

## Supplemental Material for “Generic Theory of the Dynamic Magnetic Response of Ferrofluids”

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### Abstract

This supplementary material includes two parts. The first part shows a detailed derivation of the magnetization relaxation equation for monodisperse ferrofluids, Eq. (3) in the main text. This is achieved by employing the projection operator technique in nonequilibrium statistical mechanics. The second part derives the generalized or anisotropic Debye equation and the dynamic magnetic susceptibility for monodisperse ferrofluids under a static magnetic field and subject to a weak probing AC field.

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## I. DERIVATION OF MAGNETIZATION RELAXATION EQUATION

In this section we derive the magnetization relaxation equation (MRE) for monodisperse ferrofluids, in the form of Eq. (3) in the main text. We follow Ref. [1]. Our starting point is the Smoluchowski equation (SE), or Fokker-Planck-Smoluchowski (FPS) equation, Eq. (1) in the main text. It describes the Brownian rotational motion of a rigid dipolar particle under an external magnetic field and an effective dipolar field due to all other dipolar particles.

The stochastic variable  $\mathbf{e}$  characterizing the orientational coordinate of a dipolar particle, runs through a unit sphere  $\Omega_e$  defined by  $e_1^2 + e_2^2 + e_3^2 = 1$  in three-dimensional space. For this reason, a spherical coordinate system is extensively used. However, for our purpose we find it is more convenient to employ the Cartesian coordinate system, in which the SE is recast into the following form:

$$\frac{\partial}{\partial t} W(\mathbf{e}, t) = -\frac{\partial}{\partial e_i} \mathcal{M}_{ij}(\mathbf{e}) \left[ \frac{\partial F^c(\mathbf{e})}{\partial e_j} + k_B T \frac{\partial}{\partial e_j} \right] W(\mathbf{e}, t), \quad (1)$$

where

$$\mathcal{M}_{ij} = \frac{D_r}{k_B T} (\delta_{ij} - e_i e_j), \quad (2)$$

is the mobility matrix, with  $D_r \equiv 1/2\tau_r$  the rotational self-diffusion constant. In this supplemental material summation over repeated index is always implied. We also make the vacuum magnetic permeability  $\mu_0$  implicit, without causing any confusions because it can always be absorbed into the thermal energy, just like we do in the main text.

In Eq. (1)  $F^c$  is the nonequilibrium configurational free energy functional:

$$F^c(\mathbf{e}) = -\mu \mathbf{e} \cdot [\mathbf{H} + \mathbf{H}^{dd}], \quad (3)$$

which describes the energetic part of the single-particle chemical potential. The nonequilibrium mean dipolar field  $\mathbf{H}^{dd} \equiv \mathbf{H}_e^L - \mathbf{H}_e$  is determined by Eq. (2) in the main text, with  $\mathbf{H}_e^L$  and  $\mathbf{H}_e$  respectively the nonequilibrium Langevin effective field and thermodynamic effective field associated with the instantaneous magnetization,  $\mathbf{M}(t)$ . The two effective fields points along the same direction as  $\mathbf{M}(t)$ .

For convenience, we denote  $\boldsymbol{\xi}_e^L = \mu \mathbf{H}_e^L / k_B T$  and  $\boldsymbol{\xi}_e = \mu \mathbf{H}_e / k_B T$  as the dimensionless effective fields. Given a magnetization curve specified by a dimensionless function  $G$ , we have  $L(\boldsymbol{\xi}_e^L) = G(\boldsymbol{\xi}_e)$ , where  $L(x) = \coth x - 1/x$  is the dimensionless Langevin function. Therefore, for an external magnetic field  $\mathbf{H}$  (with  $\boldsymbol{\xi} = \mu \mathbf{H} / k_B T$  its dimensionless counterpart), we

can also associate it with an auxiliary Langevin field by  $\boldsymbol{\xi}^L = L^{-1}(G(\boldsymbol{\xi}))\boldsymbol{\xi}/\xi$  and  $\mathbf{H}^L \equiv (k_B T/\mu)\boldsymbol{\xi}^L$ .

The dimensionless instantaneous magnetization is given by

$$\mathbf{m}(t) \equiv \mathbf{M}(t)/\rho\mu = \int_{\Omega_e} \mathbf{e}W(\mathbf{e}, t)d\Omega_e, \quad (4)$$

just the first-order moment of  $W(\mathbf{e}, t)$ .

