Soft Matter

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Adhesive elastocapillary force on a cantilever beam: Supplementary material

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In this supplementary material, the beam model of regimes 2 and 3 is presented in detail, as well as a strategy for solving the corresponding equations numerically. Finally, both the alternative hypothesis of a constant moment M_d and the adhesion of a dry contact are discussed in the two last sections.

1 Main model

The beam deflection is modelled as described in the schematics of Fig. 1. Points D, W and C represent positions of the right end of the beam/substrate apparent contact zone, the beam/liquid/air contact line and the clamp. In regime 2, D coincides with the beam tip while in regime 3, both are separated by a distance d. The s-axis is tangent to the beam in D, which corresponds to its origin $s = 0$. In regime 2, it makes an angle α_d with the substrate. The beam deflection $y(s)$ is measured perpendicularly to this s-axis. The local beam slope is $\varphi(s) = \arctan(\mathrm{d}y/\mathrm{d}s)$. By definition, $y(0) = \varphi(0) = 0$. The clamp is at position (s_c, y_c) and the beam slope satisfies $\varphi(s_c) = \alpha_c - \alpha_d$.

The forces that apply on the beam were described in the main text. They are recalled in the schematics of figure 1. The reaction force in D can be projected in the (s, y) coordinates: $N_0 = N_d \cos\alpha_d - T_d \sin\alpha_d$ along y and $T_0 = N_d \sin\alpha_d + T_d \cos\alpha_d$ along s. Therefore,

$$T_0 = N_0 \frac{\tan\alpha_d + \mu}{1 - \mu \tan\alpha_d}. \tag{1}$$

As the beam is a slender body, it always prefers bending instead of stretching, so its deflection $y(s)$ satisfies the Euler-Bernoulli's equation, $B\kappa(s) = M(s)$ where $\kappa(s)$ is the local beam curvature and $M(s)$ the moment at abscissa s. In the wet part between D and W,

$$M(s) = M_d + N_0 s - T_0 y - \int_0^s \frac{\sigma W}{R} \left[(s - s')\mathrm{d}s' + (y - y')\mathrm{d}y'\right] \tag{2}$$

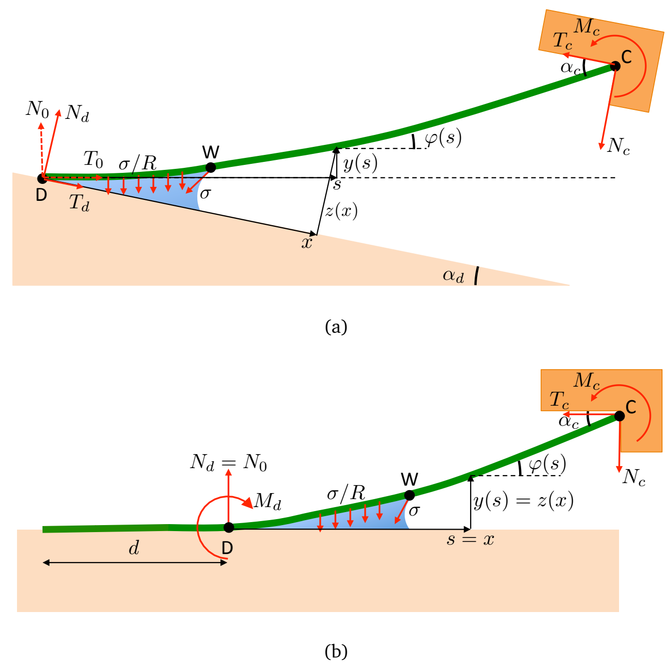

Fig. 1 Two-dimensional schematics of the beam (green), the clamp (orange), the substrate (apricot) and the capillary bridge (blue). (a) Regime 2, represented with an inclination α_d, and (b) regime 3. Forces and moments applied to the beam are represented in red. The main variables of the model are indicated.

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while in the dry part between W and C,

$$\begin{aligned} M(s) &= M_d + N_0 s - T_0 y - \int_0^{s_w} \frac{\sigma W}{R}\left[(s-s')\mathrm{d}s' + (y-y')\mathrm{d}y'\right] \\ &- \sigma W(s-s_w)\sin(\theta_b+\varphi_w) \\ &+ \sigma W(y-y_w)\cos(\theta_b+\varphi_w) \end{aligned} \tag{3}$$

where s_w, y_w and φ_w are the position, deflection and inclination at point W.

