Flexibility of clay layers: supporting information

Tulio Honorio,[*,†,‡] Laurent Brochard,[†] Matthieu Vandamme,[†] and Arthur Lebée[†]

†*Université Paris-Est, Laboratoire Navier (UMR 8205), CNRS, ENPC, IFSTTAR, 6 & 8 Avenue Blaise Pascal, 77455 Marne-la-Vallée, France*

‡*Current address: Université Paris-Est Créteil, Laboratoire Modélisation et Simulation Multi-Echelle, MSME UMR 8208 CNRS, 94010, Créteil cedex, France*

E-mail: tulio.honorio-de-faria@u-pec.fr;tuliohfaria@gmail.com

Phone: +(33) 1 64 15 36 59

Bending modulus of clay layers

Consider a periodic plate, with period L, subjected to punctual forces in a direction x and constanty distributed in a direction y perpendicular to x. The opposed forces are $F_z(x) = \begin{cases} F & \text{if } x \bmod L = -\frac{L}{4} \\ -F & \text{if } x \bmod L = \frac{L}{4} \end{cases}$, where F is given in force par length unit (in y direction). The normal efforts are zero (no in-plane loadings). Due to the symmetry of the problem, the momentum are periodic and odd in x direction and constant in y direction:

$$M_{ab}(x,y) = M_{ab}(x) = -M_{ab}(-x) = M_{ab}(x+L) \tag{1}$$

The equilibrium of moments yields:

$$\frac{\partial^2 M_{xx}}{\partial x^2} = 0 \text{ et } \left.\frac{\partial M_{xx}}{\partial x}\right|_{x^+} - \left.\frac{\partial M_{xx}}{\partial x}\right|_{x^-} = \begin{cases} F & \text{if } x \bmod L = -\frac{L}{4} \\ -F & \text{if } x \bmod L = \frac{L}{4} \end{cases} \tag{2}$$

The moments M_{yy} and M_{xy} vanish and only M_{xx} is non-zero. The expression of M_{xx} is obtained by integration of the last equation accounting for the symmetry conditions $M_{xx}\left(x = -\frac{L}{2}\right) = M_{xx}\left(x = -\frac{L}{2} + L = \frac{L}{2}\right) = -M_{xx}\left(x = -\left(-\frac{L}{2}\right) = \frac{L}{2}\right) = 0$:

$$M_{xx}(x) = \frac{F}{2} \begin{cases} -\frac{L}{2} - x & \text{if } -\frac{L}{2} < x < -\frac{L}{4} \\ x & \text{if } -\frac{L}{4} < x < \frac{L}{4} \\ \frac{L}{2} - x & \text{if } \frac{L}{4} < x < \frac{L}{2} \end{cases} \tag{3}$$

Since the displacement respects the same symmetries, we have $u_x = 0$, $u_y = 0$ and $u_z(x, y) = u_z(x) = -u_z(-x) = u_z(x + L)$ with $u_z\left(x = -\frac{L}{2}\right) = u_z\left(x = \frac{L}{2}\right) = 0$. The expression of u_z is obtained from integration of the moment M_{xx}. The constants of integration can be obtained from the conditions imposed on the boundary and from the continuity of the derivatives of displacement u_z. We have:

$$u_z(x) = \frac{1}{D_{xxxx}} \begin{cases} -\frac{F}{12}\left(\frac{L}{2} + x\right)^3 + \frac{F}{4}\left(\frac{L}{4}\right)^2\left(\frac{L}{2} + x\right) & \text{if } -\frac{L}{2} < x < -\frac{L}{4} \\ \frac{F}{12}x^3 - \frac{F}{4}\left(\frac{L}{4}\right)^2 x & \text{if } -\frac{L}{4} < x < \frac{L}{4} \\ \frac{F}{12}\left(\frac{L}{2} - x\right)^3 - \frac{F}{4}\left(\frac{L}{4}\right)^2\left(\frac{L}{2} - x\right) & \text{if } \frac{L}{4} < x < \frac{L}{2} \end{cases} \tag{4}$$

For a system controlled by a displacement δ, instead of the force F, we have:

$$\delta = u_z\left(-\frac{L}{4}\right) = -u_z\left(\frac{L}{4}\right) = \frac{F}{6D_{xxxx}}\left(\frac{L}{4}\right)^3 \Rightarrow F = 6\left(\frac{4}{L}\right)^3 D_x \delta \tag{5}$$

Finally, the free elastic energy W per units of length (with respect to y direction) is equal to the work of external applied forces: :

$$W = 2\int_0^{\delta} F d\delta = 12\left(\frac{4}{L}\right)^3 D_x \int_0^{\delta} \delta d\delta = 6\left(\frac{4}{L}\right)^3 D_{xxxx}\delta^2 = \left(\frac{L}{4}\right)^3 \frac{F^2}{6D_{xxxx}} \tag{6}$$

This result can also be found by integration of the local elastic energy over the whole plate:

$$W = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} M_{xx}\frac{\partial^2 u_z}{\partial x^2}dx = \frac{F^2}{4D_{xxxx}}\left[\frac{2}{3}\left(\frac{L}{4}\right)^3\right] = \left(\frac{L}{4}\right)^3 \frac{F^2}{6D_{xxxx}} \tag{7}$$

The dimensionless formulation is retrieved by the definiton of the dimensionless variables $x^* = \frac{4}{L}x$, $u_z^* = \frac{4}{L}u_z$, $F^* = \frac{F}{D_{xxxx}}\left(\frac{L}{4}\right)^2$, $M_{xx}^* = \frac{M_{xx}}{D_{xxxx}}\frac{L}{4}$, $\delta^* = \frac{4}{L}\delta = \frac{F^*}{6}$ and $W^* = \frac{W}{D_{xxxx}}\frac{L}{4}$. Thus, the dimensionless forms of moment, displacement and free elastic energy read, respectively:

$$\frac{M_{xx}^*(x)}{F^*} = 6\frac{M_{xx}^*(x)}{\delta^*} = \begin{cases} -\frac{1}{2}(2+x^*) & \text{if } -2 < x^* < -1 \\ \frac{1}{2}x^* & \text{if } -1 < x^* < 1 \\ \frac{1}{2}(2-x^*) & \text{if } 1 < x^* < 2 \end{cases} \tag{8}$$

$$\frac{u_z^*(x)}{F^*} = 6\frac{u_z^*(x)}{\delta^*} = \begin{cases} -\frac{1}{12}(2+x^*)^3 + \frac{1}{4}(2+x^*) & \text{if } -2 < x^* < -1 \\ \frac{1}{12}(x^*)^3 - \frac{1}{4}x^* & \text{if } -1 < x^* < 1 \\ \frac{1}{12}(2-x^*)^3 - \frac{1}{4}(2-x^*) & \text{if } 1 < x^* < 2 \end{cases} \tag{9}$$

$$W^* = 6(\delta^*)^2 = \frac{1}{6}(F^*)^2 \tag{10}$$

For the sake of concision D_{xxxx} is called D_x in the paper, with x being the direction in which the problem is treated; y, the corresponding in-plane perpendicular direction and z, the direction perpendicular to the plate.