Supplementary Information for "Modeling the relative dynamics of DNA-coated colloids"

James P. Lee-Thorp and Miranda Holmes-Cerfon*
Courant Institute of Mathematical Sciences, New York University, NY, USA

S1. DETAILS OF DISC CALCULATIONS

This section contains auxiliary details of the calculations from Section 5.

The first derivatives $\partial(l,\theta)/\partial(x,\xi)$ are calculated by implicit differentiation from the relations (20) to be

$$\frac{\partial l}{\partial x} = \sin\theta, \quad \frac{\partial l}{\partial \xi} = R\sin(\xi+\theta), \quad \frac{\partial \theta}{\partial x} = \frac{\cos\theta}{l}, \quad \frac{\partial \theta}{\partial \xi} = \frac{R}{l}\cos(\xi+\theta)\,. \tag{S1}$$

The second derivatives $\partial^2(l,\theta)/\partial(x,\xi)^2$ are

$$\begin{aligned}
\frac{\partial^2 l}{\partial x^2} &= \frac{\cos^2\theta}{l}, \qquad \frac{\partial^2 l}{\partial \xi^2} = \frac{R^2}{l}\cos(\xi+\theta)\left(\frac{l}{R}+\cos(\xi+\theta)\right), \qquad \frac{\partial^2 l}{\partial x\partial \xi} = \frac{R}{l}\cos\theta\cos(\xi+\theta),\\
\frac{\partial^2 \theta}{\partial x^2} &= \frac{\sin 2\theta}{l^2}, \qquad \frac{\partial^2 \theta}{\partial \xi^2} = \frac{R^2}{l^2}\sin(\xi+\theta)\left(\frac{l}{R}+2\cos(\xi+\theta)\right), \qquad \frac{\partial^2 \theta}{\partial x\partial \xi} = \frac{-R}{l^2}\sin(\xi+2\theta)\,.
\end{aligned} \tag{S2}$$

The second derivatives of energy, calculated by applying the chain rule to expression (19), are

$$\begin{aligned}
\partial_{xx}E_{tether} &= k_l\left(\frac{\partial l}{\partial x}\right)^2 + k_\theta\left(\frac{\partial \theta}{\partial x}\right)^2 + k_l(l-\bar{l})\frac{\partial^2 l}{\partial x^2} + k_\theta\theta\frac{\partial^2\theta}{\partial x^2}\\
&= k_l + \frac{k_\theta}{l^2}(\cos^2\theta + \theta\sin 2\theta)\\
\partial_{\xi\xi}E_{tether} &= k_l\left(\frac{\partial l}{\partial \xi}\right)^2 + k_\theta\left(\frac{\partial \theta}{\partial \xi}\right)^2 + k_l(l-\bar{l})\frac{\partial^2 l}{\partial \xi^2} + k_\theta\theta\frac{\partial^2\theta}{\partial \xi^2}\\
&= k_l R^2\left(1+\frac{l}{R}\cos(\xi+\theta)\right) + \frac{k_\theta R^2}{l^2}\left(\cos^2(\xi+\theta) + \theta\sin(2(\xi+\theta)) + \frac{l}{R}\theta\sin(\xi+\theta)\right)\\
\partial_{x\xi}E_{tether} &= k_l\left(\frac{\partial l}{\partial x}\right)\left(\frac{\partial l}{\partial \xi}\right) + k_\theta\left(\frac{\partial \theta}{\partial x}\right)\left(\frac{\partial \theta}{\partial \xi}\right) + k_l(l-\bar{l})\frac{\partial^2 l}{\partial x\partial\xi} + k_\theta\theta\frac{\partial^2\theta}{\partial x\partial\xi}\\
&= k_l R\cos\xi + \frac{k_\theta R}{l^2}\left(\cos\theta\cos(\xi+\theta) - \theta\sin(\xi+2\theta)\right)
\end{aligned}$$

S2. DERIVATION OF THE COARSE-GRAINED DYNAMICS: N TETHERS

In this section we derive the coarse-grained dynamics of the full system of N tethers. We use the same asymptotic procedure as in Section 6 for the case of one tether, only now the generator is more complicated. We start by explicitly showing the generator for $N=3$ in order to better illustrate its structure, and then outline the generator and derivation of the coarse-grained dynamics for N tethers.

