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Electronic Supplementary Information (ESI): Edge Mode Amplification in Disordered Elastic Networks†

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A. Networks with pinned particles.

To eliminate floppy translational modes, we consider networks with pinned particles. We choose these particles randomly, starting from an isostatic network[1]. To ensure the the spatial homogeneity of constraints, we divide the network into small rectangles with one to two pinned particles chosen in each. Each time a particle is pinned, we remove d bonds connecting it to its best connected neighbors, so that the resulted network is still isostatic. An example of such a network is shown in Fig. S1.

Following the formalism described in the previous section, we can compute the singular values for the pinned network. Notice that the structure matrix $\mathscr{S}$ of the pinned network includes bonds connecting the pinned and free particles. The resulting distributions are shown in the bottom panel of Fig. 3 in the main text. The cusp at $\kappa = 0$ is seems to disappear, and the distribution nicely converges to our theoretical prediction.

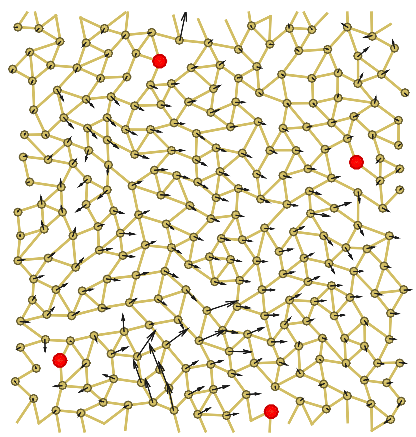

Fig. S1 An isostatic 2-dimensional network $z = 4$ with four pinned particles, shown in red. The boundaries are open on the left and right hand sides of the network, and periodic in the vertical direction. Black arrows show a typical floppy mode penetrating the network.

B. A general perturbative expansion.

In this section we specialize the general result of[2] to the case where the eigenvalues ξ are all close to unity. More precisely, we write $\xi := e^{\sqrt{\varepsilon}\zeta}$ with $\varepsilon \to 0$, and where ζ has an arbitrary even distribution $P(\zeta) = P(-\zeta)$. We will denote its variance as ζ_2 and its fourth moment as ζ_4.

The first task is to compute $\psi(z)$ as defined in the text, Eq.(5), up to order ε^2. One readily finds:

$$\psi(z) = \frac{z}{1-z} + \left[\varepsilon\frac{\zeta_2}{2} + \varepsilon^2\frac{\zeta_4}{24}\right]\frac{z(1+z)}{(1-z)^3} + \varepsilon^2\zeta_4\frac{z^2(1+z)}{(1-z)^5} + O(\varepsilon^4). \tag{S1}$$

Now, one should invert this relation to find χ such that $\chi(\psi(z)) =$

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z. Proceeding order by order in ε leads to:

$$\chi(z) = \frac{z}{1+z}\big(1 - \left[\varepsilon\frac{\zeta_2}{2} + \varepsilon^2\frac{\zeta_4}{24}\right](1+2z) + \varepsilon^2\frac{\zeta_2^2}{4}(1+z)(1+2z)\left[(1+4z) - 2\varkappa z\right]\big), \qquad \text{(S2)}$$

where we introduced the ratio $\varkappa = \zeta_4/\zeta_2^2$. From χ one deduces the S-transform as $S(z) = (1+z)\chi(z)/z$ and finally $F(z) = 1/\sqrt{S(z-1)}$ to order ε^2:

$$F(z) = 1 + \left[\varepsilon\frac{\zeta_2}{2} + \varepsilon^2\frac{\zeta_4}{24}\right](z - \frac{1}{2}) + \varepsilon^2\frac{\zeta_2^2}{4}(2z-1)\left[\varkappa z(z-1) - \frac{16z^2 - 18z + 3}{8}\right] \qquad \text{(S3)}$$

