Mesoscopic non-equilibrium measures can reveal intrinsic features of the active driving

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Derivation of $\langle \mathcal{A}^2(r)\rangle_\alpha$ in $d = 1$ and $d = 2$

In this section we derive an expression for the average area enclosing rate $\langle \mathcal{A}^2 \rangle$ as function of the distance r between two observed probes. For a one-dimensional system the area enclosing rate can be expressed in terms of the elements of the covariance matrix as:

$$\mathcal{A}_{ij} = \frac{k_B}{\gamma} \sum_z \left(\xi_z^{\mathrm{M}} \widetilde{\partial}_2^2 c_z^{\mathrm{M}} + \xi_z^{D} \widetilde{\partial}_2^2 c_z^{\mathrm{D}} \right), \tag{S1}$$

where i and j are the bead indices such, that $d_{ij} = 0$, $\widetilde{\partial}_2^2 c = c_{i,j+1} - 2c_{i,j} + c_{i,j-1}$ indicates the discrete second derivative across rows, and $\xi^{\mathrm{M/D}} = b^{\mathrm{M/D}} \alpha^{\mathrm{M/D}}$. To find how $\langle \mathcal{A}^2(r)\rangle_\alpha$ depends on the distance r between the observed probes, we use the explicit expressions for $\partial_2^2 c_z^{\mathrm{M}}(r)$ and $\partial_2^2 c_z^{\mathrm{D}}(r)$, evaluated for $i = -r/2$ and $j = r/2$, appearing in Eq. (12) and Eq. (13) in the main text.

For notational simplicity we rename: $\partial_2^2 c_z^{\mathrm{M}}(r) = f^{\mathrm{M}}(z, r)$ and $\partial_2^2 c_z^{\mathrm{D}}(r) = f^{\mathrm{D}}(z, r)$. The functions $f^{\mathrm{M}}(z, r)$ and $f^{\mathrm{D}}(z, r)$ are informative of the contribution of a monopole or dipole activity, at position z, to $\mathcal{A}^2(r)$ measured between two tracers at $i = r/2$ and $j = -r/2$. Such contributions come primarily from activities in between the two beads, as shown in Fig. S1.

