Supplementary materials for "Creasing of an everted elastomer tube"

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Appendix A. Axisymmetric deformation in an everted tube

This section reviews the equilibrium solution of an everted elastomer tube with axisymmetric deformation. As illustrated in Fig. 2a, the inner radius and outer radius of an undeformed cylindrical tube are denoted by *A* and *B*, respectively. The thickness of the tube is denoted by H=B-A. After the eversion, except for the regions close to the ends of the tube, most part of the tube remains cylindrical shape with inner radius *a* and outer radius *b*. A material particle with distance *R* from the center of an undeformed tube moves to the position with distance *r* from the center of the tube in the everted state. The deformation geometry enables us to calculate the hoop stretch: $\lambda_{\theta}=r/R$ and radial stretch: $\lambda_{r}=-dr/dR$ in the tube, respectively.

Following the literature¹, tube eversion can be regarded as a general plane strain problem, with homogenous stretch $\lambda_z = \lambda$ in the axial direction. After eversion, the inner surface of an undeformed tube becomes the outer surface of the tube in the everted state, and the outer surface of an undeformed tube becomes the inner surface of the tube in the everted state, namely,

$$r(A) = b, \ r(B) = a. \tag{A1}$$

The elastomer is taken to be incompressible, so that we have,

$$B^2 - R^2 = \lambda (r^2 - a^2),$$
 (A2)

and the deformation field can be further written as,

$$r(R) = \sqrt{\left(B^2 - R^2\right)/\lambda + a^2}.$$
 (A3)

When the axial stretch λ and the inner radius *a* in the deformed state are known, the deformation field of the everted tube *r*(*R*) can be fully determined.

Based on (A3), we can calculate the hoop stretch and radial stretch:

$$\lambda_{\theta} = \sqrt{\left(\left(B / R\right)^2 - 1\right) / \lambda + \left(a / R\right)^2}, \qquad (A4)$$

$$\lambda_r = 1/\sqrt{\left(\left(B/R\right)^2 - 1\right)\lambda + \left(a\lambda/R\right)^2} .$$
(A5)

We assume that the elastomer can be described by Neo-Hookean model². The Cauchy stress along the radial, the hoop and the axial directions of the tube can be written as

$$\sigma_r = \mu \lambda_r^2 - p \,, \tag{A6}$$

$$\sigma_{\theta} = \mu \lambda_{\theta}^2 - p , \qquad (A7)$$

$$\sigma_z = \mu \lambda_z^2 - p \,, \tag{A8}$$

where μ is the small-deformation shear modulus of the elastomer and p is hydrostatic pressure.

Plugging (A6)–(A8) into the stress balance equation (2) in the main text and using the boundary condition $\sigma_r = 0$ at r=a, we can obtain,

$$p(r) = \frac{\mu(\lambda a^2 - \lambda r^2 + B^2)}{\lambda^2 r^2} + \frac{\mu(\lambda a^2 + B^2)}{2\lambda^2} \left(\frac{1}{a^2} - \frac{1}{r^2}\right) + \frac{\mu}{\lambda} \ln \frac{aR}{Br}.$$
 (A9)

The boundary condition $\sigma_r = 0$ at r=b gives that

$$\frac{(\lambda a^2 + B^2)}{2\lambda^2} \left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{1}{\lambda} \ln \frac{aA}{Bb} = 0.$$
(A10)

Following Rivlin³, the relaxed boundary condition of zero resultant force at the end of the tube requires:

$$\int_{a}^{b} \left(\frac{\lambda a^{2} - \lambda r^{2} + B^{2}}{\lambda^{2} r} - \lambda^{2} r - \int_{a}^{b} \left(\frac{(\lambda a^{2} - \lambda r^{2} + B^{2})(b^{2} - r^{2})}{2\lambda^{2} r^{3}} - \frac{r(b^{2} - r^{2})}{2(\lambda a^{2} - \lambda r^{2} + B^{2})} \right) dr \right) dr = 0 \quad (A11)$$

Both axial stretch λ and the inner radius *a* of the inverted tube can be calculated numerically from equations (A10) and (A11). Therefore, with λ and *a* known, the deformation field *r*(*R*), the radial stretch λ_r and hoop stretch λ_{θ} can be calculated using (A3), (A4) and (A5).

In Fig. S1, the In Fig. S2a, we plot the field of radial stretch and hoop stretch in an everted tube for the thickness H=0.5B. In Fig. S2b, we plot the corresponding radial stress and hoop stress in the everted tube. Both stretch and stress in the hoop direction are compressive in the inner surface after eversion, which may result in surface instability that is shown in Fig. 1b.



