2 (n_{\mathbf{k}\sigma}^{T_1}+1)\delta(\omega_\nu-\omega_{\mathbf{k}\sigma}) = \hbar^2\gamma (n_\nu^{T_1}+1), \quad \text{when} \quad \omega_\nu>0 \end{cases} \tag{9}$$

Notice that $\gamma = \pi g_{\bar{\nu}}^2 D_{\bar{\nu}}/\hbar^2$, $D_{\bar{\nu}}$ is the density of states (DOS) and the summation over wave vector was replaced by the integral in frequency domain

$$\sum_{\mathbf{k},\sigma} \longrightarrow \int D(\omega_{\mathbf{k}\sigma})\mathrm{d}\omega_{\mathbf{k}\sigma} \tag{10}$$

Under the rotating-wave approximation (RWA), the dissipation term in Schrödinger picture then reads

$$\begin{aligned}
D^{(1)}(\rho_s) &= \frac{1}{\hbar^2}\sum_{i,j=1}^{2}\sum_{\nu,\mu=1}^{2}\Big\{\Gamma^1(\omega_\nu>0)U_{i\nu}U_{\mu j}^\dagger\left[\eta_\nu\rho_s(t)\eta_\mu^\dagger - \eta_\mu^\dagger\eta_\nu\rho_s(t)\right] \\
&\qquad + \Gamma^1(\omega_\nu<0)U_{\nu i}^\dagger U_{j\mu}\left[\eta_\nu^\dagger\rho_s(t)\eta_\mu - \eta_\mu\eta_\nu^\dagger\rho_s(t)\right]\Big\} + \text{h.c.} \\
&= \sum_{j=1}^{2}\sum_{p=1}^{2}\left[\gamma_p^{T_1,+}\left(a_p\rho_s(t)a_j^\dagger - a_j^\dagger a_p\rho_s(t)\right) + \gamma_p^{T_1,-}\left(a_p^\dagger\rho_s(t)a_j - a_j a_p^\dagger\rho_s(t)\right)\right] + \text{h.c.}
\end{aligned} \tag{11}$$

For the interaction to T_2-bath

$$\begin{aligned}
H_{int}^{(2)} &= (-i)\sum_{\nu=1}^{2}\left(U_{2\nu}\eta_\nu - U_{\nu 2}^\dagger\eta_\nu^\dagger\right)\otimes C = \sum_{\nu=1}^{2}\sum_{\omega_\nu} A_{2\nu}(\omega_\nu)\otimes C \\
C &= i\sum_{\mathbf{q},s} f_{\mathbf{q}s}\left(b_{\mathbf{q}s}^{(2)}e^{i\mathbf{q}\cdot\mathbf{r}} - b_{\mathbf{q}s}^{(2),\dagger}e^{-i\mathbf{q}\cdot\mathbf{r}}\right)
\end{aligned} \tag{12}$$

the procedures are similar as that for radiations and thus we omit the details to avoid redundancy. The dissipation term induced by T_2-bath then is of the form

$$D^{(2)}(\rho_s) = \sum_{p=1}^{2}\left[\gamma_p^{T_2,+}\left(a_p\rho_s(t)a_2^\dagger - a_2^\dagger a_p\rho_s(t)\right) + \gamma_p^{T_2,-}\left(a_p^\dagger\rho_s(t)a_2 - a_2 a_p^\dagger\rho_s(t)\right)\right] + \text{h.c.} \tag{13}$$

therefore the whole dissipation term contributed by environments reads

$$\begin{aligned}
D(\rho_s) &= D^{(1)}(\rho_s) + D^{(2)}(\rho_s) \\
&= \sum_{j=1}^{2}\sum_{p=1}^{2}\left[\gamma_p^{T_1,+}\left(a_p\rho_s(t)a_j^\dagger - a_j^\dagger a_p\rho_s(t)\right) + \gamma_p^{T_1,-}\left(a_p^\dagger\rho_s(t)a_j - a_j a_p^\dagger\rho_s(t)\right)\right] \\
&\quad + \sum_{p=1}^{2}\left[\gamma_p^{T_2,+}\left(a_p\rho_s(t)a_2^\dagger - a_2^\dagger a_p\rho_s(t)\right) + \gamma_p^{T_2,-}\left(a_p^\dagger\rho_s(t)a_2 - a_2 a_p^\dagger\rho_s(t)\right)\right] + \text{h.c.}
\end{aligned} \tag{14}$$

which gives the expression in Eq.(2) in main text. The expressions of dissipation rates in Eq.(13) are