We consider small beam deflections, i.e. $y^2 \ll s^2$ and $\tan\varphi = \mathrm{d}y/\mathrm{d}s \ll 1$, so the bending curvature can be approximated by $\mathrm{d}^2y/\mathrm{d}s^2$ and the Euler-Bernoulli equation simplifies into:

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + \frac{T_0}{B}y = \frac{M_d}{B} + \frac{N_0}{B}s - \frac{s^2}{2R\ell^2} \tag{4}$$

for the wet part, and

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}s^2} &= \left[\frac{\cos(\theta_b+\varphi_w)}{\ell^2} - \frac{T_0}{B}\right]y + \frac{M_d}{B} + \frac{N_0}{B}s + \frac{s_w^2 - 2ss_w}{2R\ell^2} \\ &- \frac{s-s_w}{\ell^2}\sin(\theta_b+\varphi_w) - \frac{y_w}{\ell^2}\cos(\theta_b+\varphi_w) \end{aligned} \tag{5}$$

for the dry part.

Besides boundary conditions, the beam is subjected to three geometrical constraints. Firstly, as it cannot stretch, its length L should satisfy

$$L = d + \int_0^{s_c}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2}\mathrm{d}s \simeq d + s_c + \frac{1}{2}\int_0^{s_c}\tan^2\varphi\,\mathrm{d}s \tag{6}$$

Secondly, the circular liquid-air interface of the capillary bridge should connect to both the substrate and the beam with contact angles θ_b and θ_s, respectively. This is satisfied if

$$z_w = s_w\sin\alpha_d + y_w\cos\alpha_d = R\left[\cos\theta_s + \cos(\theta_b+\alpha_d+\varphi_w)\right] \tag{7}$$

And thirdly, the volume of liquid per unit width $V/W = \Omega L^2$, given by

$$\begin{aligned} \Omega L^2 &\simeq \int_0^{x_w} z\mathrm{d}x + \frac{R^2}{2}\left[\theta_s + \theta_b + \alpha_d + \varphi_w - \pi\right] \\ &+ \frac{R^2}{2}\sin(\theta_b+\alpha_d+\varphi_w)\left[2\cos\theta_s + \cos(\theta_b+\alpha_d+\varphi_w)\right] \\ &- \frac{R^2}{2}\cos\theta_s\sin\theta_s \end{aligned} \tag{8}$$

should remain constant, where $x = s\cos\alpha_d - y\sin\alpha_d$ and

$$\begin{aligned} \int_0^{x_w} z\mathrm{d}x &= \frac{s_w^2 - y_w^2}{2}\sin\alpha_d\cos\alpha_d \\ &+ \int_0^{s_w}\left[y\cos^2\alpha_d - s\frac{\mathrm{d}y}{\mathrm{d}s}\sin^2\alpha_d\right]\mathrm{d}s. \end{aligned} \tag{9}$$

The reaction forces and moment at the clamp are given by

$$\begin{aligned} \frac{N_c}{B} &= \frac{N_d}{B} - \frac{x_w}{R\ell^2} - \frac{1}{\ell^2}\sin(\theta_b+\alpha_d+\varphi_w) \\ \frac{T_c}{B} &= \frac{T_d}{B} + \frac{z_w}{R\ell^2} - \frac{1}{\ell^2}\cos(\theta_b+\alpha_d+\varphi_w) \\ \frac{M_c}{B} &= \left.\frac{\mathrm{d}^2y}{\mathrm{d}s^2}\right]_{s=s_c} \end{aligned} \tag{10}$$

1.1 Wet part

1.1.1 Regime 2:

In this regime, $M_d = 0$ and, in the limit of low friction where $T_0 \ll B/s_w^2$, the wet part obeys

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} = \frac{N_0}{B}s - \frac{s^2}{2\ell^2R} \tag{11}$$

The solution to this equation that satisfies $y = \mathrm{d}y/\mathrm{d}s = 0$ in $s = 0$ is