For $N=3$ tethers, the set of possible states for the collection of tethers is $\{u,b\}^3$ so we need a vector of size $2^3=8$ to represent the collection of states:

$$\boldsymbol{f} = (f_{uuu}, f_{uub}, f_{ubu}, f_{buu}, f_{ubb}, f_{bub}, f_{bbu}, f_{bbb})^T. \tag{S3}$$

For example, the state where all tethers are unbound is identified with the component f_{uuu} and the state where only the second tether is bound to the interval is identified with the component f_{ubu}.

* holmes@cims.nyu.edu

The full generator will still have the abstract decomposition $\mathcal{L} = Q + U + B + V$ as in (24), but now the generator for each sub-process will be an 8×8 matrix of operators. We now write down each generator in turn.

The generator associated with the binding/unbinding dynamics is

$$Q^T = \begin{pmatrix} -3\lambda & \nu & \nu & \nu & 0 & 0 & 0 & 0 \\ \lambda & -2\lambda-\nu & 0 & 0 & \nu & 0 & 0 & 0 \\ \lambda & 0 & -2\lambda-\nu & 0 & \nu & 0 & \nu & 0 \\ \lambda & 0 & 0 & -2\lambda-\nu & 0 & \nu & \nu & 0 \\ 0 & \lambda & \lambda & 0 & -\lambda-2\nu & 0 & 0 & \nu \\ 0 & \lambda & 0 & \lambda & 0 & -\lambda-2\nu & 0 & \nu \\ 0 & 0 & \lambda & \lambda & 0 & 0 & -\lambda-2\nu & \nu \\ 0 & 0 & 0 & 0 & \lambda & \lambda & \lambda & -3\nu \end{pmatrix}, \tag{S4}$$

where, for display purposes, we have defined $\lambda \equiv q_{on}$, and $\nu \equiv q_{off}$.

The generator for the evolution of the unbound tether lengths, following the ordering of the states given in (S3), is the diagonal matrix

$$U = \text{diag}\left(\sum_{j=1,2,3} \mathcal{L}_j^{OU}, \sum_{j=1,2} \mathcal{L}_j^{OU}, \sum_{j=1,3} \mathcal{L}_j^{OU}, \sum_{j=2,3} \mathcal{L}_j^{OU}, \mathcal{L}_1^{OU}, \mathcal{L}_2^{OU}, \mathcal{L}_3^{OU}, 0 \right).$$

where

$$\mathcal{L}_j^{OU} = -k\gamma^{-1} l_j \partial_{l_j} + \gamma^{-1}\beta^{-1}\partial_{l_j}^2 . \tag{S5}$$

For example, the element $U_{33} = \sum_{j=1,3} \mathcal{L}_j^{OU}$ acts on f_{ubu}, corresponding to the state for which only tethers $j = 1$ and 3 are unbound.

The generator for the evolution of bound tethers is the diagonal matrix

$$B = \text{diag}\left(0, \mathcal{L}_3^{bd}, \mathcal{L}_2^{bd}, \mathcal{L}_1^{bd}, \sum_{j=2,3} \mathcal{L}_j^{bd}, \sum_{j=1,3} \mathcal{L}_j^{bd}, \sum_{j=1,2} \mathcal{L}_j^{bd}, \sum_{j=1,2,3} \mathcal{L}_j^{bd} \right)$$

where

$$\mathcal{L}_j^{bd} = v\partial_{l_j} . \tag{S6}$$

For example, the element $B_{33} = \mathcal{L}_2^{bd}$ acts on f_{ubu}, corresponding to the state for which only the $j = 2$ tether is bound.

