Appendix 6. A proof of eqn (78) :

$$L_c = (N_s - N_b)_c = \left[\frac{\sqrt{N_b} + \sqrt{N_b - \left[1 - \left(s^2(N_b) - N_b\right)\frac{k_\alpha^2}{4N_b^2}\right]\left[N_b - \frac{k_\alpha^2}{4}\left(1 + \frac{s^2(N_b)}{N_b}\right)\right]}}{\left[1 - \left(s^2(N_b) - N_b\right)\frac{k_\alpha^2}{4N_b^2}\right]} \right]^2 - N_b$$

for the computation of critical levels in the presence of an excess variance.

$$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > k_\alpha s\left(\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}}\right) \Rightarrow \sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > k_\alpha \sqrt{s^2\left(\sqrt{\frac{N_s}{t_s}}\right) + s^2\left(\sqrt{\frac{N_b}{t_b}}\right)}$$

$$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > k_\alpha \sqrt{\frac{s^2\left(\frac{N_s}{t_s}\right)}{4\left(\frac{N_s}{t_s}\right)} + \frac{s^2\left(\frac{N_b}{t_b}\right)}{4\left(\frac{N_b}{t_b}\right)}} \qquad (1)$$

The constrained maximum likelihood estimator of the mean count rate under the null hypothesis is

$$\frac{\overline{N_s}}{t_s} = \frac{\overline{N_b}}{t_b} = \frac{N_s + N_b}{t_s + t_b} \qquad (2)$$

Accordingly, we obtain: $$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > k_\alpha \sqrt{\frac{s^2\left(\frac{N_s}{t_s}\right) + s^2\left(\frac{N_b}{t_b}\right)}{4\frac{N_s + N_b}{t_s + t_b}}} \qquad (3)$$

For doubly stochastic Poisson statistics, we have:

$$s^2\left(\frac{N_s}{t_s}\right) = \frac{\overline{N}_s + p^2 s^2(M_s)}{t_s^2} = \frac{\overline{N_s}}{t_s^2} + \frac{p^2 s^2(M_s)}{t_s^2} = \frac{N_s + N_b}{t_s(t_s + t_b)} + \frac{p^2 s^2(M_s)}{t_s^2}$$

$$s^2\left(\frac{N_b}{t_b}\right) = \frac{\overline{N}_b + p^2 s^2(M_b)}{t_b^2} = \frac{\overline{N_b}}{t_b^2} + \frac{p^2 s^2(M_b)}{t_b^2} = \frac{N_s + N_b}{t_b(t_s + t_b)} + \frac{p^2 s^2(M_b)}{t_b^2} \qquad (4)$$

From inequality (3) and eqn (4), we derive:

$$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > k_\alpha \sqrt{\frac{\frac{N_s + N_b}{t_s(t_s + t_b)} + \frac{N_s + N_b}{t_b(t_s + t_b)} + \frac{p^2 s^2(M_s)}{t_s^2} + \frac{p^2 s^2(M_b)}{t_b^2}}{4\frac{N_s + N_b}{t_s + t_b}}}$$

$$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > \frac{k_\alpha}{2} \sqrt{\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \frac{t_s + t_b}{N_s + N_b}\left[\frac{p^2 s^2(M_s)}{t_s^2} + \frac{p^2 s^2(M_b)}{t_b^2}\right]} \qquad (5)$$

The maximum likelihood estimator of the mean rate for the primary population M under the null hypothesis is

$$\frac{M_s + M_b}{t_s + t_b} \qquad (6)$$

At this rate, the excess variances are estimated as follows:

$$p^2 s^2(M_s) = \left[s^2(N_b) - N_b\right]\left(\frac{t_s}{t_b}\right)\left(\frac{\frac{N_s + N_b}{t_s + t_b}}{\frac{N_b}{t_b}}\right)^2 = \left[s^2(N_b) - N_b\right]\left(\frac{t_s}{t_b}\right)\left(\frac{t_b}{t_s + t_b}\right)^2 \frac{(N_s + N_b)^2}{N_b^2} \qquad (7)$$

and

$$p^2 s^2(M_b) = \left[s^2(N_b) - N_b\right]\left(\frac{\frac{N_s + N_b}{t_s + t_b}}{\frac{N_b}{t_b}}\right)^2 = \left[s^2(N_b) - N_b\right]\left(\frac{t_b}{t_s + t_b}\right)^2 \frac{(N_s + N_b)^2}{N_b^2} \qquad (8)$$