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{N_0s^2}{2B} - \frac{s^3}{6\ell^2R} \tag{12}$$

$$y = \frac{N_0s^3}{6B} - \frac{s^4}{24\ell^2R} \tag{13}$$

Since $\mathrm{d}y/\mathrm{d}s = \tan\varphi_w$ in $s = s_w$, equations 12 and 13 can be rewritten in s_w as:

$$\frac{s_w^2}{6\ell^2R} = \frac{N_0s_w}{2B} - \frac{\tan\varphi_w}{s_w} \tag{14}$$

$$y_w = \frac{N_0s_w^3}{24B} + \frac{s_w\tan\varphi_w}{4} \tag{15}$$

Combining them with the geometrical condition on the liquid-air interface (eq. 7) yields:

$$\begin{aligned} &\left(\frac{N_0s_w}{2B}\right)^2 + 2\left(\frac{\tan\varphi_w}{s_w} + 6\frac{\tan\alpha_d}{s_w}\right)\frac{N_0s_w}{2B} \\ &- 3\frac{\tan\varphi_w}{s_w}\left(\frac{\tan\varphi_w}{s_w} + 4\frac{\tan\alpha_d}{s_w}\right) \\ &- 2\frac{\cos\theta_s + \cos(\theta_b+\alpha_d+\varphi_w)}{\ell^2\cos\alpha_d} = 0 \end{aligned}$$

This quadratic equation can be solved for $N_0s_w/(2B)$. The discriminant ρ satisfies

$$\rho^2 = \frac{4}{s_w^2}(\tan\varphi_w + 3\tan\alpha_d)^2 + 2\frac{\cos\theta_s + \cos(\theta_b+\alpha_d+\varphi_w)}{\ell^2\cos\alpha_d} \tag{16}$$

Only the solution $N_0 > 0$ is kept since the beam touches the substrate in regime 2, namely

$$\frac{N_0s_w}{2B} = \rho - \frac{\tan\varphi_w + 6\tan\alpha_d}{s_w} \tag{17}$$

Then,

$$\frac{s_w^2}{6\ell^2R} = \rho - \frac{2\tan\varphi_w + 6\tan\alpha_d}{s_w} \tag{18}$$

and

$$y_w = \frac{s_w^2}{12}\left[\rho + \frac{2\tan\varphi_w - 6\tan\alpha_d}{s_w}\right] \tag{19}$$

The liquid volume is evaluated by considering that

$$\begin{aligned}\int_0^{s_w} y\mathrm{d}s &= \frac{s_w^3}{60}\left(2\rho + \frac{\tan\varphi_w - 12\tan\alpha_d}{s_w}\right)\\ \int_0^{s_w} \frac{\mathrm{d}y}{\mathrm{d}s}s\mathrm{d}s &= \frac{s_w^3}{20}\left(\rho + \frac{3\tan\varphi_w - 6\tan\alpha_d}{s_w}\right)\end{aligned} \tag{20}$$

The wet beam length is

$$L_w = s_w + \frac{s_w^7}{504\ell^4R^2} - \frac{N_0 s_w^6}{72B\ell^2R} + \frac{N_0^2 s_w^5}{40B^2} \tag{21}$$

1.1.2 Regime 3:

In this regime, $\alpha_d = 0$ which implies $N_0 = N_d$, and $T_0 = 0$. The wet part obeys

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} = (s+md)\frac{N_d}{B} - \frac{1}{2R\ell^2}\left[s^2 + d^2(1-2m)\right] \tag{22}$$

The solution to this equation that satisfies $y = \mathrm{d}y/\mathrm{d}s = 0$ in $s = 0$ is

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{N_d}{2B}\left(s^2 + 2mds\right) - \frac{s^3 + 3d^2(1-2m)s}{6\ell^2R} \tag{23}$$

$$y = \frac{N_d}{6B}\left(s^3 + 3mds^2\right) - \frac{s^4 + 6d^2(1-2m)s^2}{24\ell^2R} \tag{24}$$

For the sake of simplifying notations, we define

$$a = \frac{md}{s_w}, \qquad b = \frac{(1-2m)d^2}{s_w^2} \tag{25}$$

Since $\mathrm{d}y/\mathrm{d}s = \tan\varphi_w$ in $s = s_w$, equations 23 and 24 can be rewritten in s_w as:

$$(1+3b)\frac{s_w^2}{6\ell^2R} = (1+2a)\frac{N_0 s_w}{2B} - \frac{\tan\varphi_w}{s_w} \tag{26}$$

$$\begin{aligned}4(1+3b)\frac{y_w}{s_w^2} &= (1+6a-6b)\frac{N_0 s_w}{6B}\\ &+ (1+6b)\frac{\tan\varphi_w}{s_w}\end{aligned} \tag{27}$$