$$
\begin{aligned}
\gamma_1^{T_1,+} &= \gamma\hbar^2\left[n_{\nu_1}^{T_1}\cos^2\theta + n_{\nu_2}^{T_1}\sin^2\theta + 1 + (n_{\nu_1}^{T_1} - n_{\nu_2}^{T_1})\sin\theta\cos\theta\right]\\
\gamma_2^{T_1,+} &= \gamma\hbar^2\left[n_{\nu_1}^{T_1}\sin^2\theta + n_{\nu_2}^{T_1}\cos^2\theta + 1 + (n_{\nu_1}^{T_1} - n_{\nu_2}^{T_1})\sin\theta\cos\theta\right]\\
\gamma_1^{T_1,-} &= \gamma\hbar^2\left[n_{\nu_1}^{T_1}\cos^2\theta + n_{\nu_2}^{T_1}\sin^2\theta + (n_{\nu_1}^{T_1} - n_{\nu_2}^{T_1})\sin\theta\cos\theta\right]\\
\gamma_2^{T_1,-} &= \gamma\hbar^2\left[n_{\nu_1}^{T_1}\sin^2\theta + n_{\nu_2}^{T_1}\cos^2\theta + (n_{\nu_1}^{T_1} - n_{\nu_2}^{T_1})\sin\theta\cos\theta\right]\\
\gamma_1^{T_2,+} &= \gamma\hbar^2(n_{\nu_1}^{T_2} - n_{\nu_2}^{T_2})\sin\theta\cos\theta, \quad \gamma_2^{T_2,+} = \gamma\hbar^2(n_{\nu_1}^{T_2}\sin^2\theta + n_{\nu_2}^{T_2}\cos^2\theta + 1)\\
\gamma_1^{T_2,-} &= \gamma\hbar^2(n_{\nu_1}^{T_2} - n_{\nu_2}^{T_2})\sin\theta\cos\theta, \quad \gamma_2^{T_2,-} = \gamma\hbar^2(n_{\nu_1}^{T_2}\sin^2\theta + n_{\nu_2}^{T_2}\cos^2\theta)
\end{aligned}
\tag{15}
$$

II. SOLVING THE DYNAMICAL EQUATION IN COHERENT SPACE

To solve the PDE in Eq.(5) in main text, we first calculate the biorthogonal eigenvectors of the drift matrix Σ and the results are

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}\left[\gamma(3+F) + i(\nu_1+\nu_2+\gamma G)\right], \quad u^{(1)} = \begin{pmatrix} 2(\epsilon+id) \\ p_+ + iq_+ \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)} = \begin{pmatrix} -\frac{p_- - iq_-}{4(\epsilon+id)(F+iG)} & \frac{1}{2(F+iG)} & 0 & 0 \end{pmatrix}\\
\lambda_2 &= \frac{1}{2}\left[\gamma(3-F) + i(\nu_1+\nu_2-\gamma G)\right], \quad u^{(2)} = \begin{pmatrix} 2(\epsilon+id) \\ p_- - iq_- \\ 0 \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \frac{p_+ + iq_+}{4(\epsilon+id)(F+iG)} & -\frac{1}{2(F+iG)} & 0 & 0 \end{pmatrix}\\
\lambda_3 &= \frac{1}{2}\left[\gamma(3+F) - i(\nu_1+\nu_2+\gamma G)\right], \quad u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 2(\epsilon-id) \\ p_+ - iq_+ \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 & 0 & -\frac{p_- + iq_-}{4(\epsilon-id)(F-iG)} & \frac{1}{2(F-iG)} \end{pmatrix}\\
\lambda_4 &= \frac{1}{2}\left[\gamma(3-F) - i(\nu_1+\nu_2-\gamma G)\right], \quad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 2(\epsilon-id) \\ p_- + iq_- \end{pmatrix}, \quad v^{(4)} = \begin{pmatrix} 0 & 0 & \frac{p_+ - iq_+}{4(\epsilon-id)(F-iG)} & -\frac{1}{2(F-iG)} \end{pmatrix}
\end{aligned}
\tag{16}
$$

then the σ-matrix is of the form

$$
\sigma_{ij}(t) = 2\sum_{\alpha,\beta\in R}\frac{1-e^{-(\lambda_\alpha+\lambda_\beta)t}}{\lambda_\alpha+\lambda_\beta}D^{(\alpha,\beta)}u_i^{(\alpha)}u_j^{(\beta)}, \quad D^{(\alpha,\beta)} = \sum_{k,l\in R}v_k^{(\alpha)}D_{kl}v_l^{(\beta)} \tag{17}
$$