The generator for the interval velocity dynamics for the 3 tether system is the diagonal matrix

$$V = \text{diag}\left(0, \mathcal{L}_3^{int}, \mathcal{L}_2^{int}, \mathcal{L}_1^{int}, \sum_{j=2,3} \mathcal{L}_j^{int}, \sum_{j=1,3} \mathcal{L}_j^{int}, \sum_{j=1,2} \mathcal{L}_j^{int}, \sum_{j=1,2,3} \mathcal{L}_j^{int} \right)$$

where

$$\mathcal{L}_j^{int} = -\frac{k}{m} l_j \partial_v . \tag{S7}$$

Now consider the generator for the N tether system, which is a natural generalization of the generator for the 3 tether system. For general $N \geq 1$, we need a vector of length 2^N to represent the possible states of the system:

$$\boldsymbol{f} = (f_{u\ldots u}, f_{u\ldots ub}, f_{u\ldots ubu}, f_{u\ldots ubuu}, \ldots, f_{b\ldots bub}, f_{b\ldots bu}, f_{b\ldots b})^T . \tag{S8}$$

Suppose the states are labelled $1, 2, \ldots, 2^N$. Let $b(i), u(i)$ be the set of tethers which are bound/unbound in state i respectively. For example, $b(3) = \{2^N - 1\}$ and $u(3) = \{1, 2, \ldots, 2^N - 2, 2^N\}$. Let $|b(i)|, |u(i)|$ be the number of bound/unbound tethers. Clearly $|b(i)| + |u(i)| = N$.

It will be convenient to define a matrix $S \in \{0,1\}^{2^N \times N}$ to be the matrix whose rows (corresponding to different states) are a set of flags indicating whether each tether (the columns) is bound (1) or unbound (0) in each state, i.e. $S_{ij} = 1$ if tether $j \in b(i)$, $S_{ij} = 0$ if tether $j \in u(i)$. The matrix which flags unbound tethers is $\underline{1} - S$, where $\underline{1}$ is the $2^N \times N$-dimensional matrix whose entries are all 1.

The generator Q for the binding and unbinding process has components

$$Q_{ij} = \begin{cases} \lambda & \text{if } |b(j)| = |b(i)| + 1 \\ \nu & \text{if } |u(j)| = |u(i)| + 1 \\ -\lambda|u(i)| - \nu|b(i)| & \text{if } i = j \end{cases} \tag{S9}$$

The N tether generalizations of the U, B, and V matrices are

$$\begin{aligned} U &= \operatorname{diag}\left(\sum_{j=1,\ldots,N} \mathcal{L}_j^{OU}, \sum_{j=1,\ldots,N-1} \mathcal{L}_j^{OU}, \ldots, \sum_{j=N,N-1} \mathcal{L}_j^{OU}, \mathcal{L}_1^{OU}, \ldots, \mathcal{L}_{N-1}^{OU}, \mathcal{L}_N^{OU}, 0 \right) \\ &= \operatorname{diag}\left((\underline{1} - S)\mathcal{L}^{OU} \right), \end{aligned} \tag{S10}$$

$$\begin{aligned} B &= \operatorname{diag}\left(0, \mathcal{L}_N^{bd}, \ldots, \mathcal{L}_1^{bd}, \sum_{j=N,N-1} \mathcal{L}_j^{bd}, \ldots, \sum_{j=1,\ldots,N-1} \mathcal{L}_j^{bd}, \sum_{j=1,\ldots,N} \mathcal{L}_j^{bd} \right) \\ &= \operatorname{diag}\left(S\mathcal{L}^{bd} \right), \end{aligned} \tag{S11}$$

$$\begin{aligned} V &= \operatorname{diag}\left(0, \mathcal{L}_N^{int}, \ldots, \mathcal{L}_1^{int}, \sum_{j=N,N-1} \mathcal{L}_j^{int}, \ldots, \sum_{j=1,\ldots,N-1} \mathcal{L}_j^{int}, \sum_{j=1,\ldots,N} \mathcal{L}_j^{int} \right), \\ &= \operatorname{diag}\left(S\mathcal{L}^{int} \right) \end{aligned} \tag{S12}$$

The operators $\mathcal{L}_j^{OU}$, $\mathcal{L}_j^{bd}$, and $\mathcal{L}_j^{int}$ are defined in (S5), (S6) and (S7), respectively, and $\mathcal{L}^{OU} = (\mathcal{L}_1^{OU}, \ldots, \mathcal{L}_N^{OU})^T$, and similarly for the other operators.