Combining them with the geometrical condition on the liquid-air interface (eq. 7) yields:

$$\begin{aligned}&(1+6a-6b)(1+2a)\left(\frac{N_d s_w}{2B}\right)^2\\ +\ &2(1+12b+18ab)\frac{\tan\varphi_w}{s_w}\frac{N_d s_w}{2B}\\ -\ &3\frac{\tan^2\varphi_w}{s_w^2}(1+6b)\\ -\ &2(1+3b)^2\frac{\cos\theta_s + \cos(\theta_b+\varphi_w)}{\ell^2} = 0\end{aligned} \tag{28}$$

This quadratic equation can be solved for $N_d s_w/(2B)$. The discriminant is then

$$\begin{aligned}\rho^2 &= 4(1+3a)^2(1+3b)^2\frac{\tan^2\varphi_w}{s_w^2}\\ &+ 2(1+3b)^2(1+2a)(1+6a-6b)\frac{\cos\theta_s + \cos(\theta_b+\varphi_w)}{\ell^2}\end{aligned} \tag{29}$$

and only the solution $N_d > 0$ is kept.

The liquid volume is evaluated by considering

$$\int_0^{s_w} y\mathrm{d}s = \frac{N_d s_w^4}{24B}(1+4a) - \frac{s_w^5}{120\ell^2R}(1+10b) \tag{30}$$

The wet beam length is

$$\begin{aligned}L_w &= s_w + \frac{s_w^7}{504\ell^4R^2} - \frac{N_d s_w^6}{72B\ell^2R}\\ &+ \left(\frac{N_d^2}{4B^2} - \frac{M_d}{3B\ell^2R}\right)\frac{s_w^5}{10} + \frac{N_d M_d s_w^4}{8B^2} + \frac{M_d^2 s_w^3}{6B^2}\end{aligned} \tag{31}$$

where we recall that

$$\frac{M_d}{B} = \frac{N_d}{B}md - \frac{d^2(1-2m)}{2R\ell^2} = a\frac{N_d s_w}{B} - b\frac{s_w^2}{2R\ell^2} \tag{32}$$

1.2 Dry part

We define

$$K = \frac{T_0}{B} - \frac{\cos(\theta_b+\varphi_w)}{\ell^2} \tag{33}$$

If $K > 0$, then $k = \sqrt{K}$ and Euler-Bernoulli's differential equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + k^2y = Gs + H \tag{34}$$

where

$$\begin{aligned}G &= \frac{N_0}{B} - \frac{s_w}{\ell^2R} - \frac{\sin(\theta_b+\varphi_w)}{\ell^2}\\ H &= \frac{M_d}{B} + \frac{s_w^2}{2\ell^2R}\\ &+ \frac{s_w\sin(\theta_b+\varphi_w)}{\ell^2} - \frac{y_w\cos(\theta_b+\varphi_w)}{\ell^2}\end{aligned} \tag{35}$$

The solution that satisfies boundary conditions at the contact line is:

$$\begin{aligned}y(s) - y_w - \frac{t}{k}\tan\varphi_w &= \frac{C_1}{K}(1-\cos t) + \frac{C_2}{kK}(t - \sin t)\\ \tan\varphi(s) - \tan\varphi_w &= \frac{C_1 k}{K}\sin t + \frac{C_2}{K}(1-\cos t)\end{aligned} \tag{36}$$

where $t = k(s - s_w)$ and

$$\begin{aligned}C_1 &= \frac{M_d}{B} + \frac{N_0}{B}s_w - \frac{T_0}{B}y_w - \frac{s_w^2}{2\ell^2R}\\ C_2 &= \frac{N_0}{B} - \frac{T_0}{B}\tan\varphi_w - \frac{s_w}{\ell^2R} - \frac{\sin\theta_b}{\ell^2\cos\varphi_w}\end{aligned} \tag{37}$$

The dry length L_d between points W and C is given by

$$L_d = \left(1+\frac{1}{2}\tan^2\varphi_w\right)\frac{t_c}{k}+\frac{C_1C_2}{2K^2}(1-\cos t_c)^2 + \frac{C_1^2}{8Kk}(2t_c-\sin 2t_c)+\frac{C_2^2}{8K^2k}(6t_c-8\sin t_c+\sin 2t_c) + \frac{C_1\tan\varphi_w}{K}(1-\cos t_c)+\frac{C_2\tan\varphi_w}{Kk}(t_c-\sin t_c) \quad (38)$$

with $t_c = k(s_c - s_w)$. The clamp position s_c is determined through the non-stretching condition $L = d + L_w + L_d$. Finally, the clamping condition $\varphi(s_c) = \alpha_c - \alpha_d$ needs to be imposed.

