Electronic Supplementary Material (ESI) for Soft Matter.

Supplementary Materials

Numerical Implementation

The governing equation of this problem is equation (24):

$$2\pi\beta\bar{v}''(\bar{x}) = \int_1^{\infty} \bar{v}''(\bar{x})\ln\left|\bar{t}^2-\bar{x}^2\right|d\bar{t} + \left(2+\bar{x}\ln\left|\frac{\bar{x}-1}{\bar{x}+1}\right| - \ln\left|\bar{x}^2-1\right|\right), \quad |\bar{x}|>1$$

Which is a Fredholm integral equation of the second kind defined on the interval $(1,\infty)$. To solve the equation numerically, we first change the interval to be finite by taking $\bar{t}=\tan(\frac{p}{2}),\ \bar{x}=\tan(\frac{q}{2})$. Now, the integral is changed to:

$$\int_1^{\infty} \bar{v}''(\bar{t})\ln\left|\bar{t}^2-\bar{x}^2\right|d\bar{t} = \int_{\pi/2}^{\pi} \frac{\bar{v}''\left(\tan\left(\frac{p}{2}\right)\right)}{2\cos^2\left(\frac{p}{2}\right)}\ln\left|\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2\right|dp \quad \text{* MERGEFORMAT (A1)}$$

Define $F(p)=\dfrac{\bar{v}''\left(\tan\left(\frac{p}{2}\right)\right)}{2\cos^2\left(\frac{p}{2}\right)}$, $G(q)=2+\tan\left(\frac{q}{2}\right)\ln\left|\dfrac{\tan\left(\frac{q}{2}\right)-1}{\tan\left(\frac{q}{2}\right)+1}\right| - \ln\left|\tan^2\left(\frac{q}{2}\right)-1\right|$, equation (24) is changed into:

$$\int_{\pi/2}^{\pi} F(p)\ln\left|\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2\right|dp - 4\pi\beta\cos^2\left(\frac{q}{2}\right)F(q) = -G(q) \quad \text{* MERGEFORMAT (A2)}$$

The integral interval is now changed to $(\pi/2,\pi)$. Divide this interval into n subinterval $(p_1,p_2],\ldots,[p_i,p_{i+1}],\ldots,[p_n,p_{n+1})$ with equal length $\Delta=\pi/(2n)$ and take $q_i=\dfrac{p_i+p_{j+1}}{2}$, where $p_i=(i-1)\Delta$. Based on this discretization, the integral $\int_{\pi/2}^{\pi} F(p)\ln\left|\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2\right|dp$ can be evaluated approximately as:

$$\int_{\pi/2}^{\pi} F(\mathrm{p})\ln\left|\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2\right|du$$

$$=\int_{\pi/2}^{\pi} F(p)\ln\left|\frac{\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2}{p-q}\right|dp+\int_{\pi/2}^{\pi} F(\mathrm{p})\ln\left|p-q\right|dp \qquad \text{* MERGEFORMAT (A3)}$$

$$\approx\sum_{j=1}^{N}F(\mathrm{q}_j)\int_{p_j}^{p_{j+1}}\ln\left|\frac{\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q}{2}\right)^2}{p-q}\right|dp+\sum_{j=1}^{N}F(\mathrm{q}_j)\int_{p_j}^{p_{j+1}}\ln\left|p-q\right|dp$$

According to * MERGEFORMAT (A3), equation * MERGEFORMAT (A2) can be changed into discretized form:

$$\sum_{i=1}^{n}K_{ij}F(\mathrm{q}_j)=-G(\mathrm{q}_i) \qquad \text{* MERGEFORMAT (A4)}$$

Where $K_{ij}=\int_{p_j}^{p_{j+1}}\ln\left|\frac{\tan\left(\frac{p}{2}\right)^2-\tan\left(\frac{q_i}{2}\right)^2}{p-q_i}\right|dp+\int_{p_j}^{p_{j+1}}\ln\left|p-q_i\right|dp-4\pi\beta\cos^2\left(\frac{q_i}{2}\right)\delta_{ij}$ (no sum on i). The integral can be easily evaluated and therefore K_{ij} can be expressed as:

$$K_{ij}=\begin{cases}(p_{j+1}-p_j)\ln\left|\frac{\tan^2(\mathrm{q}_j/2)-\tan^2(\mathrm{q}_i/2)}{q_j-q_i}\right| \\ -(\mathrm{p}_{j+1}-p_j)+(\mathrm{p}_{j+1}-q_j)\ln\left|p_{j+1}-q_i\right|+(\mathrm{q}_i-p_j)\ln\left|p_j-q_i\right|,\ \mathrm{i}\neq \mathrm{j} \\ (\mathrm{p}_{j+1}-p_j)\ln\left|\frac{\tan\left(\frac{q_i}{2}\right)}{\cos^2\left(\frac{q_i}{2}\right)}\right|-4\pi\beta\cos^2\left(\frac{q_i}{2}\right) \\ -(\mathrm{p}_{j+1}-p_j)+(\mathrm{p}_{j+1}-q_j)\ln\left|p_{j+1}-q_i\right|+(\mathrm{q}_i-p_j)\ln\left|p_j-q_i\right|,\ \mathrm{i=j}\end{cases} \qquad \text{* MERGEFORMAT (A5)}$$

Substitute equation * MERGEFORMAT (A5) into equation * MERGEFORMAT (A4), $F(\mathrm{q}_j)$ can be solved numerically and $\overline{v}''(\tan(\mathrm{q}_j/2))=2F(\mathrm{q}_j)\cos^2(\mathrm{q}_j/2)$.

