Supplementary Material of "Magnetically-actuated Artificial Cilium: A Simple Theoretical Model" – derivation of the dynamical equations of the cilium in the vicinity of a wall

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The force acting on the hard magnetic sphere is,

$$\boldsymbol{F} = F_y \boldsymbol{e}_y + F_z \boldsymbol{e}_z = F_\ell \boldsymbol{e}_r + F_\phi \boldsymbol{e}_\phi. \tag{1}$$

The velocity of the hard magnetic sphere is,

$$\boldsymbol{v} = v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z = \dot{\ell} \boldsymbol{e}_r + (\ell + r_h)\dot{\phi} \boldsymbol{e}_\phi. \tag{2}$$

The dynamic equation will be:

$$\boldsymbol{F} = \boldsymbol{Z}_{FV} \cdot [\dot{\ell}\boldsymbol{e}_r + (\ell + r_h)\dot{\phi}_c \boldsymbol{e}_\phi] + \frac{Z_{T\Omega}}{\ell + r_h}\dot{\phi}_c \boldsymbol{e}_\phi = \boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_r \dot{\ell} + \left[(\ell + r_h)\boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_\phi + \frac{Z_{T\Omega}}{\ell + r_h}\boldsymbol{e}_\phi\right]\dot{\phi}_c \tag{3}$$

where $\boldsymbol{Z}_{FV}$ is the translational friction tensor [1],

$$\boldsymbol{Z}_{FV} = 6\pi\eta r_h \begin{pmatrix} \frac{1}{1-\frac{9r_h}{16h}} & 0 \\ 0 & \frac{1}{1-\frac{9r_h}{8h}} \end{pmatrix} \tag{4}$$

and $Z_{T\Omega} = 8\pi\eta r^3$ is the rotational friction constant. In the matrix form,

$$\begin{pmatrix} F_\ell \\ F_\phi \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_r \cdot \boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \cdot \boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_r \end{pmatrix}\dot{\ell} + \begin{pmatrix} (\ell + r_h)\boldsymbol{e}_r \cdot \boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_\phi \\ (\ell + r_h)\boldsymbol{e}_\phi \cdot \boldsymbol{Z}_{FV} \cdot \boldsymbol{e}_\phi + \frac{Z_{T\Omega}}{\ell + r_h} \end{pmatrix}\dot{\phi}_c \tag{5}$$

$$= \begin{pmatrix} \frac{6\pi\eta r_h \cos^2\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \sin^2\phi_c}{1-\frac{9r_h}{8h}} \\ \frac{-6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{8h}} \end{pmatrix}\dot{\ell} + \begin{pmatrix} \frac{-6\pi\eta r_h(\ell+r_h)\sin\phi_c\cos\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h(\ell+r_h)\sin\phi_c\cos\phi_c}{1-\frac{9r_h}{8h}} \\ \frac{6\pi\eta r_h(\ell+r_h)\sin^2\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h(\ell+r_h)\cos^2\phi_c}{1-\frac{9r_h}{8h}} + \frac{8\pi\eta r_h^3}{\ell+r_h} \end{pmatrix}\dot{\phi}_c \tag{6}$$

$$= \boldsymbol{A} \cdot \begin{pmatrix} \dot{\ell} \\ (\ell + r_h)\dot{\phi}_c \end{pmatrix} \tag{7}$$

with

$$\boldsymbol{A} = \begin{pmatrix} \frac{6\pi\eta r_h \cos^2\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \sin^2\phi_c}{1-\frac{9r_h}{8h}} & \frac{-6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{8h}} \\ \frac{-6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \sin\phi_c\cos\phi_c}{1-\frac{9r_h}{8h}} & \frac{6\pi\eta r_h \sin^2\phi_c}{1-\frac{9r_h}{16h}} + \frac{6\pi\eta r_h \cos^2\phi_c}{1-\frac{9r_h}{8h}} + \frac{8\pi\eta r_h^3}{(\ell+r_h)^2} \end{pmatrix} \tag{8}$$

$$= 6\pi\eta r_h \begin{pmatrix} 1 + \frac{9r_h}{16h} + \frac{9r_h}{16h}\sin\phi_c^2 & \frac{9r_h}{16h}\sin\phi_c\cos\phi_c \\ \frac{9r_h}{16h}\sin\phi_c\cos\phi_c & 1 + \frac{9r_h}{16h} + \frac{9r_h}{16h}\cos\phi_c^2 + \frac{4r_h^2}{3(\ell+r_h)^2} \end{pmatrix} + \boldsymbol{O}\left[\left(\frac{r_h}{h}\right)^2\right] \tag{9}$$