With the generator in hand we may proceed with the asymptotic analysis. We substitute the ansatz (26) into the backward equation (23) and equate equal powers of ε to obtain a hierarchy of equations governing $\boldsymbol{f}^{(i)}(\boldsymbol{l}, v, t)$. At order $O(\varepsilon^{-2})$, we have

$$(Q + U)\, \boldsymbol{f}^{(0)} = 0. \tag{S13}$$

Equation (S13) possesses only constant in $\boldsymbol{l}$ solutions of the form:

$$\boldsymbol{f}^{(0)} = (1, 1, \ldots, 1)^T\, a(v, t). \tag{S14}$$

At order $O(\varepsilon^{-1})$, we have

$$(Q + U)\, \boldsymbol{f}^{(1)} = -(B + V)\boldsymbol{f}^{(0)} = -V\boldsymbol{f}^{(0)} = \frac{k}{m} \partial_v a\, S\boldsymbol{l}\,. \tag{S15}$$

We have used that $B\boldsymbol{f}^{(0)} = 0$ because $\boldsymbol{f}^{(0)}$ is independent of l_j, for $j = 1, \ldots, N$; see (S11) and (S6).

We claim the solution is

$$\boldsymbol{f}^{(1)} = -\frac{k}{m} \partial_v a \left(\frac{\lambda}{\nu} \frac{\gamma}{k} \underline{1} + \frac{1}{\nu} S \right) \boldsymbol{l}\,, \tag{S16}$$

To show this, it is sufficient to show that

$$(Q + U) \left(\frac{\lambda}{\nu} \frac{\gamma}{k} \underline{1} + \frac{1}{\nu} S \right) \boldsymbol{l} = -S\boldsymbol{l}\,.$$

We calculate each of the four terms in the product on the left-hand side in turn.

1. We have $Q\underline{1} = 0$, the zero matrix, since the sum of each row of Q is 0.
2. We also have that $US\boldsymbol{l} = 0$, since the ith entry is $((\underline{1} - S)\mathcal{L}^{OU})_i \cdot (S\boldsymbol{l})_i$, and $((\underline{1} - S)\mathcal{L}^{OU})_i$ only contains operators acting on tethers in set $u(i)$, while $(S\boldsymbol{l})_i$ only contains tether lengths from set $b(i)$.
3. We have that $U\underline{1}\boldsymbol{l} = -\frac{k}{\gamma}(\underline{1} - S)\boldsymbol{l}$, since the vector $\underline{1}\boldsymbol{l}$ has components identically equal to $l_1 + \cdots + l_N$, so all operators in each diagonal element have an effect, and $(U\underline{1}\boldsymbol{l})_i = \sum_{j \in u(i)} \mathcal{L}_j^{OU} l_j$.

4. The remaining term to evaluate is QS. Consider component (i,k):

$$(QS)_{ik} = \sum_j Q_{ij} S_{jk}\,.$$

Recall that $S_{jk}=1$ if $k\in b(j)$, $S_{jk}=0$ if $k\in u(j)$.

