Supplementary materials

We consider an elastic unbounded plate of thickness d, density ρ and Lamé constants λ et μ. We assume guided waves propagating along the direction x with wavenumber k_x and pulsation ω. The elastic displacement of these waves is described as a linear combination of partial longitudinal and transverse vertical waves [1]:

$$\boldsymbol{u}(x,z) = \left[\begin{pmatrix} jk_x \cos(k_{L_z} z) \\ 0 \\ k_{L_z} \sin(k_{L_z} z) \end{pmatrix} U_L^C + \begin{pmatrix} -jk_x \sin(k_{L_z} z) \\ 0 \\ k_{L_z} \cos(k_{L_z} z) \end{pmatrix} U_L^S \right. \\ \left. + \begin{pmatrix} k_{T_z} \cos(k_{T_z} z) \\ 0 \\ jk_x \sin(k_{T_z} z) \end{pmatrix} U_T^C + \begin{pmatrix} -k_{T_z} \sin(k_{T_z} z) \\ 0 \\ jk_x \cos(k_{T_z} z) \end{pmatrix} U_T^S \right] e^{j(\omega t - k_x x)}, \tag{1}$$

with

$$\begin{cases} k_{L_z}^2 = \dfrac{\omega^2}{c_L^2} - k_x^2, \\ k_{T_z}^2 = \dfrac{\omega^2}{c_T^2} - k_x^2, \end{cases} \tag{2}$$

where the coefficients U_L^C, U_L^S, U_T^C, and U_T^S have to be determined.

If the plate is in vacuum, the stress free boundary conditions at surfaces $z = \pm d/2$ lead to a 4x4 linear system of equations which can be decoupled to two 2x2 linear systems due to the symmetry of the stresses. The dispersion equations correspond then to the vanishing of the determinants of each system [1], which are given by:

$$\begin{cases} D_S = 4k_x^2 k_{L_z} k_{T_z} \tan(k_{L_z} h/2) + (k_{T_z}^2 - k_x^2)^2 \tan(k_{T_z} d/2), \\ D_A = 4k_x^2 k_{L_z} k_{T_z} \tan(k_{T_z} h/2) + \left(k_{T_z}^2 - k_x^2\right)^2 \tan(k_{L_z} d/2). \end{cases} \tag{3}$$

D_S and D_A are the determinant of symmetric and antisymmetric Lamb waves, respectively. Symmetric modes in which the longitudinal component is an even function of z and the transverse component is an odd function of z. Inversely, antisymmetric modes in which the longitudinal component is an odd function of z and the transverse component is an even function of z.

Loaded plate

If the plate is loaded by an inviscid fluid of density ρ_0 and adiabatic sound speed c_0 on its lower surface ($z = -d/2$), the displacement in the fluid is expressed as follows

$$\boldsymbol{u}_f = U_f \begin{pmatrix} -jk_x \\ 0 \\ jk_{0_z} \end{pmatrix} e^{jk_{0_z}(z+d/2)} e^{j(\omega t - k_x x)} \tag{4}$$

with

$$\frac{\omega^2}{c_0^2} = k_x^2 + k_{0_z}^2. \tag{5}$$

Due to the mass conservation equation, the acoustic pressure is given by

$$P_f = \rho_0 U_f \omega^2 e^{jk_{0_z}(z+d/2)} e^{j(\omega t - k_x x)}. \tag{6}$$

Applying boundary conditions, we obtain the following 6x6 system of linear equations:

$$\begin{cases} 2jk_x k_{L_z}\left[-\sin(\tilde{k}_L)U_L^C + \cos(\tilde{k}_L)U_L^S\right] + \left(k_{T_z}^2 - k_x^2\right)\left[-\sin(\tilde{k}_T)U_T^C + \cos(\tilde{k}_T)U_T^S\right] = 0, \\ 2jk_x k_{L_z}\left[\sin(\tilde{k}_L)U_L^C + \cos(\tilde{k}_L)U_L^S\right] + \left(k_{T_z}^2 - k_x^2\right)\left[\sin(\tilde{k}_T)U_T^C + \cos(\tilde{k}_T)U_T^S\right] = 0, \\ \left(k_{T_z}^2 - k_x^2\right)\left[\cos(\tilde{k}_L)U_L^C + \sin(\tilde{k}_L)U_L^S\right] + 2jk_x k_{T_z}\left[\cos(\tilde{k}_T)U_T^C + \sin(\tilde{k}_T)U_T^S\right] = -\dfrac{\rho_0 c_0^2}{\mu}k_0^2 U_f, \\ \left(k_{T_z}^2 - k_x^2\right)\left[\cos(\tilde{k}_L)U_L^C - \sin(\tilde{k}_L)U_L^S\right] + 2jk_x k_{T_z}\left[\cos(\tilde{k}_T)U_T^C - \sin(\tilde{k}_T)U_T^S\right] = 0, \\ -k_{L_z}\sin(\tilde{k}_L)U_L^C + k_{L_z}\cos(\tilde{k}_L)U_L^S - jk_x\sin(\tilde{k}_T)U_T^C + jk_x\cos(\tilde{k}_T)U_T^S = jk_{0_z}U_f, \end{cases} \tag{7}$$

with $\tilde{k}_L = k_{L_z}d/2$ and $\tilde{k}_T = k_{T_z}d/2$. This system is then expressed in the form

$$\begin{pmatrix} 2jk_x k_{L_z}\sin(\tilde{k}_L) & \left(k_{T_z}^2 - k_x^2\right)\sin(\tilde{k}_T) & 0 & 0 & 0 \\ \left(k_{T_z}^2 - k_x^2\right)\cos(\tilde{k}_L) & 2jk_x k_{T_z}\cos(\tilde{k}_T) & 0 & 0 & \dfrac{\rho_0\omega^2}{2\mu} \\ 0 & 0 & 2jk_x k_{L_z}\cos(\tilde{k}_L) & \left(k_{T_z}^2 - k_x^2\right)\cos(\tilde{k}_T) & 0 \\ 0 & 0 & \left(k_{T_z}^2 - k_x^2\right)\sin(\tilde{k}_L) & 2jk_x k_{T_z}\sin(\tilde{k}_T) & \dfrac{\rho_0\omega^2}{2\mu} \\ -k_{L_z}\sin(\tilde{k}_L) & -jk_x\sin(\tilde{k}_T) & k_{L_z}\cos(\tilde{k}_L) & jk_x\cos(\tilde{k}_T) & -jk_{0_z} \end{pmatrix} . \begin{pmatrix} U_L^C \\ U_T^C \\ U_L^S \\ U_T^S \\ U_f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

After some tedious manipulations, the determinant D_I of this matrix is given by

$$\begin{aligned} D_I =& \frac{\rho_0\omega^4 k_{L_z}}{2j\rho k_{0_z}c_T^4}\tan(\tilde{k}_L)\tan(\tilde{k}_T)\left[4k_x^2 k_{L_z}k_{T_z}\tan(\tilde{k}_T) + \left(k_{T_z}^2 - k_x^2\right)^2\tan(\tilde{k}_L)\right] \\ &- \frac{\rho_0\omega^4 k_{L_z}}{2j\rho k_{0_z}c_T^4}\left[4k_x^2 k_{L_z}k_{T_z}\tan(\tilde{k}_L) + \left(k_{T_z}^2 - k_x^2\right)^2\tan(\tilde{k}_T)\right] \\ &- \left[4k_x^2 k_{L_z}k_{T_z}\tan(\tilde{k}_L) + \left(k_{T_z}^2 - k_x^2\right)^2\tan(\tilde{k}_T)\right]\left[4k_x^2 k_{L_z}k_{T_z}\tan(\tilde{k}_T) + \left(k_{T_z}^2 - k_x^2\right)^2\tan(\tilde{k}_L)\right] \end{aligned}$$

and using determinants D_A and D_S given by eq. (3), it follows that

$$D_I = \frac{\rho_0 k_T^4 k_{L_z}}{2j\rho k_{0_z}}\tan(\tilde{k}_L)\tan(\tilde{k}_T)D_A - \frac{\rho_0 k_T^4 k_{L_z}}{2j\rho k_{0_z}}D_S - D_S D_A. \tag{9}$$

References

[1] D. Royer and E. Dieulesaint, *Elastic Waves in Solids I: Free and Guided Propagation*, Springer, 1999.