Electronic Supplementary Information

An analytical expression for the van der Waals interaction in oriented-attachment growth: a spherical nanoparticle and a growing cylindrical nanorod

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Eq. S1:

$$E = \int_{C-R_2}^{C+R_2} E_P q \pi \frac{R}{C}\left[R_2^2 - (C-R)^2\right] dR$$

$$= -\frac{2\pi^2 q^2 \lambda}{C} \int_{C-R_2}^{C+R_2} \left[R_2^2 - (C-R)^2\right] dR \int_{R-L/2}^{R+L/2} \frac{r - \sqrt{r^2 - R_0^2}}{r^5} dr$$

$$= -\frac{2\pi^2 q^2 \lambda}{C} \int_{C-R_2}^{C+R_2} \left[R_2^2 - (C-R)^2\right] dR \left[\sqrt{r^2 - R_0^2}\left(\frac{1}{4r^4} - \frac{1}{8R_0^2 r^2}\right) + \frac{\tan^{-1}\left(\frac{R_0}{\sqrt{r^2 - R_0^2}}\right)}{8R_0^3} - \frac{1}{3r^3} \right]_{R-L/2}^{R+L/2}$$

$$= -\frac{2\pi^2 q^2 \lambda}{C} \int_{C-R_2}^{C+R_2} \left[R_2^2 - (C-R)^2\right] dR \left\{ \begin{array}{l} \left[\sqrt{(R+L/2)^2 - R_0^2}\left(\frac{1}{4(R+L/2)^4} - \frac{1}{8R_0^2 (R+L/2)^2}\right) + \frac{\tan^{-1}\left(\frac{R_0}{\sqrt{(R+L/2)^2 - R_0^2}}\right)}{8R_0^3} - \frac{1}{3(R+L/2)^3} \right] \\ - \left[\sqrt{(R-L/2)^2 - R_0^2}\left(\frac{1}{4(R-L/2)^4} - \frac{1}{8R_0^2 (R-L/2)^2}\right) + \frac{\tan^{-1}\left(\frac{R_0}{\sqrt{(R-L/2)^2 - R_0^2}}\right)}{8R_0^3} - \frac{1}{3(R-L/2)^3} \right] \end{array} \right\}$$

$$= -\frac{2\pi^2 q^2 \lambda}{C}\frac{1}{24} \left\{ \begin{array}{l} \frac{16(L/2)(3C-2R)}{(L/2)^2 - R^2} + \sqrt{(L/2)^2 + 2R(L/2) - R_0^2 + R^2} \left[\begin{array}{l} -\frac{6((L/2)+C)}{((L/2)+R)^2} + \frac{R}{R_0^2} + \frac{5R_2^2 - 5(L/2)^2 - 5C^2 + 6R_0^2 - 10(L/2)C}{R_0^2((L/2)+R)} \\ -\frac{2(L/2)+3C}{R_0^2} + \frac{2(((L/2)+C)^2 - R_2^2)}{((L/2)+R)^3} \end{array} \right] \\ + \frac{16(L/2)((R-2C)(L/2)^2 + (C^2 - R_2^2)R)}{((L/2)^2 - R^2)^2} + \frac{1}{R_0^3}\left(-(L/2)^3 + 3C(L/2)^2 + 3R_2^2(L/2) - 3(C^2 + 4R_0^2)(L/2) + 12CR_0^2\right) \\ \tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2 - 2R(L/2) - R_0^2 + R^2}}\right) + \frac{R(-3R_2^2 + 3C^2 + R^2 - 3CR)\tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2 - 2R(L/2) - R_0^2 + R^2}}\right)}{R_0^3} \\ - \frac{1}{R_0^3}\left((L/2)^3 + 3C(L/2)^2 - 3R_2^2(L/2) + 3(C^2 + 4R_0^2)(L/2) + 12CR_0^2\right)\tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2 + 2R(L/2) - R_0^2 + R^2}}\right) \\ - \frac{R(-3R_2^2 + 3C^2 + R^2 - 3CR)\tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2 + 2R(L/2) - R_0^2 + R^2}}\right)}{R_0^3} - 8\ln(16(R-(L/2))) + 8\ln(-16(R+(L/2))) \end{array} \right\} \Bigg|_{C-R_2}^{C+R_2}$$

