1 Two disc ion detection problem

We consider the problem of an inlaid disc of radius a subjected to a constant flux of H+ ions, with no flux boundary conditions on the rest of the plane. We would like to know how the concentration varies across the plane. In particular we would like to know what the concentration is on an inlaid detector disc of radius b, whose centre is a distance L away from the centre of the first disc.

The governing equation for the concentration of H+ ions is Laplace's equation:

$$\nabla^2 C = 0, \tag{1}$$

which in cylindrical polar coordinates is given by

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} = 0. \tag{2}$$

The boundary conditions are given by

$$D\frac{\partial C}{\partial z} = -Q, \quad \text{for } r < a, \ z = 0, \tag{3}$$

$$D\frac{\partial C}{\partial z} = 0, \quad \text{for } r > a, \ z = 0, \tag{4}$$

$$C(r,\, z) \to 0, \quad \text{as } \mid \mathbf{r} \mid \to \infty. \tag{5}$$

Here Q is the constant flux of ions from the disc, and has the dimensions mol m^{-2} s^{-1}. We shall also assume that the diffusion coefficient is constant. To non-dimensionalise, we choose the following scalings:

$$r = a\hat{r}, \quad z = a\hat{z}, \quad C = \frac{Qa}{D}\hat{C}. \tag{6}$$

On dropping the hats, the equations become

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} = 0, \tag{7}$$

with boundary conditions

$$\frac{\partial C}{\partial z} = -1, \quad \text{for } r < 1, \ z = 0, \tag{8}$$

$$\frac{\partial C}{\partial z} = 0, \quad \text{for } r > 1, \ z = 0, \tag{9}$$

$$C(r,\, z) \to 0, \quad \text{as } \mid \mathbf{r} \mid \to \infty. \tag{10}$$

Using the Hankel transform, it is easy to show that the solution for the concentration is given by

$$C(r,\, z) = \int_0^\infty J_0(\alpha r) J_1(\alpha) \exp(-\alpha z)\, \frac{d\alpha}{\alpha}, \tag{11}$$

where J_0 and J_1 are Bessel functions of order zero and one. The concentration on the plane $z = 0$ is therefore given by

$$C(r,\, 0) = \int_0^\infty J_0(\alpha r) J_1(\alpha)\, \frac{d\alpha}{\alpha}. \tag{12}$$

Now to find the concentration on a disc of radius b whose centre is a distance L away, we need to write our solution in terms of a coordinate system centred on this disc. Firstly let us non-dimensionalise these distances, so that

$$\beta = \frac{b}{a}, \quad \lambda = \frac{L}{a}. \tag{13}$$

Then in terms of a cylindrical polar coordinate system, $(s,\, \theta)$, centred on the second disc, we may write r as follows:

$$r = \sqrt{\lambda^2 + s^2 + 2\lambda s \cos\theta}, \tag{14}$$

and the total concentration $C_T(\beta,\, \lambda)$ on the second disc is equal to

$$C_T(\beta,\, \lambda) = \int_0^\beta \int_0^{2\pi} \int_0^\infty J_0(\alpha\sqrt{\lambda^2 + s^2 + 2\lambda s \cos\theta}) J_1(\alpha)\, \frac{s}{\alpha}\, d\alpha\, d\theta\, ds. \tag{15}$$

Note that we must have $\lambda > 1 + \beta$, so that the discs do not overlap. Now by Graf's addition theorem (Abramowich and Stegun), it may be shown that

$$J_0(\alpha\sqrt{\lambda^2 + s^2 + 2\lambda s \cos\theta}) = \sum_{k=-\infty}^{\infty} J_k(\alpha\lambda) J_k(\alpha s) \cos(k\theta). \tag{16}$$

On using this formula and integrating over θ, only the $k = 0$ term produces a non-zero result, so that

$$C_T(\beta,\, \lambda) = 2\pi \int_0^\beta \int_0^\infty J_0(\alpha\lambda) J_0(\alpha s) J_1(\alpha)\, \frac{s}{\alpha}\, d\alpha\, ds. \tag{17}$$

Integrating with respect to s, we obtain

$$C_T(\beta,\,\lambda) = 2\pi\beta \int_0^{\infty} J_0(\alpha\lambda)J_1(\alpha\beta)J_1(\alpha)\,\frac{d\alpha}{\alpha^2}. \tag{18}$$

Using a result of Bailey, this integral may be evaulated in terms of the fourth type of Appell's hypergeometric functions of two variables, so that

$$C_T(\beta,\,\lambda) = \frac{\pi\beta^2}{2\lambda} F_4[\frac{1}{2},\,\frac{1}{2};\,2,\,2;\,\frac{\beta^2}{\lambda^2},\,\frac{1}{\lambda^2}], \tag{19}$$

noting again that this is only true if $\lambda > 1+\beta$. Appell's F_4 function is represented by the power series

$$F_4(a,\,b;c,\,c';x,\,y) = \sum_{m,n=0}^{\infty} \frac{(a,\,m+n)(b,\,m+n)}{(c,\,m)(c',\,n)\,m!\,n!} x^m y^n, \quad |\,x\,|^{\frac{1}{2}} + |\,y\,|^{\frac{1}{2}} < 1, \tag{20}$$

where

$$(a,\,m) = \Gamma(a+m)/\Gamma(a), \quad (a,\,0) = 1. \tag{21}$$

Suppose that we take $\beta = 1$, so that the detector disc is the same size as the generator disc. Then we may plot $C_T(1,\,\lambda)$ as a function of λ for $\lambda > 2$. Truncating the series at 45 terms, the profile of the concentration as λ varies is shown in Figure 1.

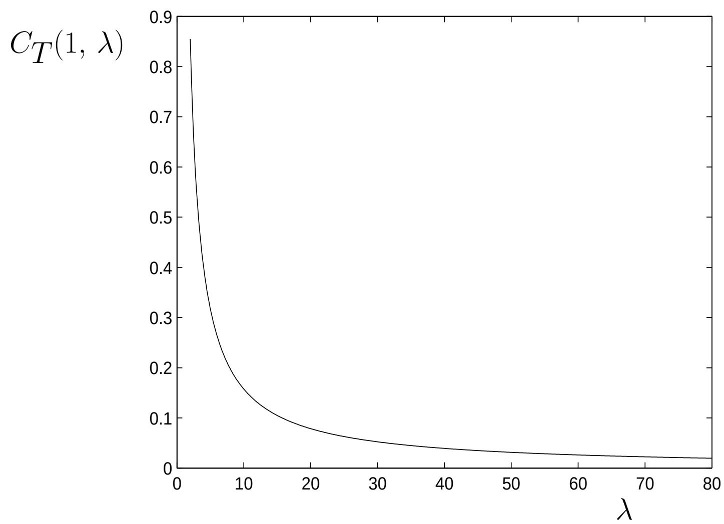

Figure 1: Graph of the total concentration on an inlaid detector disc with unit radius, as the distance λ between the detector disc and the generator disc varies.