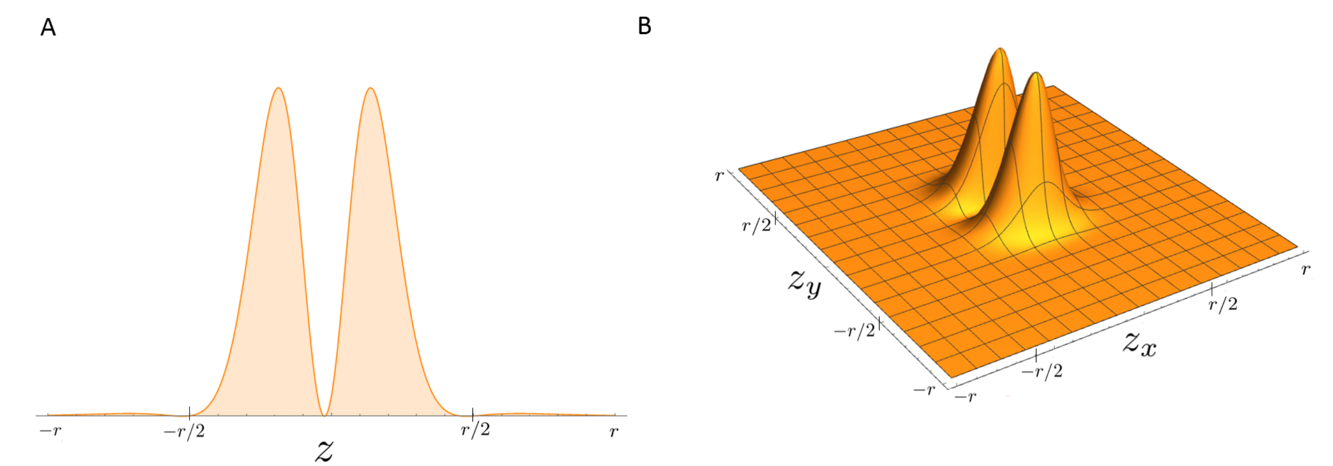

Figure S1: A) Plot of ${f^{\mathrm{D}}}^2(z)$ in $d = 1$. The value of this function represents the contribution of a dipole, at position z, to the area enclosing rate measured between two tracers beads at $i = r/2$ and $j = -r/2$ B) Plot of ${f^{\mathrm{D}}}^2(z_x, z_y)$ in $d = 2$. The value of this function represents the contribution of a dipole activity, acting between the beads at $\{z_x, z_y\}$ and $\{z_x, z_y + 1\}$, to the area enclosing rate measured from the y-displacements of two beads at $\{0, r/2\}$ and $\{0, -r/2\}$.

By taking the square of Eq. (S1) and the ensemble average over the activities we obtain:

$$\begin{aligned}\langle \mathcal{A}^2(r)\rangle_\alpha &= \frac{k_B^2}{\gamma^2}\left\langle \left[\sum_z \xi_z^{\rm M} f^{\rm M}(z,r) + \xi_z^{\rm D} f^{\rm D}(z,r)\right]^2\right\rangle \\ &= \frac{k_B^2}{\gamma^2}\sum_z \left[\langle {\xi_z^{\rm M}}^2\rangle {f^{\rm M}}^2(z,r) + \langle {\xi_z^{\rm D}}^2\rangle {f^{\rm D}}^2(z,r)\right] + \frac{k_B^2}{\gamma^2}\langle \xi^{\rm M}\rangle^2 \sum_{z,z'\neq z} f^{\rm M}(z,r) f^{\rm M}(z',r) \\ &+ \frac{k_B^2}{\gamma^2}\langle \xi^{\rm D}\rangle^2 \sum_{z,z'\neq z} f^{\rm D}(z,r) f^{\rm D}(z',r) + \frac{k_B^2}{\gamma^2} 2\langle \xi^{\rm M}\rangle\langle \xi^{\rm D}\rangle \sum_z f^{\rm M}(z,r)\sum_{z'} f^{\rm D}(z',r).\end{aligned} \tag{S2}$$

where we assumed that the noise amplitudes $\xi_z^{\rm M}$ and $\xi_{z'}^{\rm D}$ are spatially uncorrelated and that their average does not depend on z. By rewriting $\sum_{z,z'\neq z} = \sum_{z,z'} - \sum_{z,z'=z}$ we obtain:

$$\begin{aligned}\langle \mathcal{A}^2(r)\rangle_\alpha = \frac{k_B^2}{\gamma^2}\sum_z \left[\langle {\xi_z^{\rm M}}^2\rangle {f^{\rm M}}^2(z,r) + \langle {\xi_z^{\rm D}}^2\rangle {f^{\rm D}}^2(z,r)\right] + \frac{k_B^2}{\gamma^2}\langle \xi^{\rm M}\rangle^2 \left[\sum_{z,z'} f^{\rm M}(z,r) f^{\rm M}(z',r) - \sum_z {f^{\rm M}}^2(z,r)\right] \\ + \frac{k_B^2}{\gamma^2}\langle \xi^{\rm D}\rangle^2 \left[\sum_{z,z'} f^{\rm D}(z,r) f^{\rm D}(z',r) - \sum_z {f^{\rm D}}^2(z,r)\right] + \frac{k_B^2}{\gamma^2} 2\langle \xi^{\rm M}\rangle\langle \xi^{\rm D}\rangle \sum_z f^{\rm M}(z,r)\sum_{z'} f^{\rm D}(z',r).\end{aligned} \tag{S3}$$