If  $\mathbf{H}$  is a static field, the equilibrium distribution function  $W_0(\boldsymbol{\xi}, \mathbf{e})$  is determined by the Gibbs-Boltzmann distribution function  $\sim \exp(-F_0^c/k_B T)$ , where  $F_0^c \equiv \lim_{\boldsymbol{\xi}_e \rightarrow \boldsymbol{\xi}} F^c = -\mu\mathbf{e} \cdot \mathbf{H}^L$  is the equilibrium configurational free energy. After normalization we obtain

$$W_0(\boldsymbol{\xi}, \mathbf{e}) = \frac{\xi^L}{4\pi \sinh \xi^L} \exp(\boldsymbol{\xi}^L \cdot \mathbf{e}), \quad (5)$$

which, in the noninteracting limit with  $\boldsymbol{\xi}^L \rightarrow \boldsymbol{\xi}$ , reduces to Langevin's result for an ideal paramagnetic gas. Eq. (5) leads to the equilibrium magnetization curve specified by  $M_0 = \rho\mu L(\xi^L) \equiv G(\xi)$ , thereby verifying the self-consistency of our proposed form for  $\mathbf{H}^{dd}$ .

Of central importance is the generalized canonical distribution function, which maximizes the information entropy for all the nonequilibrium distribution functions satisfying the instantaneous macroscopic constraints. It is of the following form:

$$\tilde{W}(\mathbf{e}, t) = \tilde{Z}^{-1} \exp[-F^c/k_B T + \mathbf{h}(t) \cdot \mathbf{e}], \quad (6)$$

where  $\tilde{Z}$  is the normalization factor and  $\mathbf{h}$  is a vectorial parameter that parameterizes the nonequilibrium thermodynamic force. It is determined by the macroscopic constraint

$$\int_{\Omega_e} \mathbf{e}\tilde{W}(\mathbf{e}, t)d\Omega_e = \mathbf{m}(t), \quad (7)$$

from which we deduce  $\mathbf{h} = \boldsymbol{\xi}_e - \boldsymbol{\xi}$ . Therefore, we have  $\tilde{W}(\mathbf{e}, t) = W_0(\boldsymbol{\xi}_e^L(t), \mathbf{e})$ .

To progress, we rewrite the SE (1) in a compact form

$$\frac{\partial}{\partial t} W(\mathbf{e}, t) = -i\mathcal{L}W(\mathbf{e}, t), \quad (8)$$

with  $\mathcal{L}$  the FPS operator defined by

$$-i\mathcal{L}Y(\mathbf{e}) = -\frac{\partial}{\partial e_i} K_i(\mathbf{e})Y(\mathbf{e}) + \frac{\partial^2}{\partial e_i \partial e_j} \mathcal{D}_{ij}(\mathbf{e})Y(\mathbf{e}). \quad (9)$$

for an arbitrary distribution function  $Y(\mathbf{e})$  defined on  $\Omega_e$ . In Eq. (9)  $K_i$  is the FPS drift

$$K_i(\mathbf{e}) = D_r \left[ -2e_i + \tilde{\xi}_i - e_i(\tilde{\xi}_k e_k) \right] \quad (10)$$

with  $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \boldsymbol{\xi}_e^L - \boldsymbol{\xi}_e$ , and  $\mathcal{D}$  is the diffusion matrix with

$$\mathcal{D}_{ij}(\mathbf{e}) = D_r(\delta_{ij} - e_i e_j). \quad (11)$$

To derive the equation of motion for  $\mathbf{m}$  in closed form, we need to extract the relevant macroscopic information from the SE. This can be achieved via the projection operator technique in nonequilibrium statistical mechanics [2], first introduced by Zwanzig [3].

