Solution of the crack problem

Before solving these equations, the following dimensionless parameters are introduced:

$$(\tilde{r},\tilde{z})=(r,z)/c\,,\ (\tilde{u}_r,\tilde{u}_z)=(u_r,u_z)/c\,,\ \lambda=a/c\,,\text{ and }\tilde{\delta}=\delta/c \tag{S1}$$

Using the axisymmetry of the crack problem and the properties of the Bessel functions, Ξ can be expressed as

$$\Xi=\int_0^\infty P(k,\tilde{z})kJ_0(k\tilde{r})dk \tag{S2}$$

Applying the Hankel transform of the first order and the zeroth order on $\tilde{u}_r(\tilde{r},\ \tilde{z})$ and $\tilde{u}_z(\tilde{r},\ \tilde{z})$, respectively, one has

$$R(k,\ \tilde{z})=\int_0^\infty \tilde{u}_r(\tilde{r},\ \tilde{z})\tilde{r}J_1(k\tilde{r})d\tilde{r} \tag{S3}$$

$$Z(k,\ \tilde{z})=\int_0^\infty \tilde{u}_z(\tilde{r},\ \tilde{z})\tilde{r}J_0(k\tilde{r})d\tilde{r} \tag{S4}$$

$$\tilde{u}_r(\tilde{r},\ \tilde{z})=\int_0^\infty R(k,\ \tilde{z})kJ_1(k\tilde{r})dk \tag{S5}$$

$$\tilde{u}_z(\tilde{r},\ \tilde{z})=\int_0^\infty Z(k,\ \tilde{z})kJ_0(k\tilde{r})dk \tag{S6}$$

The stresses in terms of the auxiliary functions are (S7)

$$\sigma_{rz}=\mu\int_0^\infty\left(\frac{\partial R}{\partial\tilde{z}}-kZ\right)kJ_1(k\tilde{r})dk \tag{S8}$$

$$\sigma_{zz}=2\mu\int_0^\infty\left(\frac{\nu}{1-2\nu}P+\frac{\partial Z}{\partial\tilde{z}}\right)kJ_0(k\tilde{r})dk \tag{S9}$$

Substituting Eqs. (S2), (S5) and (S6) into Eqs. (1)-(3) yields

$$\frac{\partial Z(k,\tilde{z})}{\partial\tilde{z}}+kR(k,\tilde{z})=P(k,\tilde{z}) \tag{S10}$$

$$\left(\frac{\partial^2}{\partial\tilde{z}^2}-k^2\right)R(k,\tilde{z})=\frac{k}{1-2\nu}P(k,\tilde{z}) \tag{S11}$$

$$\left(\frac{\partial^2}{\partial\tilde{z}^2}-k^2\right)Z(k,\tilde{z})=-\frac{1}{1-2\nu}\frac{\partial P(k,\tilde{z})}{\partial\tilde{z}} \tag{S12}$$

$$\left(\frac{\partial^2}{\partial\tilde{z}^2}-k^2\right)P(k,\tilde{z})=0 \tag{S13}$$

The formal solutions of Eqs. (S10)-(S13) for $\tilde{z}<0$ are

$$P(k,\tilde{z}) = 2(1-2\nu)kAe^{k\tilde{z}} \tag{S14}$$

$$R(x,\tilde{z}) = [A(3-4\nu) + Ak\tilde{z} + B]e^{k\tilde{z}} \tag{S15}$$

$$Z(k,\tilde{z}) = -(Ak\tilde{z} + B)e^{k\tilde{z}} \tag{S16}$$

where A and B are to be determined. Substituting Eqs. (S14)-(S16) in Eqs. (S8) and (S9), one obtains

$$\sigma_{rz} = 2\mu\int_0^\infty [B + 2A(1-\nu) + Ak\tilde{z}]e^{k\tilde{z}}\ k^2 J_1(k\tilde{r})dk \tag{S17}$$

$$\sigma_{zz} = -2\mu\int_0^\infty [B + A(1-2\nu) + Ak\tilde{z}]\ k^2 e^{k\tilde{z}} J_0(k\tilde{r})dk \tag{S18}$$

Using the boundary condition of (10), the relationship between A and B is found to be

$$A = -\frac{B}{2(1-\nu)} \tag{S19}$$

and the stress components are expressed as

$$\sigma_{rz} = 2\mu\int_0^\infty A\tilde{z}e^{k\tilde{z}}\ k^3 J_1(k\tilde{r})dk \tag{S20}$$

$$\sigma_{zz} = 2\mu\int_0^\infty A(1-k\tilde{z})k^2 e^{k\tilde{z}} J_0(k\tilde{r})dk \tag{S21}$$

