Supplementary Information (ESI) for Soft Matter

Extracting Shape from Curvature Evolution in Moving Surfaces

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Appendix A

The purpose of this appendix is to introduce the surface tensor decomposition and applied to express four basic curvatures introduced in section 2: H,K, D, and C.

The four independent basis surface tensors are [23]:

$$\left\{\mathbf{I}_s,\ \mathbf{q},\ \boldsymbol{\varepsilon}_s, \mathbf{q}_1 = \mathbf{q}\cdot\boldsymbol{\varepsilon}_s\ \right\} \tag{A.1}$$

The matrix representations of the basis vectors in the principal frame are [23]:

$$\mathbf{I}_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\ \mathbf{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\varepsilon}_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{q}_1 = \mathbf{q}\cdot\boldsymbol{\varepsilon}_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.2}$$

where $\boldsymbol{\varepsilon}_s$ is the surface alternator tensor. The tensor basis orthonormality yields the following results:

$$\mathbf{I}_s : \mathbf{I}_s = \mathbf{q} : \mathbf{q} = \boldsymbol{\varepsilon}_s : \boldsymbol{\varepsilon}_s = \mathbf{q}_1 : \mathbf{q}_1 = 2, \mathbf{I}_s : \mathbf{q} = \mathbf{I}_s : \boldsymbol{\varepsilon}_s = \mathbf{I}_s : \mathbf{q}_1 = \mathbf{q} : \boldsymbol{\varepsilon}_s = \mathbf{q} : \mathbf{q}_1 = \boldsymbol{\varepsilon}_s : \mathbf{q}_1 = 0,$$

$$\mathbf{I}_s \cdot \mathbf{q} = \mathbf{q}, \mathbf{I}_s \cdot \mathbf{q}_1 = \mathbf{q}_1, \mathbf{I}_s \cdot \boldsymbol{\varepsilon}_s = \boldsymbol{\varepsilon}_s, \mathbf{q}\cdot\mathbf{q}_1 = \boldsymbol{\varepsilon}_s, \mathbf{q}\cdot\boldsymbol{\varepsilon}_s = \mathbf{q}_1, \mathbf{q}_1 \cdot \boldsymbol{\varepsilon}_s = \mathbf{q} \tag{A.3}$$

Any 2×2 tensor **Z** can be expanded as:

$$\mathbf{Z} = \underbrace{\frac{1}{2}(\mathbf{Z}:\mathbf{I}_s)\mathbf{I}_s}_{trace} + \underbrace{\frac{1}{2}(\mathbf{Z}:\mathbf{q})\mathbf{q}}_{diagonal\ traceless} + \underbrace{\frac{1}{2}(\mathbf{Z}:\boldsymbol{\varepsilon}_s)\boldsymbol{\varepsilon}_s}_{antisymmetric} + \underbrace{\frac{1}{2}(\mathbf{Z}:\mathbf{q}_1)\mathbf{q}_1}_{symmetric\ \text{off-diagonal}} \tag{A.4}$$

where the subtext identifies the nature of the tensor. A symmetric 2×2 tensor diagonal in the principal coordinate frame simplifies to:

$$\mathbf{Z} = \frac{1}{2}(\mathbf{Z}:\mathbf{I}_s)\mathbf{I}_s + \frac{1}{2}(\mathbf{Z}:\mathbf{q})\mathbf{q} \tag{A.5}$$

Using eqn.(A.6), the curvature tensor $\mathbf{b}$ can be decomposed into a trace and a deviatoric curvature tensor: $\mathbf{b} = \mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q}$.

Next we use these results to derive the results in Table 1.

$$\mathbf{b} = \mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q};\ 2\mathrm{H} = \kappa_1 + \kappa_2;\ 2\mathrm{D} = \kappa_1 - \kappa_2 \tag{A.6}$$

The magnitude of the deviatoric curvature D is a useful non-sphericity index, since for a sphere D=0. Defining $\mathbf{f} = \mathrm{H}\mathbf{I}_s - \mathbf{b} = -\mathrm{D}\mathbf{q}$, we find, using $\mathbf{b}$ and $\mathbf{f}$, the average H, Gaussian K, square deviatoric D^2, and Casorati C curvatures:

$$\mathrm{H} = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\mathbf{I}_s \cdot \boldsymbol{\varepsilon}_s \cdot \mathbf{b}) = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\boldsymbol{\varepsilon}_s \cdot \mathbf{b}) = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\mathrm{H}\boldsymbol{\varepsilon}_s \cdot \mathbf{I}_s + \mathrm{D}\boldsymbol{\varepsilon}_s \cdot \mathbf{q}) = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\mathrm{H}\boldsymbol{\varepsilon}_s - \mathrm{D}\mathbf{q}_1) = \mathrm{H}$$

$$\mathrm{K} = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\mathbf{b} \cdot \boldsymbol{\varepsilon}_s \cdot \mathbf{b}) = \frac{1}{2}\boldsymbol{\varepsilon}_s : ((\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q}) \cdot \boldsymbol{\varepsilon}_s \cdot (\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q})) = \frac{1}{2}\boldsymbol{\varepsilon}_s : (\mathrm{H}^2\boldsymbol{\varepsilon}_s - \mathrm{D}^2\boldsymbol{\varepsilon}_s) = \mathrm{H}^2 - \mathrm{D}^2$$

