Supplementary material

Daniel Midtvedt[1,*] and Alexander Croy[2,†]

[1] Department of Physics, Chalmers University of Technology, Gothenburg, Sweden
[2] Max-Planck-Institute for the Physics of Complex Systems, Dresden, Germany

I. PHONON DISPERSION

A. Analytic expressions for optical frequencies

The phonon frequencies at the Gamma point are related to the VFM parameters as follows

$$\omega^2_{B_{1g}}/s = 4.6505 K'_\theta \ , \tag{S1a}$$
$$\omega^2_{B_{1u}}/s = 7.5540 K'_\theta \ , \tag{S1b}$$
$$\omega^2_{B_{2g}}/s = 2.2264(K_r - K_{rr'}) - 5.4889 K'_{r\theta} + 6.7662 K'_\theta \ , \tag{S1c}$$
$$\omega^2_{A_u}/s = 2.2264(K_r - K_{rr'}) - 0.9383 K'_{r\theta} + 0.1977 K'_\theta \ , \tag{S1d}$$
$$\omega^2_{B_{2u}}/s = 1.7735(K_r + K_{rr'}) + 0.9383 K'_{r\theta} + 0.2482 K'_\theta - 7.9485 K_{r\theta} + 8.9056 K_\theta \ , \tag{S1e}$$

where the factor $s = 8785.5\,\mathrm{cm}^{-2}/\mathrm{eV}$ converts the frequencies to have units of cm^{-1} if the VFM parameters are given in eV. Moreover, the two B_{3g} frequencies are obtained as

$$\omega^2_{B^1_{3g}}/s = \frac{1}{2}\left(c_{B_{3g}} - \sqrt{d_{B_{3g}}}\right); \quad \omega^2_{B^2_{3g}}/s = \frac{1}{2}\left(c_{B_{3g}} + \sqrt{d_{B_{3g}}}\right) \ , \tag{S2a}$$

where

$$c_{B_{3g}} = 2K'_r - 7.3917 K''_{r\theta} + 14.0188 K'_\theta \ , \tag{S2b}$$
$$d_{B_{3g}} = 57.5135({K''_{r\theta}}^2 - K'_r K'_\theta) + c^2_{B_{3g}} \ . \tag{S2c}$$

Finally, the frequencies of the two A_g modes are given by

$$\omega^2_{A^1_g}/s = \frac{1}{2}\left(c_{A_g} - \sqrt{d_{A_g}}\right); \quad \omega^2_{A^2_g}/s = \frac{1}{2}\left(c_{A_g} + \sqrt{d_{A_g}}\right) \ , \tag{S3a}$$

where

$$\begin{aligned} c_{A_g} &= 1.7735(K_r + K_{rr'}) + 2K'_r - 1.6496 K'_{rr'} - 7.9485 K_{r\theta} \\ &\quad - 2.3391 K'_{r\theta} - 0.4363 K''_{r\theta} + 8.9056 K_\theta + 1.8636 K'_\theta \ , \end{aligned} \tag{S3b}$$
$$\begin{aligned} d_{A_g} &= (16.3805 K'_{rr'} - 2.2787 K_r - 2.2787 K_{rr'} - 14.7189 K'_r + 10.2123 K_{r\theta} - 11.4419 K_\theta) K'_\theta \\ &\quad + (-38.8655 K'_{rr'} - 8.19024 K'_{r\theta} + 10.8132 K_r + 10.8132 K_{rr'} - 48.4608 K_{r\theta} + 54.2957 K_\theta + 14.7189 K''_{r\theta}) K''_{r\theta} \\ &\quad + (19.4327 K'_{r\theta} - 12.8281 K_r - 12.8281 K_{rr'} + 57.4907 K_{r\theta} - 64.4129 K_\theta) K'_r \\ &\quad + 25.6562 {K'_{rr'}}^2 - 10.8132 K'_{rr'} K'_{r\theta} + 1.13935 {K'_{r\theta}}^2 + c^2_{A_g} \ . \end{aligned} \tag{S3c}$$

Using the expressions above, the phonon frequencies can be calculated from the VFM parameters listed in the main text (Table 1). The resulting frequencies are shown in Table ??.