From the conditions of (9), (11) and (13), the boundary value problem is reduced to the following solution of dual integral equations as

$$\int_0^\infty AkJ_0(k\tilde{r})dk = 0 \qquad \text{for } \tilde{r} > 1 \tag{S22}$$

$$\int_0^\infty Ak^2 J_0(k\tilde{r})dk = \frac{-\sigma_0 + f(\tilde{r})}{2\mu} \qquad \text{for } \tilde{r} < 1 \tag{S23}$$

The solution of the dual integral equations has been extensively studied [1], and A can be expressed as

$$A = \frac{1}{\pi\mu}\frac{1}{k}\int_0^1 t\sin kt dt\int_0^1 \frac{[-\sigma_0 + f(t\rho)]\rho}{\sqrt{1-\rho^2}}d\rho \tag{S24}$$

Using the following equation,

$$\int_0^\infty \sin tkJ_0(\rho t)dt = \begin{cases} 0 & \rho > k \\ (k^2-\rho^2)^{-1/2} & \rho < k \end{cases} \tag{S25}$$

the crack opening is found as

$$\Delta = u_z(\tilde{r},0^+) - u_z(\tilde{r},0^-) = \frac{8(1-\nu^2)c}{\pi E}\int_{\tilde{r}}^1 \frac{tdt}{\sqrt{t^2-\tilde{r}^2}}\int_0^1 \frac{[\sigma_0 - f(txc)]xdx}{\sqrt{1-x^2}} \qquad \text{for } \tilde{r} < 1 \tag{S26}$$

with $tx = \tilde{r}$. The normal component of the stress field in front of the crack tip is

$$\sigma_{zz}(\tilde{r},0) = -\frac{2}{\pi}\int_0^\infty \varsigma J_0(\varsigma\tilde{r})d\varsigma\int_0^1 t\sin(\varsigma t)dt\int_0^1 \frac{x[\sigma_0 - f(txc)]}{\sqrt{1-x^2}}dx \qquad \text{for } \tilde{r} > 1 \tag{S27}$$

Using Eq. (S25) and the following relationships

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x) \text{ and } \int_0^\infty \sin tkJ_1(\rho t)dt = \begin{cases} 0 & \rho > k \\ \rho / k(k^2 - \rho^2)^{1/2} & \rho < k \end{cases} \tag{S28}$$

Eq. (S27) can be further simplified to

$$\sigma_{zz}(\tilde{r},0) = -\frac{2}{\pi}\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}\int_0^1 \frac{t^2 q(t)}{\sqrt{\tilde{r}^2 - t^2}}dt \qquad \text{for } \tilde{r} > 1 \tag{S29}$$

where

$$q(t) = \int_0^1 \frac{[\sigma_0 - f(txc)]x}{\sqrt{1-x^2}}dx \tag{S30}$$

To derive both the normal component of the stress field and the crack opening with the function *f*(*r*) of (13), we first consider the auxiliary problem that a penny crack of dimensionless radius 1 in an infinite elastic space is only subjected to uniform pressure of p_0 on the crack surface over a circular area of dimensionless radius *b* (*b*<1), i.e.

$$\sigma_{zz} = 0 \qquad \text{for } |z| \to \infty \tag{S31}$$

$$\sigma_{zz}(r,0) = f(r) = \begin{cases} -p_0 \\ 0 \end{cases} \qquad \text{for } \begin{matrix} \tilde{r} < b \\ b < \tilde{r} < 1 \end{matrix} \tag{S32}$$

Using Eq. (S32), one obtains

$$q(t) = \begin{cases} p_0 \\ p_0(1 - \sqrt{1 - b^2 / t^2}) \end{cases} \qquad \text{for } \begin{matrix} t < b \text{ and } x > \tilde{r}/b \\ t > b \text{ and } x < \tilde{r}/b \end{matrix} \tag{S33}$$

Define

$$\Delta_0 = \frac{8(1-\nu^2)cp_0}{\pi E} \tag{S34}$$

The crack opening is

$$\frac{\Delta^a}{\Delta_0} = \int_{\tilde{r}}^b \frac{xdx}{\sqrt{x^2 - \tilde{r}^2}} + \int_b^1 \frac{(x - \sqrt{x^2 - b^2})dx}{\sqrt{x^2 - \tilde{r}^2}} \qquad \text{for } \tilde{r} < b \tag{S35}$$

$$\frac{\Delta^a}{\Delta_0} = \int_{\tilde{r}}^1 \frac{(x - \sqrt{x^2 - b^2})dx}{\sqrt{x^2 - \tilde{r}^2}} \qquad \text{for } b < \tilde{r} < 1 \tag{S36}$$