If $K < 0$, then $k = \sqrt{-K}$ and

$$\frac{d^2y}{ds^2} - k^2 y = Gs + H \quad (39)$$

The solution that satisfies boundary conditions at the contact line is:

$$y(s) - y_w - \frac{t}{k}\tan\varphi_w = \frac{C_1}{K}(1-\cosh t)+\frac{C_2}{Kk}(t-\sinh t)$$
$$\tan\varphi(s) - \tan\varphi_w = -\frac{C_1 k}{K}\sinh t + \frac{C_2}{K}(1-\cosh t) \quad (40)$$

where again $t = k(s - s_w)$.

The dry length is then given by

$$L_d = \left(1+\frac{1}{2}\tan^2\varphi_w\right)\frac{t_c}{k}+\frac{C_1C_2}{2K^2}(\cosh t_c-1)^2 + \frac{C_1^2}{8Kk}(2t_c-\sinh 2t_c)+\frac{C_2^2}{8K^2k}(6t_c-8\sinh t_c+\sinh 2t_c) + \frac{C_1\tan\varphi_w}{K}(1-\cosh t_c)+\frac{C_2\tan\varphi_w}{Kk}(t_c-\sinh t_c) \quad (41)$$

1.3 Solving the system of equations

In the previous section, the model of beam deflection has been progressively reduced from a system of differential equations with boundary conditions to a system of non-linear algebraic equations. This latter has been solved iteratively in Matlab according to the following procedure:

1. For a given value of φ_w (here chosen within $[-5^\circ, 12^\circ]$),
   (a) Consider a range of values for s_w (here chosen between $10^{-4}L$ and L),
   (b) Calculate $\rho(s_w)$, $N_0(s_w)$, $R(s_w)$, $y_w(s_w)$ and $V(s_w)$ according to section 1.1. In regime 3, both solutions $\pm\rho$ should be considered, and only the one yielding $V > 0$ and $y_w > 0$ is kept.
   (c) Find s_w that yields the desired liquid volume V [the function $V(s_w)$ is monotonic].
   (d) Find the clamp position s_c that yields the desired beam length $L = d + L_w + L_d$ [the function $L(s_c)$ is monotonic]. Deduce y_c and φ_c.
2. Loop on φ_w (i.e., repeat step 1.) until the boundary condition $\varphi_c = \alpha_c - \alpha_d$ at the clamp is satisfied. A bisection method was adopted, as there may be several solutions and no guarantee of convergence. Only the less deformed solution (i.e. the solution of smallest $|\varphi_w|$) was kept.
3. Calculate the reaction forces at the clamp N_c, T_c and M_c.

This process involves one main loop (φ_w) and two independent secondary loops (s_w and s_c). It is therefore much simpler and less time consuming to solve than the initial model based on the Euler-Bernoulli differential equation with several boundary conditions and geometrical constraints.

2 Alternative hypothesis of constant M_d

The model of regime 3 is based on the hypothesis of a distributed reaction force through the parameter m. In the previous models of elastocapillary adhesion, the reaction force was assumed to be localized in D and constant (possibly equal to zero). Following a similar approach to section 1.1 with this new hypothesis on M_d, we find

$$\rho^2 = \left(\frac{M_d}{B}+2\frac{\tan\varphi_w}{s_w}\right)^2 + 2\frac{\cos\theta_s+\cos(\theta_b+\varphi_w)}{\ell^2}$$
$$\frac{N_0 s_w}{2B} = \rho - \frac{2M_d}{B} - \frac{\tan\varphi_w}{s_w}$$
$$\frac{s_w^2}{6R\ell^2} = \rho - \frac{M_d}{B} - 2\frac{\tan\varphi_w}{s_w} \quad (42)$$

and a deflection

$$y = \frac{M_d}{2B}s^2 + \frac{N_0}{6B}s^3 - \frac{s^4}{24R\ell^2} \quad (43)$$

in the wet zone.