where D is the diffusion matrix defined in Eq.(6) in main text. Hence from the literature the full solution to the PDE with respect to the initial condition $P(\alpha_\mu, \alpha_\mu^*, 0) = \delta^{(2)}(\alpha_1)\delta^{(2)}(\alpha_2)$ reads

$$
P(\alpha_\mu, \alpha_\mu^*, t) = \frac{1}{\sqrt{(2\pi)^4\det[\sigma(t)]}}\exp\left[-\frac{1}{2}\sum_{i,j\in R}\sigma_{ij}^{-1}(t)x_i x_j\right] \tag{18}
$$

where $R = \{1, 2, 1^*, 2^*\}$ and $x = \{\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*\}$. The expectation value of observable within normal order is

$$
\langle(a_1^\dagger)^m(a_2^\dagger)^n a_1^k a_2^l\rangle = \int(\alpha_1^*)^m(\alpha_2^*)^n\alpha_1^k\alpha_2^l P(\alpha_\mu, \alpha_\mu^*, t)d^2\alpha_1 d^2\alpha_2 \tag{19}
$$

The coefficients $A_{...}$ are

$$
\begin{aligned}
A^{13}_{11} &= \frac{1}{F^2+G^2}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}\frac{p_-^2+q_-^2}{2}+\frac{1-e^{-\gamma(3-F)t}}{3-F}\frac{p_+^2+q_+^2}{2}-\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}(p_+-iq_+)(p_--iq_-)\right]\right)\\
A^{13}_{22} &= \frac{2(\epsilon^2+d^2)}{F^2+G^2}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}+\frac{1-e^{-\gamma(3-F)t}}{3-F}-2\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\right]\right)\\
A^{13}_{1221} &= -\frac{\epsilon(\epsilon^2+d^2)}{F^2+G^2}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right]+\frac{1-e^{-\gamma(3-F)t}}{3-F}\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right]\right.\\
&\qquad\left.-\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\left(\frac{p_+-iq_+}{\epsilon-id}+\frac{p_--iq_-}{\epsilon+id}\right)\right]\right)\\
A^{14}_{11} &= \frac{\epsilon+id}{4(\epsilon^2+d^2)(F^2+G^2)}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}(p_-^2+q_-^2)(p_+-iq_+)+\frac{1-e^{-\gamma(3-F)t}}{3-F}(p_+^2+q_+^2)(p_-+iq_-)\right.\\
&\qquad\left.-\frac{1-e^{-\gamma(3+iG)t}}{3+iG}(p_-^2+q_-^2)(p_+-iq_+)-\frac{1-e^{-\gamma(3-iG)t}}{3-iG}(p_+^2+q_+^2)(p_-+iq_-)\right)\\
A^{14}_{22} &= \frac{\epsilon+id}{F^2+G^2}\left(\left[\frac{1-e^{-\gamma(3+F)t}}{3+F}-\frac{1-e^{-\gamma(3-iG)t}}{3-iG}\right](p_+-iq_+)+\left[\frac{1-e^{-\gamma(3-F)t}}{3-F}-\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\right](p_-+iq_-)\right)\\
A^{14}_{1221} &= -\frac{\epsilon+id}{2(F^2+G^2)}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right](p_+-iq_+)+\frac{1-e^{-\gamma(3-F)t}}{3-F}\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right](p_-+iq_-)\right.\\
&\qquad\left.-\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\left(\frac{p_+-iq_+}{\epsilon-id}+\frac{p_--iq_-}{\epsilon+id}\right)(p_-+iq_-)-\frac{1-e^{-\gamma(3-iG)t}}{3-iG}\left(\frac{p_++iq_+}{\epsilon+id}+\frac{p_-+iq_-}{\epsilon-id}\right)(p_+-iq_+)\right)\\
A^{24}_{11} &= \frac{(p_+^2+q_+^2)(p_-^2+q_-^2)}{8(\epsilon^2+d^2)(F^2+G^2)}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}+\frac{1-e^{-\gamma(3-F)t}}{3-F}-2\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\right]\right)\\
A^{24}_{22} &= \frac{1}{2(F^2+G^2)}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}(p_+^2+q_+^2)+\frac{1-e^{-\gamma(3-F)t}}{3-F}(p_-^2+q_-^2)-2\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}(p_++iq_+)(p_-+iq_-)\right]\right)\\
A^{24}_{1221} &= -\frac{\epsilon}{4(F^2+G^2)}\left(\frac{1-e^{-\gamma(3+F)t}}{3+F}\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right](p_+^2+q_+^2)+\frac{1-e^{-\gamma(3-F)t}}{3-F}\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right](p_-^2+q_-^2)\right.\\
&\qquad\left.-\mathrm{Re}\left[\frac{1-e^{-\gamma(3+iG)t}}{3+iG}\left(\frac{p_+-iq_+}{\epsilon-id}+\frac{p_--iq_-}{\epsilon+id}\right)(p_++iq_+)(p_-+iq_-)\right]\right)
\end{aligned}
\tag{20}
$$