Suppose $k\in b(i)$. Then term $Q_{ij}S_{jk}=\lambda$ iff $|b(j)|=|b(i)|+1$ and $k\in b(j)$. The number of such states, is the number of ways of adding a bound tether, since $k\in b(i)$ so we automatically have $k\in b(j)$. The contribution to the sum is $\lambda|u(i)|$. Now consider the number of js such that $Q_{ij}S_{jk}=\nu$. We must have $|u(j)|=|u(i)|+1$, and $k\in b(j)$. Without this last condition on k, we would have $|b(i)|$ such terms, the number of tethers that can be flipped from bound to unbound, however one of these flips is k itself, which would make $S_{jk}=0$. Therefore the contribution to the sum is $\nu(|b(i)|-1)$. There is also a contribution from the diagonal, $Q_{ii}S_{ik}=-\nu|b(i)|-\lambda|u(i)|$. Putting this all together shows that

$$k\in b(i) \quad \Rightarrow \quad (QS)_{ik} = \lambda|u(i)| + \nu(|b(i)|-1) - \nu|b(i)| - \lambda|u(i)| = -\nu\,.$$

Now suppose $k\in u(i)$. The number of js that contribute a λ equals 1, since to contribute we must have $k\in b(j)$, and so only k can be flipped. The number of js that contribute ν is 0, since a tether unbinds to go from $i\to j$ so $k\in u(j)$. Therefore

$$k\in u(i) \quad \Rightarrow \quad (QS)_{ik} = \lambda\,.$$

Putting all ks together shows that

$$QS = -\nu S + \lambda(\underline{1} - S)\,.$$

Now we put all the four terms together to find

$$(Q+U)\left(\frac{\lambda}{\nu}\frac{\gamma}{k}\underline{1} + \frac{1}{\nu}S\right)\boldsymbol{l} = \left(-S + \frac{\lambda}{\nu}(\underline{1}-S)\right)\boldsymbol{l} - \frac{\lambda}{\nu}(\underline{1}-S)\boldsymbol{l} = -S\boldsymbol{l}\,,$$

so (S16) holds, as claimed.

Now we consider the $O(\varepsilon^0)$ equation:

$$(Q+U)\,\boldsymbol{f}^{(2)} = -(B+V)\boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)}. \tag{S17}$$

Solvability of (S17) requires that for each π in the nullspace of $(Q+U)^*$, we have

$$\left\langle \boldsymbol{\pi}, -(B+V)\boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)} \right\rangle = 0. \tag{S18}$$

One can verify (in a similar way to the calculations in section S3, see e.g. Eqs (S23), (S24)) that the nullspace of $(Q+U)^*$ is spanned by the vector

$$\boldsymbol{\pi} = \left(\left(\frac{\nu}{\lambda}\right)^N, \underbrace{\left(\frac{\nu}{\lambda}\right)^{N-1}, \ldots, \left(\frac{\nu}{\lambda}\right)^{N-1}}_{N \text{ elements}}, \underbrace{\left(\frac{\nu}{\lambda}\right)^{N-2}, \ldots, \left(\frac{\nu}{\lambda}\right)^{N-2}}_{\binom{N}{2} \text{ elements}}, \ldots, 1 \right)^T \times e^{-\beta k \sum_{j=1}^N l_j^2/2}\,. \tag{S19}$$

We now compute each of the terms in the inner product.

Letting $\mathbf{b} = (|b(1)|, \ldots, |b(2^N)|)^T$, we have

$$\begin{aligned}
\left\langle \boldsymbol{\pi}, -B\boldsymbol{f}^{(1)} \right\rangle &= \left\langle \boldsymbol{\pi}, -\frac{k}{m}\mathrm{diag}(S\mathcal{L}^{bd}) \left[\partial_v a \left(\frac{\lambda}{\nu}\frac{\gamma}{k}\underline{1} + \frac{1}{\nu}S\right)\boldsymbol{l}\right]\right\rangle \\
&= \left(\frac{\gamma\lambda + k}{m\nu}\right) v\partial_v a \left\langle \boldsymbol{\pi}, \mathbf{b}\right\rangle \\
&= Z\,\left(\frac{\gamma\lambda + k}{m\nu}\right) v\partial_v a\,\sum_{k=0}^{N} k\binom{N}{k}\left(\frac{\nu}{\lambda}\right)^{N-k} \\
&= Z\,\left(\frac{\gamma\lambda + k}{m\nu}\right) v\partial_v a\,\sum_{k=1}^{N} N\binom{N-1}{k-1}\left(\frac{\nu}{\lambda}\right)^{N-k} \\
&= Z\,\left(\frac{\gamma\lambda + k}{m\nu}\right) v\partial_v a\,N\,\sum_{j=0}^{N-1} \binom{N-1}{j}\left(\frac{\nu}{\lambda}\right)^{N-1-j} \\
&= Z\,\left(\frac{\gamma\lambda + k}{m\nu}\right) v\partial_v a\,N\,\left(1 + \frac{\nu}{\lambda}\right)^{N-1}
\end{aligned}$$