and approximating the sum by an integral yields

$$\begin{aligned}\langle \mathcal{A}^2(r)\rangle_\alpha = \frac{k_B^2}{\gamma^2}\int dz \left[\langle {\xi_z^{\rm M}}^2\rangle {f^{\rm M}}^2(z,r) + \langle {\xi_z^{\rm D}}^2\rangle {f^{\rm D}}^2(z,r)\right] + \frac{k_B^2}{\gamma^2}\langle \xi^{\rm M}\rangle^2 \left[\int dz dz' f^{\rm M}(z,r) f^{\rm M}(z',r) - \int dz {f^{\rm M}}^2(z,r)\right] \\ + \frac{k_B^2}{\gamma^2}\langle \xi^{\rm D}\rangle^2 \left[\int dz dz' f^{\rm D}(z,r) f^{\rm D}(z',r) - \int dz f^{\rm D}(z,r)^2\right] + \frac{k_B^2}{\gamma^2} 2\langle \xi^{\rm M}\rangle\langle \xi^{\rm D}\rangle \int dz f^{\rm M}(z,r)\int dz' f^{\rm D}(z',r).\end{aligned} \tag{S4}$$

Considering that $\int f^{\rm M/D}(z,r) dz = 0$:

$$\langle \mathcal{A}^2(r)\rangle_\alpha = \left(\frac{k_B}{\gamma}\right)^2 \left(\sigma^2_{\xi^{\rm M}} \int dz {f^{\rm M}}^2(z,r) + \sigma^2_{\xi^{\rm D}}\int dz {f^{\rm D}}^2(z,r)\right). \tag{S5}$$

Since $\sigma^2_{\xi^{\rm M/D}}$ is the variance of a stochastic variable $\xi^{\rm M/D} = b^{\rm M/D}\alpha^{\rm M/D}$, where $b \in \{0,1\}$ with $\langle b\rangle = \rho^{\rm M/D}$ and $\langle b^2\rangle = \rho^{\rm M/D}$, we have $\sigma^2_{\xi^{\rm M/D}} = \rho^{\rm M/D}\sigma^2_{\alpha^{\rm M/D}}$. By solving the integral, and keeping only the leading terms in the limit $r \gg 1$, we obtain

$$\langle \mathcal{A}^2(r)\rangle_\alpha = \frac{1}{\pi}\left(\frac{k_B}{\gamma}\right)^2 \left(\rho^{\rm M}\sigma^2_{\alpha^{\rm M}}\frac{1}{2r^3} + \rho^D \sigma^2_{\alpha^D}\frac{45}{r^7}\right). \tag{S6}$$

Similar calculations can be performed in d=2, and lead to the integral form of the area enclosing rate:

$$\langle \mathcal{A}^2(r)\rangle_\alpha = \frac{k_B^2}{\gamma^2}\left[\rho^{\rm M}\sigma^2_{\alpha^{\rm M}}\int_{-\infty}^{\infty} dz_x dz_y {f^{\rm M}}^2(z_x,z_y,r) + \rho^{\rm D}\sigma^2_{\alpha^D}\int_{-\infty}^{\infty} dz_x dz_y {f^D}^2(z_x,z_y,r)\right]. \tag{S7}$$

where $f^{\rm M}(z_x,z_y,r)$ and $f^{\rm D}(z_x,z_y,r)$ are defined in Eq. (20) and Eq. (21) in the main text. The second integral in Eq. (S7) is arduous to calculate analytically. Therefore, we estimate the integral numerically for different values of the distance r. The result is reported in Fig. S2, together with the result of a linear interpolation of such numerical data $\int_{-\infty}^{\infty} dz_x dz_y {f^{\rm D}}^2(z_x,z_y,r) \simeq \frac{4}{\pi^4} a r^{-b}$ with $a \simeq 529$ and $b \simeq 10$. Finally, for $d = 2$ we obtain :

$$\langle \mathcal{A}^2(r)\rangle_\alpha \simeq \left(\frac{2k_{\rm B}}{\pi^2\gamma}\right)^2 \left[\rho^{\rm M}\sigma^2_{\alpha^{\rm M}}\frac{2\pi}{5r^6} + \rho^{\rm D}\sigma^2_{\alpha^{\rm D}}\frac{529}{r^{10}}\right]. \tag{S8}$$