This allows us to determine the edges of the distribution as $t_{\min} = F(0)$ and $t_{\max} = F(1)$, which are found to be at a distance $O(\varepsilon)$ of $t = 1$. The cumulative distribution $P(t)$ of the eigenvalues of the asymptotic product as the inverse of $F(z)$:

$$F(P(t)) = t. \qquad \text{(S4)}$$

Taking the derivative of this expression with respect to t and introducing the pdf $f(t) = P'(t)$ one gets:

$$f(t)\left.\frac{\partial F}{\partial z}\right|_{z=P(t)} = 1 \qquad \text{(S5)}$$

From the above expression of $F(z)$, one derives, setting $z := (1+u)/2$

$$\frac{\partial F}{\partial z} = \left[\varepsilon\frac{\zeta_2}{2} + \varepsilon^2\frac{\zeta_4}{24}\right] + \varepsilon^2\frac{\zeta_2^2}{8}\left[\varkappa(3u^2-1) - (6u^2 - u + 1)\right]. \qquad \text{(S6)}$$

To order ε^2, one therefore only needs the relation between t and u at order ε, which reads:

$$t = 1 + \frac{1}{4}\varepsilon\zeta_2 u. \qquad \text{(S7)}$$

It turns out to be more convenient to work with the variable x such that $\varepsilon\zeta_2 x := 4\ln t$. In particular, due to the symmetry $t \to 1/t$, one must find that the distribution of x is even. The final result is, to leading order:

$$f(x) = 2 + \varepsilon\frac{\zeta_2}{2}\left[(\varkappa - 1) + \frac{\zeta_2}{4}(6 - 3\varkappa)x^2\right], \qquad \text{(S8)}$$

for $x \in [-1 + \varepsilon\zeta_2/4, 1 - \varepsilon\zeta_2/4]$, and zero otherwise. By inspection, this distribution is indeed even in x.
The conclusion of this computation is that:

- To lowest order in ε, the distribution of eigenvalues t is universal and uniform around unity, in a symmetric interval of size $\varepsilon\zeta_2/2$. This ties up with the result obtained in the context of 1-d disordered conductors in the corresponding limit, see[3], section III.B.
- To next order, we find that the distribution acquires a parabolic shape, with a curvature that depends on $\varkappa$, i.e. the kurtosis of the distribution of $\zeta \propto \ln\xi$. For a bimodal distribution, one finds $\varkappa = 1$ and a positive curvature, such that $t = 1$ is a minimum of the distribution. As soon as $\varkappa > 2$ (for example for a Gaussian distribution of ζ), the curvature is negative and $t = 1$ is the mode of the distribution. The case of a Cauchy distribution for $\lambda = \sqrt{\xi}$, as in the text, corresponds to $\varkappa = 5$, such that the distribution indeed is expected to be bell shaped in this case.
- One also concludes that if freeness holds, the only way to obtain a cusp in the distribution at $t = 1$ is to have an infinite kurtosis $\varkappa$ at the microscopic level. This is clearly not the case of our "slab" matrices, see Fig. 5.