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* midtvedt@chalmers.se
† croy@pks.mpg.de

TABLE S1. Comparison of phonon frequencies given in cm^{-1}

mode	this work	Ref. [**?**] (VFM)	Ref. [**?**] (DFT)
A_g^1	363	362.4	343.2
A_g^2	468	472.6	450.5
B_{1g}	194.5	194.9	188.5
B_{2g}	440	440	426.4
B_{3g}^1	227	223.4	223.8
B_{3g}^2	438	429.1	412.7
B_{1u}	247.9	248.4	144.7
B_{2u}	483	463.5	458.4
A_u	466.3	411	422.6

II. SOLUTION TO THE AIRY STRESS EQUATION FOR THE SUSPENDED PHOSPHORENE DRUM

In dimensionless form, the Airy stress equation for an orthotropic drum is (see Eq. (9) in the main text)

$$q\partial_{\tilde{x}}^4\tilde{\chi} + (1/q)\partial_{\tilde{y}}^4\tilde{\chi} + \left((\tilde{Y}/G_{xy}) - 2\tilde{\nu}\right)\partial_{\tilde{x}}^2\partial_{\tilde{y}}^2\tilde{\chi} = (w_0/R)^2\left[(\partial_{\tilde{x}}\partial_{\tilde{y}}\tilde{w})^2 - (\partial_{\tilde{x}}^2\tilde{w})(\partial_{\tilde{y}}^2\tilde{w})\right] \;, \tag{S4}$$

where $q = \sqrt{Y_x/Y_y}$, $\tilde{\chi} = (\chi R^2)/\tilde{Y}$, and $\tilde{w} = w/w_0$. Inserting the Ansätze for the Airy stress function, $\tilde{\chi} = A\tilde{x}^4 + B\tilde{y}^4 + C\tilde{x}^2\tilde{y}^2 + D\tilde{x}^2 + E\tilde{y}^2 + \chi_0$, and the deformation, $\tilde{w} = 1 - (\tilde{x}^2 + \tilde{y}^2)$ given in the main text, the Airy stress equation reduces to

$$24Aq + (24/q)B + 4C\left[(\tilde{Y}/G_{xy}) - 2\tilde{\nu}\right] = -4(w_0/R)^2 \;. \tag{S5}$$

In addition to this equation, there are four boundary conditions that need to be fulfilled for the in-plane displacements, namely $\tilde{u}(\tilde{x}=0,\tilde{y}=0)=0$, $\tilde{v}(\tilde{x}=0,\tilde{y}=0)=0$, $\tilde{u}(\tilde{x}^2+\tilde{y}^2=1)=0$, $\tilde{v}(\tilde{x}^2+\tilde{y}^2=1)=0$. Using the relations between the Airy stress function and the stresses, in combination with the stress-strain relations one obtains for the strains

$$\begin{aligned}\epsilon_{xx} &= \left(1-\tilde{\nu}^2\right)\left[(1/q)\partial_{\tilde{y}}^2\tilde{\chi} - \tilde{\nu}\partial_{\tilde{x}}^2\tilde{\chi}\right] \;,\\ \epsilon_{yy} &= \left(1-\tilde{\nu}^2\right)\left[q\partial_{\tilde{x}}^2\tilde{\chi} - \tilde{\nu}\partial_{\tilde{y}}^2\tilde{\chi}\right] \;,\\ \epsilon_{xy} &= -(\tilde{Y}/(2G_{xy}))\partial_{\tilde{x}}\partial_{\tilde{y}}\tilde{\chi} \;.\end{aligned} \tag{S6}$$

The displacement fields are then obtained by integration as

$$\begin{aligned}\tilde{u}(\tilde{x},\tilde{y}) =& (R/w_0)^2\int_0^{\tilde{x}} d\tilde{x}\epsilon_{xx} \;,\\ \tilde{v}(\tilde{x},\tilde{y}) =& (R/w_0)^2\int_0^{\tilde{y}} d\tilde{y}\epsilon_{yy} \;.\end{aligned} \tag{S7}$$

Note that these solutions for the displacement fields immediately satisfy the boundary condition at the center of the drum. In fact, they imply that $\tilde{u}$ and $\tilde{v}$ vanish along the lines $\tilde{x}=0$ and $\tilde{y}=0$ respectively. Thus, we write $\tilde{u}(\tilde{x},\tilde{y}) = \tilde{x}f(\tilde{x},\tilde{y})$ and $\tilde{v}(\tilde{x},\tilde{y}) = \tilde{y}g(\tilde{x},\tilde{y})$ where f and g are second order polynomials in $\tilde{x},\tilde{y}$ satisfying $f(\tilde{x}^2+\tilde{y}^2=1)=0$ and $g(\tilde{x}^2+\tilde{y}^2=1)=0$. This implies that the displacement fields must be given by $\tilde{u}(\tilde{x},\tilde{y}) = \tilde{u}_0\tilde{x}\left(1-\tilde{x}^2-\tilde{y}^2\right)$ and $\tilde{v}(\tilde{x},\tilde{y}) = \tilde{v}_0\tilde{x}\left(1-\tilde{x}^2-\tilde{y}^2\right)$. The coefficients A, B, C, D and E in the Airy equation and the coefficients $\tilde{u}_0$ and $\tilde{v}_0$ in the displacement fields are then obtained by inserting the Ansatz for the Airy function into Eq. (**??**) and matching coefficients, while also imposing Eq. (**??**).