Calculation of $\overline{\Lambda}_H$:

First, we need to calculate the Compliance C:

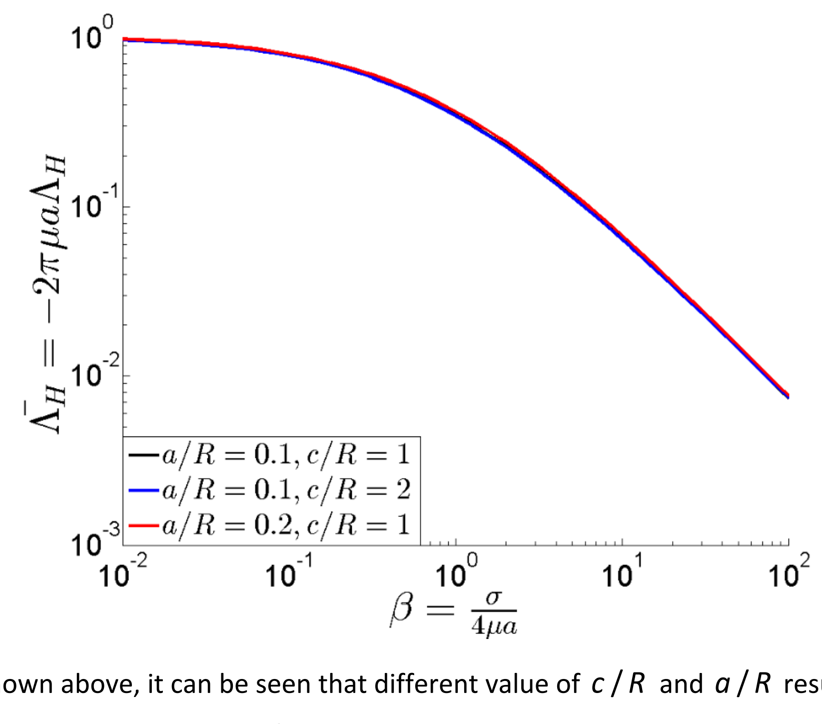
$$C=\frac{d\delta_H}{dP_H}=\frac{d\delta_H\ /\ da}{dP_H\ /\ da}$$
$$=\frac{d(\overline{\delta_H}a^2\ /\ R)\ /\ \mathrm{d}(\overline{a}\mathrm{R})}{d(\overline{P_H}\pi\mu a^2\ /\ \mathrm{R})\ /\ \mathrm{d}(\overline{a}\mathrm{R})}=\frac{1}{\pi\mu}\frac{\overline{a}d\overline{\delta_H}\ /\ d\overline{a}+2\overline{\delta_H}}{\overline{a}d\overline{P_H}\ /\ d\overline{a}+2\overline{P_H}} \quad \text{(A6)}$$

Where $\overline{a}=a\ /\ R$. It is easy to show that $\overline{\Lambda_H}=2\pi\mu a\Lambda_H$ is calculated as:

$$\overline{\Lambda_H}=-2\frac{(2\overline{P_H}-\beta\frac{d\overline{P_H}}{d\beta})(\beta^2\frac{\partial^2\overline{\delta_H}}{\partial\beta^2}+2\beta\overline{c}\frac{\partial^2\overline{\delta_H}}{\partial\beta\partial\overline{c}}+\overline{c}^2\frac{\partial^2\overline{\delta_H}}{\partial\overline{c}^2}-\beta\frac{\partial\overline{\delta_H}}{\partial\beta}-\overline{c}\frac{\partial\overline{\delta_H}}{\partial\overline{c}})+(\beta\frac{\partial\overline{\delta_H}}{\partial\beta}+\overline{c}\frac{\partial\overline{\delta_H}}{\partial\overline{c}}-2\overline{\delta_H})(\beta^2\frac{d^2\overline{P_H}}{d\beta^2}-\beta\frac{d\overline{P_H}}{d\beta})}{(2\overline{P_H}-\beta\frac{d\overline{P_H}}{d\beta})^2}$$

(A7)

Since $\overline{P_H}$ is only a function of β and $\overline{\delta_H}$ is a function of β and $\overline{c}$, $\overline{\Lambda_H}$ is at most a function of β and $\overline{c}$ (or equivalently, $\beta, c\ /\ R, \overline{a}$). However, since the general JKR solution can't depend on where we select our dictum plane, $\overline{\Lambda_H}$ can only be a function of β. To prove this point, different values of $c\ /\ R$ and $a\ /\ R$ are taken to compute $\overline{\Lambda_H}$:



From the plot shown above, it can be seen that different value of $c\ /\ R$ and $a\ /\ R$ result in almost same $\overline{\Lambda_H}-\beta$ curve. This curve can be fitted by:

$$\overline{\Lambda}_H(\beta)=\frac{\pi\beta^2+9.102\beta+1}{4\beta^3+20.82\beta^2+12.75\beta+1} \quad \text{(A8)}$$