where $Z = \int_{\mathbb{R}^N} e^{-\beta k\sum_{j=1}^{N} l_j^2/2} d\boldsymbol{l}$.

For the second term, we use that

$$\int_{\mathbb{R}^N} Z^{-1} l_m l_n e^{-\beta k\sum_{i=1}^{N} l_i^2/2} d\boldsymbol{l} = \frac{1}{k\beta}\delta_{m,n},$$

and calculate the second inner product to be, substituting the calculation of $\langle \boldsymbol{\pi}, \mathbf{b}\rangle$ from above,

$$\begin{aligned}
\left\langle \boldsymbol{\pi}, -V\boldsymbol{f}^{(1)} \right\rangle &= \left\langle \boldsymbol{\pi}, \frac{k}{m}\mathrm{diag}(S\mathcal{L}^{int}) \left[\partial_v a \left(\frac{\lambda}{\nu}\frac{\gamma}{k}\underline{1} + \frac{1}{\nu}S\right)\boldsymbol{l}\right]\right\rangle \\
&= -Z\,\frac{k}{m}\left(\frac{\gamma\lambda + k}{m\nu}\right)\partial_v^2 a\,\frac{1}{\beta k}\left\langle \boldsymbol{\pi}, \mathbf{b}\right\rangle \\
&= -Z\,\left(\frac{\gamma\lambda + k}{\beta m^2\nu}\right)\partial_v^2 a\,N\,\left(1 + \frac{\nu}{\lambda}\right)^{N-1}
\end{aligned}$$

The term involving $\boldsymbol{f}^{(0)}$ in (S18) are straightforward to simplify. We have

$$\begin{aligned}
\left\langle \boldsymbol{\pi}, \partial_t\boldsymbol{f}^{(0)} \right\rangle &= \left\langle \boldsymbol{\pi}, (1, 1, \ldots, 1)^T\right\rangle \partial_t a \\
&= Z\,\sum_{k=0}^{N} \binom{N}{k}\left(\frac{\nu}{\lambda}\right)^{N-k} \\
&= Z\,\left(1 + \frac{\nu}{\lambda}\right)^{N}\,\partial_t a
\end{aligned}$$

Substituting the above relations into the solvability condition (S18) and grouping terms, (S18) reduces to:

$$\partial_t a = \frac{N\left(\frac{\gamma\lambda+k}{m\nu}\right)}{1 + \frac{\nu}{\lambda}}\,v\partial_v a + \frac{1}{\beta m}\frac{N\left(\frac{\gamma\lambda+k}{m\nu}\right)}{1 + \frac{\nu}{\lambda}}\,\partial_v^2 a\,. \tag{S20}$$

This is the backward equation for the process that solves (8), as claimed (recall $\lambda = q_{on}$, $\nu = q_{off}$, $\lambda/\nu = e^{\beta e_0}$.)

S3. STATIONARY DISTRIBUTION FOR THE N TETHER DYNAMICS

In this section we verify that the stationary distribution for the N tether dynamics is the Boltzmann distribution, $\pi \propto e^{-\beta E}$, where Z is a normalizing constant and E is the energy, given in (1). Specifically,

$$\pi(\boldsymbol{l}, \boldsymbol{s}, v) = Z^{-1} e^{-\frac{\beta m v^2}{2}} \prod_{j=1}^{N} e^{-\frac{\beta k l_j^2}{2}} \left(e^{\beta e_0}\delta_{s_j,b} + \delta_{s_j,u}\right)\,. \tag{S21}$$

The normalization constant Z is chosen to ensure that π is a probability measure. From this formula, one can verify by direct integration that the probabilities a tether is bound or unbound at any length in equilibrium are those given in (3).