Accordingly, under the radical of inequality (5), we obtain:

$$\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \frac{t_s + t_b}{N_s + N_b}\left[\frac{p^2 s^2(M_s)}{t_s^2} + \frac{p^2 s^2(M_b)}{t_b^2}\right] =$$

$$= \left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \frac{t_s + t_b}{N_s + N_b}\left[s^2(N_b) - N_b\right]\left[\frac{1}{t_s^2}\left(\frac{t_s}{t_b}\right)\left(\frac{t_b}{t_s + t_b}\right)^2 \frac{(N_s + N_b)^2}{N_b^2} + \frac{1}{t_b^2}\left(\frac{t_b}{t_s + t_b}\right)^2 \frac{(N_s + N_b)^2}{N_b^2}\right] = \qquad (9)$$

$$= \left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \left[s^2(N_b) - N_b\right]\frac{N_s + N_b}{N_b^2}\frac{1}{t_s}$$

Inequality (5) becomes as follows:

$$\sqrt{\frac{N_s}{t_s}} - \sqrt{\frac{N_b}{t_b}} > \frac{k_\alpha}{2}\sqrt{\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \left[s^2(N_b) - N_b\right]\frac{N_s + N_b}{N_b^2}\frac{1}{t_s}} \qquad (10)$$

The same is now convenient to re-write in the intensity-based notation:

$$\sqrt{I_s} - \sqrt{I_b} > \frac{k_\alpha}{2}\sqrt{\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \left[t_b^2 s^2(I_b) - t_b I_b\right]\frac{t_s I_s + t_b I_b}{t_b^2 I_b^2}\frac{1}{t_s}}$$

$$\sqrt{I_s} - \sqrt{I_b} > \frac{k_\alpha}{2}\sqrt{\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \frac{\left[t_b s^2(I_b) - I_b\right]}{t_b I_b^2} I_s + \frac{\left[t_b s^2(I_b) - I_b\right]}{t_s I_b}} \qquad (11)$$

At L_c, the left-hand and the right-hand part of this inequality are equal to each other, resulting in a quadratic equation relative to the argument $\sqrt{I_s}$:

$$I_s - 2\sqrt{I_b}\sqrt{I_s} + I_b = \frac{k_a^2}{4}\left[\left(\frac{1}{t_s} + \frac{1}{t_b}\right) + \frac{\left[t_b s^2(I_b) - I_b\right]}{t_b I_b^2} I_s + \frac{\left[t_b s^2(I_b) - I_b\right]}{t_s I_b}\right] \qquad (12)$$

i.e.,

$$\left[1 - \frac{k_a^2}{4}\frac{\left[t_b s^2(I_b) - I_b\right]}{t_b I_b^2}\right] I_s - 2\sqrt{I_b}\sqrt{I_s} + I_b - \frac{k_a^2}{4}\left[\frac{1}{t_b} + \frac{t_b s^2(I_b)}{t_s I_b}\right] = 0 \qquad (13)$$

Solving this equation, we obtain:

$$\sqrt{I_s} = \frac{\sqrt{I_b} + \sqrt{I_b - \left[1 - \frac{k_a^2}{4}\frac{\left[t_b s^2(I_b) - I_b\right]}{t_b I_b^2}\right]\left[I_b - \frac{k_a^2}{4}\left(\frac{1}{t_b} + \frac{t_b s^2(I_b)}{t_s I_b}\right)\right]}}{\left[1 - \frac{k_a^2}{4}\frac{\left[t_b s^2(I_b) - I_b\right]}{t_b I_b^2}\right]} \qquad (14)$$

or

$$L_c = (I_s - I_b)_c = \left[\frac{\sqrt{I_b} + \sqrt{I_b - \left[1 - \frac{k_\alpha^2}{4}\frac{\left(t_b s^2(I_b) - I_b\right)}{t_b I_b^2}\right]\left[I_b - \frac{k_\alpha^2}{4}\left(\frac{1}{t_b} + \frac{t_b s^2(I_b)}{t_s I_b}\right)\right]}}{\left[1 - \frac{k_\alpha^2}{4}\frac{\left(t_b s^2(I_b) - I_b\right)}{t_b I_b^2}\right]}\right]^2 - I_b$$