For this purpose we introduce a projection operator  $\mathcal{P}(t)$  by

$$\mathcal{P}(t)Y(\mathbf{e}) = \tilde{W}(\mathbf{e}, t) \int d\Omega_e Y(\mathbf{e}) + \frac{\partial \tilde{W}(\mathbf{e}, t)}{\partial e_j} \int d\Omega_e [e_j - m_j(t)] Y(\mathbf{e}) \quad (12)$$

for an arbitrary  $Y(\mathbf{e})$ . Obviously,  $\mathcal{P}(t)$  projects out  $W(\mathbf{e}, t)$  to its relevant part:

$$\mathcal{P}(t)W(\mathbf{e}, t) = \tilde{W}(\mathbf{e}, t). \quad (13)$$

The initial distribution function is determined by our experimental preparation and usually assumes the form  $W(\mathbf{e}, 0) = \tilde{W}(\mathbf{e}, 0)$ , describing an equilibrium state with a definite macroscopic magnetization. Utilizing the properties of  $\mathcal{P}(t)$ , we obtain

$$W(\mathbf{e}, t) = \tilde{W}(\mathbf{e}, t) - \int_0^t ds G(t, s) [1 - \mathcal{P}(s)] i\mathcal{L}\tilde{W}(\mathbf{e}, s), \quad (14)$$

with

$$G(t, s) = T_+ \exp \left( - \int_s^t du [1 - \mathcal{P}(u)] i\mathcal{L} \right) \quad (15)$$

in which  $T_+$  denotes operators are ordered from right to left as time increases. Hence, the time rates of change for  $m_i(t) \equiv \int_{\Omega_e} e_i W(\mathbf{e}, t) d\Omega_e = \int_{\Omega_e} e_i \tilde{W}(\mathbf{e}, t) d\Omega_e$  can be obtained as

$$\dot{m}_i(t) = \int_{\Omega_e} d\Omega_e K_i(\mathbf{e}) \tilde{W}(\mathbf{e}, t) - \int_0^t ds \int_{\Omega_e} d\Omega_e K_i(\mathbf{e}) G(t, s) [1 - \mathcal{P}(s)] i\mathcal{L}\tilde{W}(\mathbf{e}, s), \quad (16)$$

Eq. (16) is a closed equation for the evolution of  $m_i(t)$ , because  $\tilde{W}(\mathbf{e}, t)$ ,  $\mathcal{P}(t)$ , and  $G(t, s)$  can be explicitly expressed in terms of  $m_i(t)$ . We can further recast Eq. (16) in a standard form of nonequilibrium thermodynamics, i. e., expressing its right-hand side in terms of thermodynamic driving forces  $\mathbf{h}(t) \equiv \boldsymbol{\xi} - \boldsymbol{\xi}_e$  multiplied by transport coefficients. First, we write

$$\int d\Omega_e K_i(\mathbf{e}) \tilde{W}(\mathbf{e}, t) = \tilde{\mathcal{M}}_{ij}(t) h_j(t), \quad (17)$$

with the renormalized mobility matrix defined by

$$\tilde{\mathcal{M}}_{ij}(t) = \int d\Omega_e \tilde{W}(\mathbf{e}, t) \mathcal{M}_{ij}(\mathbf{e}). \quad (18)$$

Second, noting that

$$-i\mathcal{L}\tilde{W}(\mathbf{e}, t) = \frac{\partial}{\partial e_i} \mathcal{M}_{ij}(\mathbf{e}) \tilde{W}(\mathbf{e}, t) h_j(t), \quad (19)$$

we have

$$\int d\Omega_e K_i(\mathbf{e}) G(t, s) [1 - \mathcal{P}(s)] i\mathcal{L}\tilde{W}(\mathbf{e}, s) = \Lambda_{ij}(t, s) h_j(s), \quad (20)$$

where

$$\Lambda_{ij}(t, s) = - \int d\Omega_e K_i(\mathbf{e}) G(t, s) [1 - \mathcal{P}(s)] \frac{\partial}{\partial e_k} \mathcal{M}_{kj} \tilde{W}(\mathbf{e}, s). \quad (21)$$

Using Eqs. (17) and (20), the equation of motion (16) is recast into

$$\dot{m}_i(t) = -\tilde{\mathcal{M}}_{ij}(t) h_j(t) - \int_0^t ds \Lambda_{ij}(t, s) h_j(s). \quad (22)$$