$$\mathrm{D}^2 = \frac{1}{2}\mathbf{I}_s : (\mathbf{f} \cdot \mathbf{I}_s \cdot \mathbf{f}) = \frac{1}{2}\mathbf{I}_s : (\mathrm{D}\mathbf{q} \cdot \mathbf{I}_s \cdot \mathrm{D}\mathbf{q}) = \frac{1}{2}\mathbf{I}_s : (\mathrm{D}\mathbf{q} \cdot \mathrm{D}\mathbf{q}) = \frac{1}{2}\mathbf{I}_s : (\mathrm{D}^2\mathbf{I}_s) = \mathrm{D}^2$$

$$C^2 = \frac{1}{2}\mathbf{I}_s : ((\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q}) \cdot \mathbf{I}_s \cdot (\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q})) = \frac{1}{2}\mathbf{I}_s : ((\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q}) \cdot (\mathrm{H}\mathbf{I}_s + \mathrm{D}\mathbf{q}))$$
$$= \frac{1}{2}\mathbf{I}_s : (\mathrm{H}^2\mathbf{I}_s + \mathrm{D}^2\mathbf{I}_s + \mathrm{HD}\mathbf{q}) = \mathrm{H}^2 + \mathrm{D}^2$$

Appendix B

The purpose of this appendix is to derive equation (13). The co-rotational derivative $\overset{\circ}{\mathbf{b}}$ of the curvature tensor is [29]:

$$\overset{\circ}{\mathbf{b}} = (\mathbf{U} \cdot (\nabla_s \mathbf{b})) \cdot \mathbf{I}_s + \mathbf{b} \cdot \mathbf{b}\mathrm{V} + (\nabla_s \nabla_s \mathrm{V}) + (\mathbf{W} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{W}) \tag{B.1}$$

where **U** is the tangential velocity vector , V the normal speed, and W the surface vorticity tensor. Under uniform normal motion $\left(\mathbf{U}=0, \mathbf{W}=0, \nabla_s \mathrm{V}=\mathbf{0}\right)$we decompose $\mathring{\mathbf{b}}$ and find the kinematics of the three curvatures [14,29]:

$$\dot{\mathrm{H}}=\left(\mathrm{H}^2+\mathrm{D}^2\right)\mathrm{V}, \quad \dot{\mathrm{D}}=2\mathrm{HDV}, \dot{\mathrm{K}}=2\mathrm{HKV} \tag{B.2 a-c}$$

To find the astigmatic trajectories we work with $\{\dot{\mathrm{H}};\dot{\mathrm{D}}\}$. Dividing eqn.((B.2a) with (B.2b):

$$\frac{\mathrm{dH}}{\mathrm{dD}}=\frac{\mathrm{H}}{2\mathrm{D}}+\frac{\mathrm{D}}{2\mathrm{H}} \tag{B.3}$$

Introducing $H=uD$, leads to:

$$\frac{\mathrm{dH}}{\mathrm{dD}}=\frac{\mathrm{du}}{\mathrm{dD}}\mathrm{D}+\mathrm{u};\frac{\mathrm{du}}{\mathrm{dD}}\mathrm{D}=\left(\frac{\mathrm{u}}{2}+\frac{1}{2\mathrm{u}}-\mathrm{u}\right);\frac{2\mathrm{udu}}{\left(1-\mathrm{u}^2\right)}=\frac{\mathrm{dD}}{\mathrm{D}} \tag{B.4}$$

whose solution is:

$$\left(1-\mathrm{u}^2\right)=\frac{\mathrm{m}}{\mathrm{D}} \tag{B.5}$$

where m is a shape constant. Replacing H=uD we find:

$$H^2-D^2=K=-mD \tag{B.6}$$

Using curvatures$\left(\kappa_1,\kappa_2\right)$, the last equality yields:

$$\kappa_1\kappa_2=\frac{m}{2}\left(\kappa_2-\kappa_1\right) \tag{B.7}$$

Dividing by $\kappa_1\kappa_2$ we find the astigmatism equation [30,31]:

$$\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)=\frac{2}{m}\rightarrow \mathrm{R}_2=\mathrm{R}_1+\frac{2}{m} \tag{B.8}$$

Appendix C

The purpose of this appendix is to derive the shape-curvedness kinematics and related trajectories under astigmatic curvature flow. Using $\mathrm{H} = \mathrm{C}\sin\left(\mathrm{S}\pi/2\right)$; $\mathrm{D} = \mathrm{C}\cos\left(\mathrm{S}\pi/2\right)$ computing the partial derivatives $\left(\partial\mathrm{C}/\partial\mathrm{H}, \partial\mathrm{C}/\partial\mathrm{D}, \partial\mathrm{S}/\partial\mathrm{H}, \partial\mathrm{S}/\partial\mathrm{D}\right)$ and replacing in the total time rate of changes:

$$\dot{\mathrm{C}} = \frac{\partial\mathrm{C}}{\partial\mathrm{H}}\dot{\mathrm{H}} + \frac{\partial\mathrm{C}}{\partial\mathrm{D}}\dot{\mathrm{D}};\ \dot{\mathrm{S}} = \frac{\partial\mathrm{S}}{\partial\mathrm{H}}\dot{\mathrm{H}} + \frac{\partial\mathrm{S}}{\partial\mathrm{D}}\dot{\mathrm{D}} \tag{C.1}$$

we find eqns. (15 a-b) for the time evolution of Casorati's curvature and shape. Using $K = -mD$ as defined in Section 3, the astigmatic flow in the SC morphological space is:

$$-C^2\cos\left(S\pi\right) = -mC\cos\left(\frac{S\pi}{2}\right) \tag{C.2}$$

which leads to the sought after result presented in Section 3.