It will be convenient to write π in the following form:

$$\pi = Z^{-1}\pi_{v,l}\, \boldsymbol{\pi_s} \tag{S22}$$

where

$$\pi_{v,l} = e^{-\frac{\beta m v^2}{2}} e^{-\frac{\beta k}{2}\sum_{j=1}^{N} l_j^2} \tag{S23}$$

and

$$\boldsymbol{\pi_s} = \left(\left(\frac{\nu}{\lambda}\right)^N, \underbrace{\left(\frac{\nu}{\lambda}\right)^{N-1}, \dots, \left(\frac{\nu}{\lambda}\right)^{N-1}}_{N \text{ elements}}, \underbrace{\left(\frac{\nu}{\lambda}\right)^{N-2}, \dots, \left(\frac{\nu}{\lambda}\right)^{N-2}}_{\binom{N}{2} \text{ elements}}, \dots, 1 \right)^T . \tag{S24}$$

The ordering of states is the same as that in (S8).

The generator is $\mathcal{L} = Q + U + B + V$ with Q, U, B, V defined in (S9), (S10), (S11), (S12) respectively. We must show that

$$\mathcal{L}^*\pi = (Q + U + B + V)^*\pi = 0\,,$$

where * denotes the formal adjoint.

First we show that $(B + V)^*\pi_{v,l}\, \boldsymbol{c} = 0$, where $\boldsymbol{c}$ is any vector with the right dimensions which doesn't depend on $v, \boldsymbol{l}$. We have that

$$(\mathcal{L}_j^{bd})^*\pi_{v,l} = -\partial_{l_j}(v\, \pi_{v,l}) = \frac{\beta k v l_j}{2}\pi_{v,l}$$

and

$$(\mathcal{L}_j^{int})^*\pi_{v,l} = \frac{k}{m}\partial_v(l_j\, \pi_{v,l}) = -\frac{\beta k v l_j}{2}\pi_{v,l}\,.$$

Therefore $(\mathcal{L}_j^{bd})^*\pi_{v,l} + (\mathcal{L}_j^{int})^*\pi_{v,l} = 0$, so $((B + V)^*\pi_{v,l}\boldsymbol{c})_i = \boldsymbol{c}_i \sum_{j\in b(i)} \left((\mathcal{L}_j^{bd})^*\pi_{v,l} + (\mathcal{L}_j^{int})^*\pi_{v,l}\right) = 0$, so the result follows.

Next we show that $U^*\pi_{v,l} = 0$. We have that

$$(\mathcal{L}_j^{OU})^*\pi_{v,l} = \frac{k}{\gamma}\partial_{l_j}(l_j\pi_{v,l}) + \frac{\beta^{-1}}{\gamma}\partial_{l_j}^2\pi_{v,l} = 0\,,$$

and therefore any sum $\sum_{j\in u(i)}(\mathcal{L}_j^{OU})^*\pi_{v,l} = 0$.

Finally we show that $Q^*\boldsymbol{\pi_s} = 0$. We have

$$\begin{aligned}
(Q^*\boldsymbol{\pi_s})_i &= \sum_j Q_{ij}(\boldsymbol{\pi_s})_i \\
&= |b(j)|\lambda\left(\frac{\lambda}{\nu}\right)^{|u(j)|-1} + |u(j)|\nu\left(\frac{\lambda}{\nu}\right)^{|u(j)|+1} - (\lambda|u(j)| + \nu|b(j)|)\left(\frac{\lambda}{\nu}\right)^{|u(j)|} \\
&= 0
\end{aligned}$$

Combining these calculations shows the desired result.