Here, the first term on the right hand side describes the instantaneous reaction of  $m_i$  to the corresponding thermodynamic force ( $-\mathbf{h} \equiv \boldsymbol{\xi} - \boldsymbol{\xi}_e$ ), while the second term represents a retarded response due to forces in the past. The memory kernel  $\Lambda_{ij}(t, s)$  describes the coupling of  $\mathbf{m}$  to the mesoscopic fluctuations of high-order moments of the distribution function. These are supposed to decay in a time much shorter than the time scale for an appreciable change of  $m_i$ . Therefore, the retardation effect may be discarded if we are interested in the dynamics on a sufficiently slow time scale that validates  $\mathbf{m}$  as the sole slow variable. Hence, we have the following instantaneous response equation:

$$\dot{m}_i(t) = -\tilde{\mathcal{M}}_{ij}(t) h_j(t), \quad (23)$$

which is in Onsager's standard form. However, the transport coefficients,  $\tilde{\mathcal{M}}_{ij}$ , is in general time-dependent through its dependence on  $\mathbf{m}(t)$ .

The instantaneous transport coefficients  $\tilde{\mathcal{M}}_{ij}(t)$  can be calculated by substituting into Eq. (18) the expressions for  $\mathcal{M}_{ij}$  and  $\tilde{W}$ , respectively. To perform integration over the domain of  $\mathbf{e}$ , we set up a spherical coordinate system with the polar axis pointing along the direction of  $\mathbf{m}(t)$  or  $\boldsymbol{\xi}_e^L(t)$ . Then we write  $\boldsymbol{\xi}_e^L(t) = \xi_e^L(1, 0, 0)$  and parameterize  $\mathbf{e} = (e_1, e_2, e_3)$  with  $(\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$  ( $0 \leq \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$ ). We thus have

$$\tilde{\mathcal{M}}_{ij}(t) = \frac{D_r}{k_B T} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta (\delta_{ij} - e_i e_j) \frac{\xi_e^L}{4\pi \sinh \xi_e^L} \exp(\xi_e^L \cos \theta), \quad (24)$$

from which we deduce that the transport matrix is of the following diagonal form:

$$\tilde{\mathcal{M}}(t) = \frac{2D_r}{k_B T} \begin{bmatrix} \gamma_{\parallel} & 0 & 0 \\ 0 & \gamma_{\perp} & 0 \\ 0 & 0 & \gamma_{\perp} \end{bmatrix}, \quad (25)$$

with the longitudinal coefficient

$$\gamma_{\parallel} = L(\xi_e^L)/\xi_e^L \quad (26)$$

and the transverse coefficient

$$\gamma_{\perp} = \frac{1}{2} [1 - L(\xi_e^L)/\xi_e^L]. \quad (27)$$

Notably, due to  $e_1^2 + e_2^2 + e_3^2 = 1$ , we have  $\text{tr}\tilde{\mathcal{M}} = \text{tr}\mathcal{M} = 2D_r/k_B T$  so that

$$\gamma_{\parallel} + 2\gamma_{\perp} = 1. \quad (28)$$

Arising from symmetry properties of the diffusion matrix, such a general relation between the longitudinal and transverse transport coefficients holds irrespective of microscopic details.

It should be noted that the longitudinal and transverse components are defined with respect to the direction of the instantaneous magnetization  $\mathbf{M}(t) = \rho\mu\mathbf{m}(t)$ . This is the ‘‘canonical decomposition’’ in the framework of nonequilibrium thermodynamics for an anisotropic thermodynamic state characterized by  $\mathbf{M}(t)$ .

To this end, we may write down the vectorial magnetization relaxation equation for  $\mathbf{M}$ , which reads

$$\tau_r \frac{d\mathbf{M}}{dt} = \gamma_{\parallel} 3\chi_L [\mathbf{H} - \mathbf{H}_e]_{\parallel} + \gamma_{\perp} 3\chi_L [\mathbf{H} - \mathbf{H}_e]_{\perp}, \quad (29)$$

exactly in the same form as Eq. (3) in the main text.

## II. DYNAMIC SUSCEPTIBILITY AT FINITE STATIC MAGNETIC FIELD

The MRE (29) is nonlinear with respect to the instantaneous magnetization, arising from both the equation of state and transport coefficients. In some situations, e. g., in the simultaneous presence of a large constant field and a tiny time-independent field, it may be reduced to a mathematically simpler form.