Small displacement approximation in $d = 2$

In a two-dimensional network, we assume that the dipole forces act along the directions of the springs at each point in time. Here we show that in the limit of small displacements we can consider the action of the dipole forces to be directed along the principal axes of the network at rest, as done in the main text. For simplicity, let's consider a square lattice of size $N = n \times n$, with zero-rest length springs. We index the coordinates as $\mathbf{x} = \{x_1, \ldots, x_N, y_1, \ldots, y_N\}$, and we assume the presence of only two dipole activities: one of intensity $\eta_{i,i+n}$ acting vertically between the beads of index i and $i+n$, and the other one of

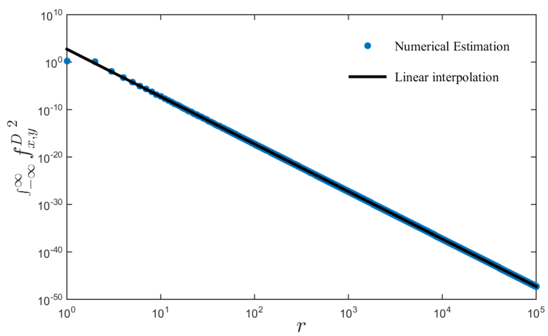

Figure S2: Numerical estimation of the second integral in Eq. (S7) as function of the distance (blue), and result of the linear interpolation $f(x) = 6,27 - 9.99\ln(r)$ (black).

intensity $\eta_{i,i+1}$ acting horizontally between the beads of index i and $i+1$. The Langevin equations for the y-displacements of the i_{th} bead reads:

$$\frac{dy_i}{dt} = \frac{k}{\gamma} a_{i,l} y_l + \eta_i^T + \eta_{i,i+n}^{\mathrm{D}} \cos\left(\frac{\Delta x_{i,i+n}}{\ell}\right) + \eta_{i,i+1}^{\mathrm{D}} \sin\left(\frac{\Delta y_{i,i+1}}{\ell}\right) \tag{S9}$$

where we defined $\Delta x_{i,i+n} = x_i - x_{i+n}$, $\Delta y_{i,i+1} = y_i - y_{i+1}$ and ℓ is the lattice spacing. In the limit $\frac{\Delta x_{i,i+n}}{\ell} \ll 1$ and $\frac{\Delta y_{i,i+1}}{\ell} \ll 1$, the last term is negligible and the second last is $\simeq \eta_{i,i+n}^{\mathrm{D}}$. Therefore in the limit of small displacements we can consider the action of the dipole forces to be directed along the principal axes of the network.

To check the validity of this approximation for the non-equilibrium measure, we explicitly simulated the dynamics of the network. We employed the Euler-Maruyama method to numerically integrate the Langevin equation of a square lattice where both vertical and horizontal dipoles are distributed randomly along the network and act along the spring direction. When the standard deviation of displacements is small compared to the rest length of the springs ($\sigma_x/\ell < 1$), our theoretical prediction is in good agreement with the simulation, as shown in Fig. S3. However, also in the case of $\sigma_x/\ell > 1$, the simulation results are slightly shifted respect to our prediction, but the scaling exponent remains the same.

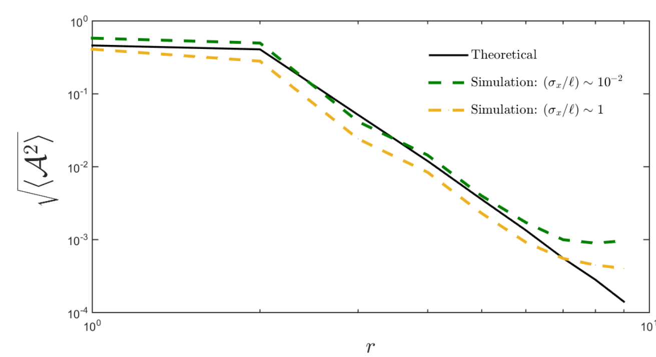

Figure S3: Theoretical prediction (black) of the average area enclosing rate (obtained from Eq. 7 in the main text by numerically solving the Lyapunov equation), compared with simulation results for different values of the ratio σ_x/ℓ (green and yellow). For computational convenience the ensemble average over the activity distributions has been evaluated by performing a spatial average over the lattice.