3 Dry adhesion in regime 3

In the absence of a liquid bridge, the aforementioned equations are greatly simplified. If we further assume that there is still no friction, the Euler-Bernoulli equation becomes

$$B\frac{d^2y}{ds^2} = N_d s + M_d. \quad (44)$$

The beam deflection is then

$$y(s) = (-2y_c+\alpha_c s_c)\left(\frac{s}{s_c}\right)^3 + (3y_c-\alpha_c s_c)\left(\frac{s}{s_c}\right)^2, \quad (45)$$

where $s_c = L - d$. It satisfies $y(0) = y'(0) = 0$, $y(s_c) = y_c$ and $y'(s_c) = \alpha_c$. The curvature is

$$y'' = \left(-12\frac{y_c}{s_c^2}+6\frac{\alpha_c}{s_c}\right)\frac{s}{s_c} + \left(6\frac{y_c}{s_c^2}-2\frac{\alpha_c}{s_c}\right), \quad (46)$$

from which we infer

$$N_d = \frac{B}{s_c}\left(-12\frac{y_c}{s_c^2}+6\frac{\alpha_c}{s_c}\right) \text{ and } M_d = B\left(6\frac{y_c}{s_c^2}-2\frac{\alpha_c}{s_c}\right). \quad (47)$$

As the beam shall not penetrate the underlying substrate, $y''(0) \geq 0$, which yields

$$s_c \leq \frac{3y_c}{\alpha_c}. \quad (48)$$

We may here attempt to determine s_c (and so d) through energy arguments. The internal bending energy of the beam is

$$U = \int_0^{s_c} \frac{B}{2}(y'')^2 dx = \frac{2B}{s_c}\left[3\frac{y_c^2}{s_c^2} - 3\frac{y_c\alpha_c}{s_c} + \alpha_c^2\right]. \quad (49)$$

The external work of the loads is

$$W = -\alpha_c M_c + y_c N_c. \quad (50)$$

If the beam is free to slide along the substrate, there is no external force in the s direction so W is independent of s_c. We may consider an additional adhesive energy $E_a = -\xi(L - s_c)$.

The total potential energy is therefore

$$\begin{aligned}\Pi(y_c,\alpha_c,s_c) &= U + W + E_a \quad (51)\\ &= \frac{2B}{s_c}\left[3\frac{y_c^2}{s_c^2} - 3\frac{y_c\alpha_c}{s_c} + \alpha_c^2\right] - \alpha_c M_c + y_c N_c - \xi(L - s_c).\end{aligned}$$

It should be minimum regarding the three possible displacements y_c, α_c and s_c:

$$\left.\frac{\partial \Pi}{\partial y_c}\right]_{\alpha_c,s_c} = 0 \quad \Rightarrow \quad \frac{N_c}{B} = -12\frac{y_c}{s_c^3} + 6\frac{\alpha_c}{s_c^2} \quad (52)$$

$$\left.\frac{\partial \Pi}{\partial \alpha_c}\right]_{y_c,s_c} = 0 \quad \Rightarrow \quad \frac{M_c}{B} = -6\frac{y_c}{s_c^2} + 4\frac{\alpha_c}{s_c} \quad (53)$$

$$\left.\frac{\partial \Pi}{\partial s_c}\right]_{y_c,\alpha_c} = 0 \quad \Rightarrow \quad \sqrt{\frac{\xi}{2B}}s_c^2 + \alpha_c s_c - 3y_c = 0. \quad (54)$$

This latter equation can only be satisfied in the adhesive regime ($\xi > 0$). If the solid-solid interaction is not energetically favourable ($\xi < 0$), then $\frac{\partial \Pi}{\partial s_c} < 0$ and the minimum is found when $s_c = 1$, i.e. when the contact area between the beam and the substrate is reduced to 0.

For $\xi > 0$, an equilibrium in $s_c < 1$ satisfying equation (54) is necessarily stable since

$$\frac{\partial^2 \Pi}{\partial s_c^2} = \frac{6B}{s_c^5}\left[(3y_c - \alpha_c s_c)^2 + 3y_c^2\right] > 0. \quad (55)$$

The solution s_c to Eq. (54) is in the range $]0, L[$ when

$$y_c < \frac{\alpha_c L}{3} + \frac{L^2}{3}\sqrt{\frac{\xi}{2B}}. \quad (56)$$

We note that $M_d = \sqrt{2\xi B}$ is constant.