In the regime defined by  $|\xi_e - \xi| \ll 1$ , the magnetization relaxes in the neighborhood of a hypothetical quasi-equilibrium state sustained by the external magnetic field  $\mathbf{H}$ . This hypothetical quasi-equilibrium state is described by the magnetization  $\mathbf{M}_H = \rho\mu G(\xi)\mathbf{H}/H$ , which may be time-dependent if  $\mathbf{H}$  is. Treating  $\boldsymbol{\nu} \equiv \xi_e(t) - \xi$  as a small parameter we can simplify our MRE (29) to the first order of  $\nu$ .

To proceed, we notice that [4, 5], to the first order of  $\nu$ ,

$$\xi_e = \xi + \frac{\boldsymbol{\nu} \cdot \boldsymbol{\xi}}{\xi}, \quad \frac{\xi_e}{\xi} = \frac{\boldsymbol{\xi}}{\xi} + \frac{\boldsymbol{\xi} \times (\boldsymbol{\nu} \times \boldsymbol{\xi})}{\xi^3} \quad (30)$$

and

$$G(\xi_e) = G(\xi) + \frac{\boldsymbol{\nu} \cdot \boldsymbol{\xi}}{\xi} \frac{dG(\xi)}{d\xi}, \quad (31)$$

with which we can express the driving force  $\boldsymbol{\nu}$  linearly in terms of  $\mathbf{M} - \mathbf{M}_H$ . On the other hand, we can simply replace the state-dependent transport coefficients at  $\mathbf{M}$  by their corresponding values at  $\mathbf{M}_H$ .

Using Eqs. (30) and (31), after some algebra we obtain the following anisotropic Debye equation [Eq. (4) in the main text]:

$$\frac{d\mathbf{M}}{dt} = -\frac{(\mathbf{M}_{\parallel} - \mathbf{M}_H)}{\tau_{\parallel}} - \frac{\mathbf{M}_{\perp}}{\tau_{\perp}}, \quad (32)$$

where  $\mathbf{M}_{\parallel, \perp}$  denotes the components of  $\mathbf{M}$  parallel or perpendicular to  $\mathbf{H}$ , respectively, while  $\tau_{\parallel}$  and  $\tau_{\perp}$  are the corresponding field-dependent relaxation times:

$$\frac{\tau_{\parallel}(\xi)}{\tau_r} = \frac{d\xi^L}{d\xi} \frac{d \ln L(\xi^L)}{d \ln \xi^L}; \quad \frac{\tau_{\perp}(\xi)}{\tau_r} = \frac{\xi^L}{\xi} \frac{2L(\xi^L)}{\xi^L - L(\xi^L)}. \quad (33)$$

Now, suppose the total external magnetic field  $\mathbf{H}(t)$  is comprised of a static uniform field  $\mathbf{H}_0 = H_0 \mathbf{e}_{\parallel}$  and a probing AC field  $\overline{\mathbf{H}} \exp(i\omega t) = \overline{H} \exp(i\omega t) \mathbf{e}_{\parallel, \perp}$ . The AC field can be either parallel (along  $\mathbf{e}_{\parallel}$ ) or perpendicular (along  $\mathbf{e}_{\perp}$ ) to  $\mathbf{H}_0$ .

In the longitudinal (parallel) situation, the instantaneous magnetization  $\mathbf{M}(t)$  keeps along the  $\mathbf{e}_{\parallel}$  axis and the dimensionless magnetic field is  $\xi = \xi_0 + \overline{h} \exp(i\omega t)$ , with  $\overline{h} = \mu \overline{H} / k_B T$ . Thus we have

$$\frac{dm(t)}{dt} = -\frac{m(t) - m_H}{\tau_{\parallel}(\xi)}, \quad (34)$$

in which  $m_H \equiv M_H / \rho \mu = G(\xi)$  is the (dimensionless) magnetization for the instantaneous hypothetical quasi-equilibrium state. With  $\overline{h} \ll \xi_0$  we may replace  $\tau_{\parallel}(\xi)$  with  $\tau_{\parallel}(\xi_0)$  and write

$$m_H = G(\xi_0) + \frac{dG(\xi_0)}{d\xi_0} \overline{h} \exp(i\omega t), \quad (35)$$

to the first order of  $\overline{h}/\xi_0$ . On the other hand, the instantaneous nonequilibrium magnetization assumes the following form:

$$m(t) = G(\xi_0) + \frac{\chi_{\parallel}(\omega, \xi_0)}{3\chi_L} \overline{h} \exp(i\omega t), \quad (36)$$

where  $\chi_{\parallel}$  is the longitudinal frequency-dependent susceptibility, also depending on the static magnetic field strength.