Fig. S1 The inner radius a and the outer radius b of the tube are functions of the tube thickness H of the undeformed state. In the calculation, we ignore all possible mechanical instabilities. The comparisons between our analytical predictions and experimental measurements of the inner radius a (square dots) and outer radius b (circle dots) of everted tubes are shown.



Fig. S2 (a) Distribution of radial stretch and hoop stretch in an everted tube with the thickness H=0.5B. (b) Distribution of radial stress and hoop streess in the everted tube with the thickness H=0.5B.

Appendix B. Linear stability analyses of an everted tube

The section summarizes the linear stability analysis of the everted tube. The stretch and stress field of an everted tube with axisymmetric deformation have been obtained in *Appendix A*. Each material point of a tube in the undeformed state is described by the reference coordinate \mathbf{X} , which moves to current coordinate \mathbf{x} after the eversion. The axisymmetric deformation filed of the elastomer is given by $\mathbf{x}^{\circ}(\mathbf{X})$, with deformation gradient defined as,

$$F_{iK}^{0} = \frac{\partial x_{i}^{0}(\mathbf{X})}{\partial X_{K}}.$$
(B1)

To obtain the critical conditions of wrinkling of the elastomeric tube after eversion, we adopt linear perturbation analysis⁴, by perturbing the equilibrium solution $x_i^0(\mathbf{X})$ with a state of infinitesimal displacement $\mathbf{u}(\mathbf{x})$. Using the Neo-Hookean material model², we can obtain the corresponding perturbation of Cauchy stress as

$$\widetilde{\sigma}_{ij} = \mu F_{jK} F_{pK} L_{ip} + p L_{ji} - \widetilde{p} \,\delta_{ij} \,, \tag{B2}$$

where $\tilde{\sigma}_{ij}$ and \tilde{p} are the perturbed true stress and hydrostatic pressure, and $L_{ij} = \partial u_i / \partial x_j$. The incompressible condition can be expressed as $L_{ii} = 0$.

The perturbation of Cauchy stress also needs to satisfy force balance equation, namely,

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} = 0.$$
 (B3)

In general, the displacement perturbation $\mathbf{u}(\mathbf{x})$ can be decomposed into the radial component $u_r(r,\theta)$ and the hoop component $u_{\theta}(r,\theta)$. Before the perturbation, the

deformation field of the tube is axisymmetric and under the generalized plane strain condition. Consequently, the gradient of the displacement perturbation is given by,

$$L_{rr} = \frac{\partial u_r}{\partial r}, L_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}, L_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_{\theta}}{r}, L_{\theta r} = \frac{\partial u_{\theta}}{\partial r}.$$
 (B4)

The incompressible condition is,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} = 0.$$
 (B5)

The perturbed stress $\tilde{\sigma}_{ii}$ in (B2) takes the form,

$$\begin{split} \tilde{\sigma}_{rr} &= \left(\mu\lambda_r^2 + p\right)L_{rr} - \tilde{p},\\ \tilde{\sigma}_{\theta\theta} &= \left(\mu\lambda_{\theta}^2 + p\right)L_{\theta\theta} - \tilde{p},\\ \tilde{\sigma}_{r\theta} &= \mu\lambda_{\theta}^2 L_{r\theta} + pL_{\theta r},\\ \tilde{\sigma}_{\theta r} &= \mu\lambda_r^2 L_{\theta r} + pL_{r\theta}. \end{split}$$
(B6)

The perturbed stress balance equation (B3) can be expressed in the polar coordinate as,

$$\frac{\partial \tilde{\sigma}_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\sigma}_{r\theta}}{\partial \theta} + \frac{\tilde{\sigma}_{rr} - \tilde{\sigma}_{\theta\theta}}{r} = 0,$$

$$\frac{\partial \tilde{\sigma}_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\sigma}_{\theta\theta}}{\partial \theta} + \frac{\tilde{\sigma}_{\theta r} + \tilde{\sigma}_{r\theta}}{r} = 0.$$
(B7)