$$
-\frac{2\pi^2 q^2 \lambda}{C}\frac{1}{24}\left\{
\begin{array}{l}
\frac{1}{R_0^3}\left((L/2)^3-3C(L/2)^2-3R_2^2(L/2)+3\left(C^2+4R_0^2\right)(L/2)-12CR_0^2\right)\\
\left[\begin{array}{l}\tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2-2R(L/2)-R_0^2+R^2}}\right)-\\
i\ln\left(\frac{2R_0^2\left(\sqrt{(L/2)^2-2R(L/2)-R_0^2+R^2}-iR_0\right)}{\left((L/2)^3-3C(L/2)^2-3R_2^2(L/2)+3\left(C^2+4R_0^2\right)(L/2)-12CR_0^2\right)(R-(L/2))}\right)\end{array}\right]\\
+8\ln\left(2\left(-(L/2)+R+\sqrt{(L/2)^2-2R(L/2)-R_0^2+R^2}\right)\right)+\\
\frac{1}{R_0^3}\left((L/2)^3+3C(L/2)^2-3R_2^2(L/2)+3\left(C^2+4R_0^2\right)(L/2)+12CR_0^2\right)\\
\left[\begin{array}{l}\tan^{-1}\left(\frac{R_0}{\sqrt{(L/2)^2+2R(L/2)-R_0^2+R^2}}\right)-\\
i\ln\left(\frac{2R_0^2\left(\sqrt{(L/2)^2+2R(L/2)-R_0^2+R^2}-iR_0\right)}{\left((L/2)^3+3C(L/2)^2-3R_2^2(L/2)+3\left(C^2+4R_0^2\right)(L/2)+12CR_0^2\right)(R+(L/2))}\right)\end{array}\right]\\
-8\ln\left(2\left((L/2)+R+\sqrt{(L/2)^2+2R(L/2)-R_0^2+R^2}\right)\right)+\\
\sqrt{(L/2)^2-2R(L/2)-R_0^2+R^2}\left[\begin{array}{l}\frac{2\left(((L/2)-C)^2-R_2^2\right)}{((L/2)-R)^3}-\frac{R}{R_0^2}+\\
\frac{5R_2^2-5(L/2)^2-5C^2+6R_0^2+10(L/2)C}{R_0^2((L/2)-R)}+\\
\frac{3C-2(L/2)}{R_0^2}+\frac{6(C-(L/2))}{((L/2)-R)^2}\end{array}\right]
\end{array}
\right\}\Bigg|_{C-R_2}^{C+R_2}
$$

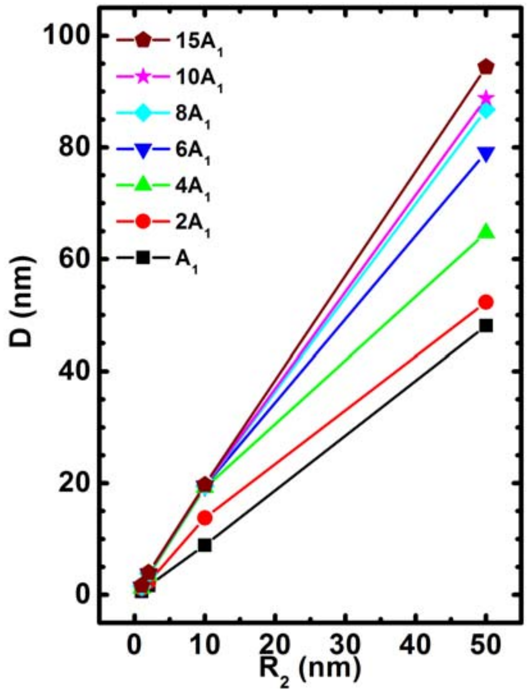


Fig. S1. Plots of D vs. R_2 at different As and fixed AR of 10. $A1=10^{-20}$ J. The plots give linear expressions between D and R2: $D = x+y\ R_2$, where the slope S=y.

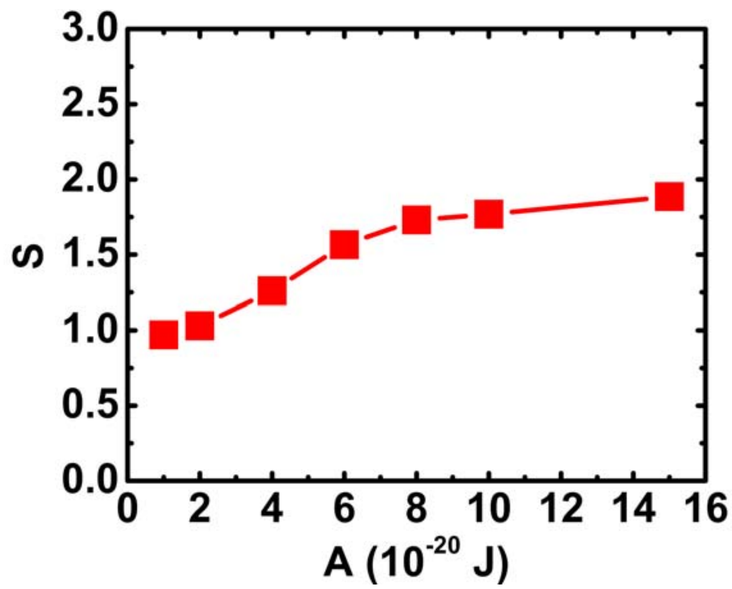


Fig. S2. Plots of S vs. A with fixed AR based on the expressions $D = x+y\ R_2$, where the slope S=y.