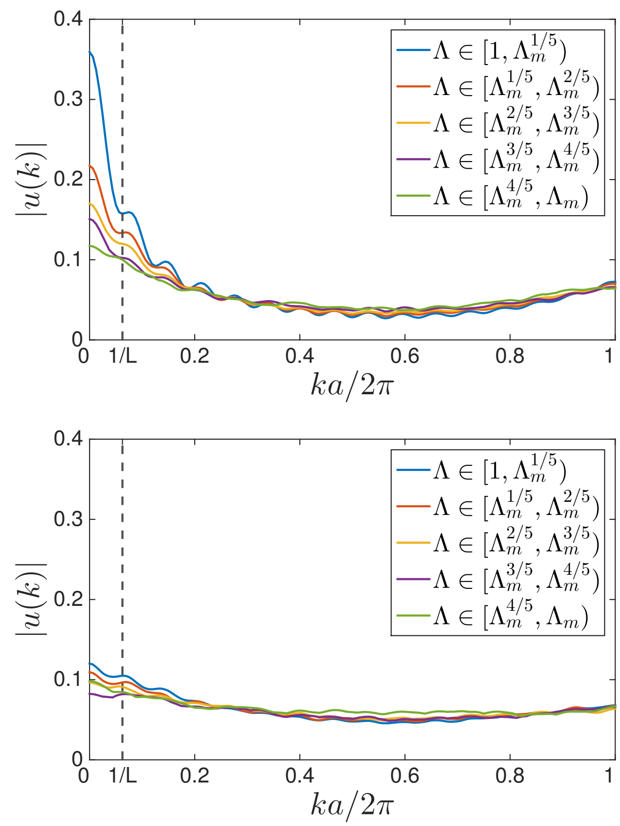

Fig. S2 Fourier transform of the displacement fields $u(k)$ (defined in the text) corresponding to different singular value range Λ for free isostatic networks (top) and networks with pinned particles (bottom). The fraction of pinned particles is 0.12. $L = 16$, and a is the typical distance between the particles.

C. Overlap with sinusoidal modes.

To check that the coupling between translational (or long wavelength) modes and floppy modes is responsible for the apparent cusp in the distribution $g(\kappa)$, we measure the Fourier components of the floppy modes near $\Lambda = 1$. To do so, we use the following recipe. Let us consider that the x-axis is in the horizontal direction which connects the two open boundaries, and that the y-axis is the vertical direction which is periodic. 1. Noticing that the amplitude of the floppy modes varies exponentially along the x direction, we first rescale the field by $\Lambda^{x/L}$, so that the displacements are now of the similar magnitude when x varies, so that each slab contributes similarly to the Fourier transform. 2. We normalize the mode, and get $\delta\vec{r} = \delta r_x \hat{x} + \delta r_y \hat{y}$ at each node j. 3. We make the Fourier transformation along the direction parallel to the open boundaries by $\vec{u}(k) = \sum_j \delta r_x(j) e^{-iky_j}\hat{x} + \delta r_y(j) e^{-iky_j}\hat{y}$. 4. Finally, we compute the absolute value of $\vec{u}(k)$.

The results averaged over various Λ are shown in Fig. S2. In the original networks (no pinned particles), for modes near $\Lambda = 1$ we clearly see a strong peak at $k = 0$ and $k = 2\pi n/La$ for small integers n. The closer to $\Lambda = 1$, the stronger is the effect. When a fraction of the particles are pinned, this strong coupling with translational modes and sinusoidal modes disappear. The Fourier decomposition is then close to flat, as it should be in the free matrix approximation where eigenvectors are random.

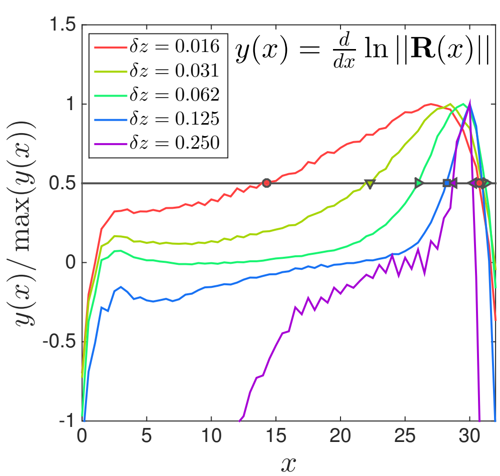

Fig. S3 Growth rate of the response magnitude per unit distance $y(x) \equiv \frac{1}{||\mathbf{R}(x)||} \frac{d||\mathbf{R}(x)||}{dx}$ rescaled by its peak value, as a function of position x. The width of the amplification zone ℓ_{edge} is determined as the distance between the free boundary and the position of the half maximum of that plot, indicated by the symbols at the intersection with the black line.

D. Width of the amplification zone near a free boundary.

To define in a robust way the width ℓ_{edge} over which edge modes amplify the response, we consider the growth rate of the response magnitude per unit distance $\frac{1}{||\mathbf{R}(x)||} \frac{d||\mathbf{R}(x)||}{dx}$ as shown in Fig. S3. ℓ_{edge} is then defined as the distance between the free boundary and the position of the half maximum of that quantity (where curves cross the black horizontal line on that plot).

References

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