The boundary condition for the perturbed stress can be expressed by,

$$\tilde{\sigma}_{rr} = 0, \tilde{\sigma}_{\theta r} = 0. \tag{B8}$$

By setting the perturbed displacement field as

$$u_{r}(r,\theta) = f(r)\cos(m\theta),$$

$$u_{\theta}(r,\theta) = g(r)\sin(m\theta),$$

$$\tilde{p}(r,\theta) = k(r)\cos(m\theta),$$

(B9)

where f(r), g(r) and k(r) are real function and m is the wave number. Substituting (B9) into (B4)-(B7), we obtain that,

$$\frac{\mu R^{2}}{m^{2}\lambda^{2}}f^{W} + \frac{2\mu(R^{2} - 2\lambda r^{2})}{m^{2}\lambda^{2}r}f''' - \left(\frac{p'r}{m^{2}} + \frac{\mu((4 + m^{2})R^{2} + 10\lambda r^{2})}{m^{2}\lambda^{2}r^{2}} + \frac{(m^{2} + 1)\mu r^{2}}{m^{2}R^{2}}\right)f'' - \left(\frac{p''r}{m^{2}} + \frac{2}{m^{2}}p' - \frac{\mu((4 + m^{2})R^{2} + 2(m^{2} + 2)\lambda r^{2})}{m^{2}\lambda^{2}r^{3}} + \frac{3\mu(m^{2} + 1)rR^{2} + 2\mu(m^{2} + 1)\lambda r^{3}}{m^{2}R^{4}}\right)f' \quad (B10) + \left(\frac{\mu(m^{4} - 1)}{m^{2}R^{2}} + \frac{2\mu(m^{2} - 1)\lambda r^{2}}{m^{2}R^{4}} + \frac{m^{2} - 1}{m^{2}}p''\right)f = 0,$$

where r=r(R) and p=p(R) can be obtained from (A3) and (A9) in *Appendix A*. Substituting (B9) into the boundary condition (B8) yields,

$$\frac{R^{2}}{\lambda^{2}m^{2}}f''' + \frac{2(R^{2} - \lambda r^{2})}{\lambda^{2}m^{2}}f'' - \left(\frac{r^{2}}{R^{2}} + \frac{2R^{2}}{\lambda^{2}r^{2}} + \frac{(R^{2} + 2\lambda r^{2})}{\lambda^{2}m^{2}r^{2}}\right)f' - \frac{(m^{2} - 1)(R^{2} + 2\lambda r^{2})}{\lambda^{2}m^{2}r^{3}}f = 0,$$
(B11)
$$r^{2}f'' + rf' + (m^{2} - 1)f = 0,$$
(B12)

for both r=a and b.

The differential equation (B10) accompanied with the boundary (B11) and (B12) consist an eigenvalue problem for the loading parameter *H*. The result can be resolved numerically by compound matrix method⁵.

In Fig. S3, we plot the critical thickness *H* for wrinkling instability with respect to wave number *m*. The smallest tube thickness for the onset of wrinkling instability in the everted tube is defined as critical thickness, which is $H_{cirt}=0.58B$.



Fig. S3 Critical tube thickness for the wrinkling instability with respect to the wavenumber *m*. The red dot represents the critical thickness for the wrinkling instability in the an everted tube, which is H_{cirt} =0.58*B*.

Appendix C. Finite element simulation of crease formation in an everted tube

Creases are localized folds with a singular region of self-contact, around which the strain field is concentrated^{6, 7}. Consequently, the critical condition for the onset of creases and the subsequent growth of creases cannot be predicted by linear stability analyses. Following our previous studies⁸, in this letter, we conduct finite element simulation using commercial software ABAQUS to predict the critical condition of crease initiation and study the growth of creases in an everted cylindrical tube.

To avoid simulating the complex finite deformation of tube eversion process, the equilibrium stress state after eversion is introduced as initial stress through the user subroutine SIGINI in ABAQUS⁹. The stress field is obtained from the equilibrium analysis in *Appendix A* and the subroutine is called at the start of the crease analysis.

We assume that the multiple creases periodically distribute in the inner surface of an everted tube. Therefore, only half crease with symmetric boundary condition is adopted in the simulation as shown in the inset of Fig. 3, with θ being the sector angle. The number of creases is determined by π/θ . To calculate the strain energy of the everted tube with creases, we apply a radial displacement *d* at a point in the inner surface of the everted tube to induce the formation of a crease. The inner and outer surfaces of the tube are set to be traction free.

In the finite element simulation, the plane strain hybrid element CPE6MH is adopted. To resolve stress/strain field around the crease tip, the mesh size in our simulation is set to be much smaller than the crease depth d. In addition, mesh-to-mesh solution mapping⁹ is adopted in our simulation when the elements deform significantly from their original configurations and become severely distorted during the crease formation. The old, deformed mesh is replaced by a new mesh of better quality. The

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solutions are mapped from the old mesh to the new mesh so that the analysis can continue.

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