which can be expressed as

$$\frac{\Delta^a}{\Delta_0} = \sqrt{1-\tilde{r}^2}(1-\sqrt{1-b^2}) + b(\mathbf{E}[\frac{\tilde{r}^2}{b^2}] - \mathbb{E}[\sin^{-1} b, \frac{\tilde{r}^2}{b^2}]) \qquad \text{for } \tilde{r} < b \tag{S37}$$

$$\frac{\Delta^a}{\Delta_0} = \sqrt{1-\tilde{r}^2}(1-\sqrt{1-b^2}) \qquad \text{for } b < \tilde{r} < 1 \tag{S38}$$

$$+\tilde{r}\left(\mathbf{E}[\frac{b^2}{\tilde{r}^2}] - \mathbb{E}[\sin^{-1}\tilde{r}, \frac{b^2}{\tilde{r}^2}] + \left(1-\frac{b^2}{\tilde{r}^2}\right)\left(\mathbb{F}[\sin^{-1}\tilde{r}, \frac{b^2}{\tilde{r}^2}] - \mathbf{K}[\frac{b^2}{\tilde{r}^2}]\right)\right)$$

Here $\mathbb{F}(\bullet,\bullet)$ and $\mathbb{E}(\bullet,\bullet)$ are the first and second elliptical integrals, respectively, and $\mathbf{E}(\bullet)$ and $\mathbf{K}(\bullet)$ are the first and the second complete elliptical integrals, respectively. Note that the $\tilde{r}/b$ and $b/\tilde{r}$ in the elliptical integrals in the work of Parihar and Krishna Rao [2] need to be replaced by $\tilde{r}^2/b^2$ and $b^2/\tilde{r}^2$, respectively. For $b \ll 1$, there is

$$x - \sqrt{x^2 - b^2} \approx \frac{1}{2}\frac{b^2}{x} \tag{S39}$$

Eq. (S36) gives

$$\Delta^a = \frac{4(1-\nu^2)cp_0}{\pi E}\frac{b^2}{\tilde{r}}\cos^{-1}\tilde{r} = \frac{4(1-\nu^2)c}{\pi^2 E}\frac{P}{\tilde{r}}\cos^{-1}\tilde{r} \qquad \text{for } b \ll \tilde{r} < 1 \tag{S40}$$

with P being the applied load. For $\tilde{r} \to 1$, there is

$$\lim_{\tilde{r}\to 1}\frac{\Delta^a}{\Delta_0} = \sqrt{1-\tilde{r}^2}(1-\sqrt{1-b^2}) \tag{S41}$$

which reduces to the result for b=1.

The normal component of the stress field in the front of the crack tip is

$$\frac{\sigma_{zz}^a(\tilde{r},0)}{p_0} = -\frac{2}{\pi}\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}\left(\int_0^b \frac{t^2}{\sqrt{\tilde{r}^2-t^2}}dt + \int_b^1 \frac{t(t-\sqrt{t^2-b^2})}{\sqrt{\tilde{r}^2-t^2}}dt\right) \quad \text{for } \tilde{r} > 1 \tag{S42}$$

$$= \frac{2}{\pi}\left(\frac{1-\sqrt{1-b^2}}{\sqrt{\tilde{r}^2-1}} + \frac{\pi}{4} - \cot^{-1}(\sqrt{\tilde{r}^2-1}) + \frac{1}{2}\tan^{-1}\frac{2-b^2-\tilde{r}^2}{2\sqrt{1-b^2}\sqrt{\tilde{r}^2-1}}\right)$$

which gives the same stress distribution as that by Parihar and Krishna Rao [2]. For $b \ll 1$, there is

$$\frac{\sigma_{zz}^a(\tilde{r},0)}{p_0} = \frac{2}{\pi}\left(\frac{1}{2}\frac{b^2}{\tilde{r}^2}\left(\frac{1}{\sqrt{\tilde{r}^2-1}} - \frac{b}{\sqrt{\tilde{r}^2-b^2}}\right) + \frac{b}{\sqrt{\tilde{r}^2-b^2}} - \tan^{-1}\frac{b}{\sqrt{\tilde{r}^2-b^2}}\right) \tag{S43}$$

Equation (S42) gives the stress intensity factor, K_I^a, as

$$K_I^a = \lim_{\tilde{r}\to 1}\sqrt{2\pi c(\tilde{r}-1)}\sigma_{zz}(\tilde{r},0) = 2p_0(1-\sqrt{1-b^2})\sqrt{\frac{c}{\pi}} \tag{S44}$$

Here, the superscript *a* represents the auxiliary problem.

References:

1. I. N. Sneddon, *Int. J. Eng. Sci.*, 1965, **3**, 47-57.
2. K. Parihar and J. K. Rao, *Engineering fracture mechanics*, 1991, **39**, 1067-1095.