Due to the radial symmetry of the boundary conditions, the displacement fields and the strains are more conveniently expressed in polar coordinates. The stresses on the other hand are also influenced by the orthotropic symmetry of the underlying lattice, and are therefore better expressed in Cartesian coordinates. In polar coordinates $\tilde{x} = r\cos\theta$ and $\tilde{y} = r\sin\theta$ with $0<r<1$ and $0<\theta<2\pi$ the corresponding displacement fields are $\tilde{u}_r = \tilde{u}\cos\theta + \tilde{v}\sin\theta$, $\tilde{u}_\theta = -\tilde{u}\sin\theta + \tilde{v}\cos\theta$. We find

$$\begin{aligned}\tilde{u}_r &= r(1-r^2)\frac{G_{xy}(1-\tilde{\nu}^2)^2\left(2q\tilde{\nu} - 3(1+q^2) - 3(1-q^2)\cos 2\theta\right)}{q\tilde{Y}\left(\tilde{\nu}^2-9\right) + G_{xy}\left(2q\tilde{\nu} - 3(1+q^2)\right)(1-\tilde{\nu^2})}\\ \tilde{u}_\theta &= r(1-r^2)\frac{3G_{xy}(1-\tilde{\nu}^2)^2(q^2-1)\sin 2\theta}{q\tilde{Y}\left(\tilde{\nu}^2-9\right) + G_{xy}\left(2q\tilde{\nu} - 3(1+q^2)\right)(1-\tilde{\nu^2})} \;.\end{aligned} \tag{S8}$$

The strains are obtained from the displacements as

$$
\begin{aligned}
\epsilon_{rr} =&(w_0/R)^2 \left(\partial_r \tilde{u}_r + (1/2)(\partial_r \tilde{w})^2\right) = \\
&(w_0/R)^2 \left(2r^2 + (1-3r^2)\frac{G_{xy}(1-\tilde{\nu}^2)^2 \left(2q\tilde{\nu} - 3(1+q^2) - 3(1-q^2)\cos 2\theta\right)}{q\tilde{Y}\left(\tilde{\nu}^2 - 9\right) + G_{xy}\left(2q\tilde{\nu} - 3(1+q^2)\right)\left(1-\widetilde{\nu^2}\right)}\right) , \\
\epsilon_{\theta\theta} =&(w_0/R)^2 \left(\partial_\theta \tilde{u}_\theta / r + \tilde{u}/r + (1/2r^2)(\partial_\theta \tilde{w})^2\right) = \\
&(w_0/R)^2 \left((1-r^2)\frac{G_{xy}(1-\tilde{\nu}^2)^2 \left(2q\tilde{\nu} - 3(1+q^2) + 3(1-q^2)\cos 2\theta\right)}{q\tilde{Y}\left(\tilde{\nu}^2 - 9\right) + G_{xy}\left(2q\tilde{\nu} - 3(1+q^2)\right)\left(1-\widetilde{\nu^2}\right)}\right) , \\
\epsilon_{r\theta} =&(1/2)(w_0/R)^2 \left[(\partial_r - 1/r)\,\tilde{u}_\theta + (1/r)\partial_\theta \tilde{u} + (1/r)(\partial_\theta \tilde{w})(\partial_r \tilde{w})\right] = \\
&(w_0/R)^2 \left((1-2r^2)\frac{3G_{xy}(1-\tilde{\nu}^2)^2 \left((1-q^2)\sin 2\theta\right)}{q\tilde{Y}\left(\tilde{\nu}^2 - 9\right) + G_{xy}\left(2q\tilde{\nu} - 3(1+q^2)\right)\left(1-\widetilde{\nu^2}\right)}\right) .
\end{aligned}
\tag{S9}
$$

We note that by setting $q = 1$ the limit of an isotropic drum is obtained. In particular, the displacements become purely radial, the shear strain vanishes and the remaining strain components are functions of the distance from the center of the drum only. For $q \neq 1$ this is no longer true.