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Analogy between compressible 3D Kirchhoff's rod and relativistic rotor

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In this supplementary document we derive the equations of equilibrium for a 3D compressible elastic rod under the assumptions of Kirchhoff's model [1, 2]. It is shown that these equations are mathematically analogous to the 3D Euler's equations of rigid-body rotation, where the non-relativistic angular momentum, $\mathbf{L}$, is replaced by, $\gamma\mathbf{L}$, γ being the Lorentz factor. This analogy is derived while keeping in mind the known difficulties in the latter, relativistic problem [3, 4].

Following the formulation in ref. [5], we consider a space curve described by the position vector, $\mathbf{R}(s)$, where s is the arclength parameter of the relaxed configuration. Defining $\hat{s}$ as the arclength parameter of the deformed configuration, the unit tangent vector to $\mathbf{R}(s)$ is given by, $\mathbf{d}_3 = d\mathbf{R}/d\hat{s}$. In addition, let $\{\mathbf{d}_1, \mathbf{d}_2\}$ be a pair of unit vectors perpendicular to $\mathbf{d}_3$ and parallel to the principal axes of the filament's cross-section. The triad $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ form a co-moving coordinate frame attached to the rod's mid-axis. These vectors satisfy the relations,

$$\frac{d\mathbf{d}_i}{ds} = \gamma\boldsymbol{\kappa} \times \mathbf{d}_i, \tag{1}$$

where $i = 1, 2, 3$, $\gamma = d\hat{s}/ds$ is the strain field, and $\boldsymbol{\kappa} = \kappa_1\mathbf{d}_1 + \kappa_2\mathbf{d}_2 + \kappa_3\mathbf{d}_3$ is the curvature vector. The local bending moment is given by,

$$\mathbf{M}(s) = B_1(\gamma\kappa_1)\mathbf{d}_1 + B_2(\gamma\kappa_2)\mathbf{d}_2 + B_3(\gamma\kappa_3)\mathbf{d}_3, \tag{2}$$

where B_i are bending rigidities.

In correspondence with eqn (1) of the main text, the energy functional of a 3D elastic filament, confined by a boundary constant force $\mathbf{P}$, is given by $E = \int_0^L e[\boldsymbol{\kappa}(s), \gamma(s)]ds$, where

$$e[\boldsymbol{\kappa}(s), \gamma(s)] = \frac{1}{2}\mathbf{M}\cdot(\gamma\boldsymbol{\kappa}) + \frac{Y}{2}(\gamma - 1)^2 + \gamma\mathbf{d}_3\cdot\mathbf{P}. \tag{3}$$

The appearance of the $\gamma\boldsymbol{\kappa}$ term in the bending energy is a consequence of the requirement to keep bending and compression contributions independent. Minimizing eqn (3) with respect to γ and κ_i (keeping in mind that κ_i are not independent) gives [5],

$$\frac{d\mathbf{M}}{d\hat{s}} - \mathbf{d}_3 \times \mathbf{P} = 0, \tag{4}$$

$$Y(\gamma - 1) + \mathbf{d}_3\cdot\mathbf{P} = 0. \tag{5}$$

In the incompressible limit eqn (5) is redundant and equations (4) become analogous to the non-relativistic Euler equations of a 3D rigid body, fixed at one point and rotating under the influence of an external force (such as gravity) [6, p. 200]. In this analogy, the bending moment takes the role of angular momentum, $\mathbf{M} \leftrightarrow \mathbf{L}$, and the boundary force is analogous to an external torque, $\mathbf{d}_3 \times \mathbf{P} \leftrightarrow \mathbf{N}$. Turning on compressibility effects, we have by eqn (2) that $\mathbf{M} \to \gamma\mathbf{M}$. Thus, it is left to show that γ coincides with the Lorentz factor. First integration of equations (4) and (5) gives,

$$\mathcal{H} = \frac{1}{2}\mathbf{M}\cdot(\gamma\boldsymbol{\kappa}) - \frac{Y}{2}(\gamma - 1)^2 - \gamma\mathbf{d}_3\cdot\mathbf{P} = \text{const}. \tag{6}$$

Eliminating $\mathbf{d}_3\cdot\mathbf{P}$ from eqn (6) and substituting in eqn (5) gives,

$$\gamma = \frac{\sqrt{1 + 2\mathcal{H}/Y}}{\sqrt{1 + \frac{B_1}{Y}\kappa_1^2 + \frac{B_2}{Y}\kappa_2^2 + \frac{B_3}{Y}\kappa_3^2}}. \tag{7}$$

This expression indeed resembles the Lorentz factor up to the constant prefactor, $\gamma_0 \equiv \sqrt{1 + 2\mathcal{H}/Y}$, which appears also in the 2D problem, and which can be absorbed in the force $\mathbf{P}$, (see discussion in the main text).

References

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