The expressions of $\mathrm{I}^{\cdots}_{\cdots}$ in Eq.(18) in main text are

$$
\begin{aligned}
\mathcal{I}^{13}_{11} &= -\frac{1}{2(F^2+G^2)}\left(\frac{p_-^2+q_-^2}{(3+F)^2}+\frac{p_+^2+q_+^2}{(3-F)^2}-2\mathrm{Re}\left[\frac{(p_+-iq_+)(p_--iq_-)}{(3+iG)^2}\right]\right)\\
\mathcal{I}^{13}_{22} &= -\frac{2(\epsilon^2+d^2)}{F^2+G^2}\left[\frac{1}{(3+F)^2}+\frac{1}{(3-F)^2}-\frac{18-2G^2}{(9+G^2)^2}\right]\\
\mathcal{I}^{13}_{1221} &= \frac{\epsilon(\epsilon^2+d^2)}{F^2+G^2}\left(\frac{\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right]}{(3+F)^2}+\frac{\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right]}{(3-F)^2}-\mathrm{Re}\left[\frac{\frac{p_+-iq_+}{\epsilon-id}-\frac{p_--iq_-}{\epsilon+id}}{(3+iG)^2}\right]\right)\\
\mathcal{I}^{14}_{11} &= -\frac{1}{4(\epsilon-id)(F^2+G^2)}\left[(p_-^2+q_-^2)(p_+-iq_+)\left(\frac{1}{(3+F)^2}-\frac{1}{(3+iG)^2}\right)\right.\\
&\qquad\left.+(p_+^2+q_+^2)(p_-+iq_-)\left(\frac{1}{(3-F)^2}-\frac{1}{(3-iG)^2}\right)\right]\\
\mathcal{I}^{14}_{22} &= -\frac{\epsilon+id}{F^2+G^2}\left[\left(\frac{1}{(3+F)^2}-\frac{1}{(3-iG)^2}\right)(p_+-iq_+)+\left(\frac{1}{(3-F)^2}-\frac{1}{(3+iG)^2}\right)(p_-+iq_-)\right]\\
\mathcal{A}^{14}_{1221} &= \frac{\epsilon+id}{2(F^2+G^2)}\left(\frac{p_+-iq_+}{(3+F)^2}\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right]+\frac{p_-+iq_-}{(3-F)^2}\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right]\right.\\
&\qquad\left.-\frac{p_-+iq_-}{(3+iG)^2}\left(\frac{p_+-iq_+}{\epsilon-id}+\frac{p_--iq_-}{\epsilon+id}\right)-\frac{p_+-iq_+}{(3-iG)^2}\left(\frac{p_++iq_+}{\epsilon+id}+\frac{p_-+iq_-}{\epsilon-id}\right)\right)\\
\mathcal{I}^{24}_{11} &= -\frac{(p_+^2+q_+^2)(p_-^2+q_-^2)}{8(\epsilon^2+d^2)(F^2+G^2)}\left[\frac{1}{(3+F)^2}+\frac{1}{(3-F)^2}-\frac{18-2G^2}{(9+G^2)^2}\right]\\
\mathcal{I}^{24}_{22} &= -\frac{1}{2(F^2+G^2)}\left(\frac{p_+^2+q_+^2}{(3+F)^2}+\frac{p_-^2+q_-^2}{(3-F)^2}-2\mathrm{Re}\left[\frac{(p_++iq_+)(p_-+iq_-)}{(3+iG)^2}\right]\right)\\
\mathcal{I}^{24}_{1221} &= \frac{\epsilon}{4(F^2+G^2)}\left(\frac{p_+^2+q_+^2}{(3+F)^2}\mathrm{Re}\left[\frac{p_--iq_-}{\epsilon+id}\right]+\frac{p_-^2+q_-^2}{(3-F)^2}\mathrm{Re}\left[\frac{p_++iq_+}{\epsilon+id}\right]\right.\\
&\qquad\left.-\mathrm{Re}\left[\frac{(p_++iq_+)(p_-+iq_-)}{(3+iG)^2}\left(\frac{p_+-iq_+}{\epsilon-id}+\frac{p_--iq_-}{\epsilon+id}\right)\right]\right)
\end{aligned}
\tag{21}
$$