We substitute Eqs. (35) and (36) into Eq. (34) and obtain

$$\chi_{\parallel}(\omega, \xi_0) = \frac{\chi_L}{1 + i\omega\tau_{\parallel}(\xi_0)} \frac{3dG(\xi_0)}{d\xi_0} \equiv \frac{\chi_{\parallel}^0(\xi_0)}{1 + i\omega\tau_{\parallel}(\xi_0)}, \quad (37)$$

where  $\chi_{\parallel}^0(\xi_0) \equiv \chi_{\parallel}(0, \xi_0) = 3\chi_L dG(\xi_0)/d\xi_0$  is the finite-field static (zero-frequency) susceptibility along the direction ( $\mathbf{e}_{\parallel}$ ) parallel to  $\mathbf{H}$ . Thus the peak frequency of the imaginary-part susceptibility spectra lies at  $\tau_{\parallel}^{-1}(\xi_0)$ .

In the transverse (perpendicular) situation, the total magnetic field is specified by  $\boldsymbol{\xi} = \xi_0 \mathbf{e}_{\parallel} + \bar{h} \exp(i\omega t) \mathbf{e}_{\perp}$ . Thus  $\xi = \xi_0 + O((\bar{h}/\xi_0)^2)$ . To the first order of  $\bar{h}/\xi_0$ , we have

$$\mathbf{m}_H = G(\xi_0) [\mathbf{e}_{\parallel} + (\bar{h}/\xi_0) \exp(i\omega t) \mathbf{e}_{\perp}] \quad (38)$$

as the (dimensionless) magnetization of the hypothetical quasi-equilibrium state.

On the other hand, the true instantaneous nonequilibrium magnetization assumes the following form:

$$\mathbf{m}(t) = G(\xi_0) \mathbf{e}_{\parallel} + \frac{\chi_{\perp}(\omega, \xi_0)}{3\chi_L} \bar{h} \exp(i\omega t) \mathbf{e}_{\perp}, \quad (39)$$

where  $\chi_{\perp}$  is the transverse frequency-dependent susceptibility, also depending on the static magnetic field strength.

Now, we substitute Eqs. (38) and (39) into Eq. (32) and find the longitudinal relaxation is of higher order and we are left with the transverse relaxation. We obtain

$$\chi_{\perp}(\omega, \xi_0) = \frac{\chi_L}{1 + i\omega\tau_{\perp}(\xi_0)} \frac{3G(\xi_0)}{\xi_0} \equiv \frac{\chi_{\perp}^0(\xi_0)}{1 + i\omega\tau_{\perp}(\xi_0)}, \quad (40)$$

where  $\chi_{\perp}^0(\xi_0) \equiv \chi_{\perp}(0, \xi_0) = 3\chi_L G(\xi_0)/\xi_0$  is the finite-field static (zero-frequency) magnetic susceptibility along the direction ( $\mathbf{e}_{\perp}$ ) perpendicular to  $\mathbf{H}$ . Similar to the longitudinal case, the peak frequency of the imaginary-part susceptibility spectra lies at  $\tau_{\perp}^{-1}(\xi_0)$ .

It is amazing that our simple analytical expressions for the field-dependent dynamic susceptibility spectra, Eqs. (37) and (40) along with Eq. (33), can quantitatively match the corresponding results from Brownian dynamics simulations, for all the studied samples, with different particle concentrations and interaction strength.

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[1] A. Fang, *Phys. Fluids* **31**, 122002 (2019).

[2] H. Grabert, *Projection Operator Techniques in Nonequilibrium Statistical Mechanics* (Springer Verlag, Berlin, 1982).

- [3] R. Zwanzig, Phys. Rev. **124**, 983 (1961).
- [4] M. A. Martsenyuk, Y. L. Raikher, and M. I. Shliomis, J. Exp. Theor. Phys. **38**, 413 (1974).
- [5] Y. L. Raikher and M. I. Shliomis, Adv. Chem. Phys. **87**, 595 (1994).