Alternatively,

$$\begin{pmatrix} \dot{\ell} \\ (\ell + r_h)\dot{\phi}_c \end{pmatrix} = \boldsymbol{A}^{-1} \cdot \begin{pmatrix} F_\ell \\ F_\phi \end{pmatrix} \tag{10}$$

with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \frac{1}{6\pi\eta r_h[1+\frac{4r_h^2}{3(\ell+r_h)^2}]}\left[1-\frac{1}{1+\frac{4r_h^2}{3(\ell+r_h)^2}}\left(\frac{27r_h}{16h}+\frac{9r_h}{16h}(1+\sin\phi_c^2)\frac{4r_h^2}{3(\ell+r_h)^2}\right)\right] \\
&\quad \begin{pmatrix} 1+\frac{9r_h}{16h}+\frac{9r_h}{16h}\cos\phi_c^2+\frac{4r_h^2}{3(\ell+r_h)^2} & -\frac{9r_h}{16h}\sin\phi_c\cos\phi_c \\ -\frac{9r_h}{16h}\sin\phi_c\cos\phi_c & 1+\frac{9r_h}{16h}+\frac{9r_h}{16h}\sin\phi_c^2 \end{pmatrix} + \boldsymbol{O}\left[\left(\frac{r_h}{h}\right)^2\right] \\
&\simeq \frac{1}{6\pi\eta r_h[1+\frac{4r_h^2}{3(\ell+r_h)^2}]} \\
&\quad \begin{pmatrix} 1+\frac{9r_h}{16h}+\frac{9r_h}{16h}\cos\phi_c^2+\frac{4r_h^2}{3(\ell+r_h)^2} & -\frac{9r_h}{16h}\sin\phi_c\cos\phi_c \\ -\frac{9r_h}{16h}\sin\phi_c\cos\phi_c & 1+\frac{9r_h}{16h}+\frac{9r_h}{16h}\sin\phi_c^2 \end{pmatrix} \\
&- \frac{1}{6\pi\eta r_h[1+\frac{4r_h^2}{3(\ell+r_h)^2}]}\frac{1}{1+\frac{4r_h^2}{3(\ell+r_h)^2}}\left(\frac{27r_h}{16h}+\frac{9r_h}{16h}(1+\sin\phi_c^2)\frac{4r_h^2}{3(\ell+r_h)^2}\right) \\
&\quad \begin{pmatrix} 1+\frac{4r_h^2}{3(\ell+r_h)^2} & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{11}
$$

By approximating the above equation to the leading order in r_h/h, we obtain

$$
\dot{\ell} = \left[1-\frac{9}{16}\left(\frac{r_h[1+\sin\phi_c^2]}{h_0+\ell\sin\phi_c}\right)\right]\left(\frac{F_\ell}{\zeta_\ell}\right)-\frac{9}{16}\left(\frac{r_h\sin\phi_c\cos\phi_c}{h_0+\ell\sin\phi_c}\right)\left(\frac{F_\phi}{\zeta_\phi}\right), \tag{12}
$$

$$
(\ell+r_h)\dot{\phi}_c = -\frac{9}{16}\left(\frac{r_h\sin\phi_c\cos\phi_c}{h_0+\ell\sin\phi_c}\right)\left(\frac{F_\ell}{\zeta_\phi}\right)+\left[1-\frac{9}{16}\left(\frac{r_h[1+\cos\phi_c^2]}{h_0+\ell\sin\phi_c}\right)\frac{1}{1+\frac{4r_h^2}{3(\ell+r_h)^2}}\right]\left(\frac{F_\phi}{\zeta_\phi}\right) \tag{13}
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