Derivation of analytical expression of vdw force between a nanorod and nanosphere with a random configuration:

The general equation of vdw force between two objects is:

$$E = -\int_{V_1} dv_1 \int_{V_2} dv_2 \frac{q^2\lambda}{r^6}$$

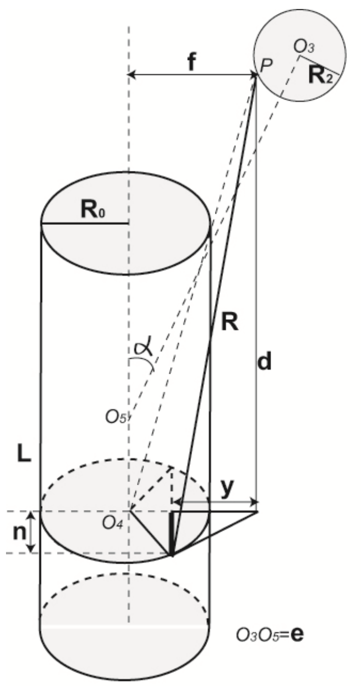
The vdw force between a nanorod and a nanosphere which is slightly off the axis of the nanorod has been derived as follows. First the total contribution of a nanorod to a point on the nanosphere can be written as:

$$E_{point} = \int_V dV \frac{q^2\lambda}{R^6}$$

and expanded the integration in detail:

$$E_{point} = \int dl \int dy \int dn \frac{q^2\lambda}{R^6}$$

In the above equation, n stands for the string on the circle, which is the horizontal cross-section of the cylinder, and y stands for the horizontal distance from the observed point P to the string and l stands for the height of the nanorod. The configuration shows as follows:



$$R=\sqrt{y^2+d^2+n^2}$$

$$n_{\max}=\sqrt{b^2-(f-y)^2}$$

$$E_{po\text{int}}=q^2\lambda\int dl\int dy\cdot 2\int_0^{\sqrt{b^2-(f-y)^2}}dn\frac{1}{(y^2+d^2+n^2)^3}$$

$$\int_0^{\sqrt{b^2-(f-y)^2}}dn\frac{1}{(d^2+y^2+n^2)^3}=\frac{1}{4}\left.\left(\frac{3\arctan(\frac{n}{\sqrt{d^2+y^2}})}{(d^2+y^2)^{\frac{5}{2}}}+\frac{n\cdot(5d^2+5y^2+3n^2)}{(d^2+y^2)^2\cdot(d^2+y^2+n^2)^2}\right)\right|_0^{\sqrt{b^2-(f-y)^2}}$$

$$=\frac{1}{4}\left(\frac{3\arctan(\frac{\sqrt{b^2-(f-y)^2}}{\sqrt{d^2+y^2}})}{(d^2+y^2)^{\frac{5}{2}}}+\frac{\sqrt{b^2-(f-y)^2}\cdot(5d^2+5y^2+3\cdot(b^2-(f-y)^2))}{(d^2+y^2)^2\cdot(d^2+y^2+(b^2-(f-y)^2))^2}\right)$$

R can be expressed in terms of *y*, *d*, *n*, and we get:

The maxixum of the string is (*b* is the radius of the rod):

So the total contribution of a string on the circle is:

The factor 2 is given due to the symmetry contribution of the string. This gives us the result:

$$E_{point} = q^2\lambda\int dl\int dx\frac{1}{4}\left(\frac{3\arctan(\frac{\sqrt{b^2-(f-x)^2}}{\sqrt{d^2+x^2}})}{(d^2+x^2)^{\frac{5}{2}}}+\frac{\sqrt{b^2-(f-x)^2}\cdot(5d^2+5x^2+3\cdot(b^2-(f-x)^2))}{(d^2+x^2)^2\cdot(d^2+x^2+(b^2-(f-x)^2))^2}\right)$$

Subsitute it back to the integration, we get:

$$E_{point} = q^2\lambda\int dl\int dx\frac{1}{4}\left(\frac{3\arctan(\frac{\sqrt{b^2-f^2+2fx}}{d})}{d^5}+\frac{\sqrt{b^2-f^2+2fx}\cdot(5d^2+3\cdot(b^2-f^2+2fx))}{d^4\cdot(d^2+b^2-f^2+2fx)^2}\right)$$

The next step is to calculate the vdw force contribution of the circle. The y direction has been chosen to integrate to get the contribution of the the total area of the circle. The integration is: (We use mathematica to calculate the intergral, to facilitate the calculation we set y=x, dy=dx)

$$E_{point} = q^2\lambda\int dl\int dy\frac{1}{4}\left(\frac{3\arctan(\frac{\sqrt{b^2-(f-y)^2}}{\sqrt{d^2+y^2}})}{(d^2+y^2)^{\frac{5}{2}}}+\frac{\sqrt{b^2-(f-y)^2}\cdot(5d^2+5y^2+3\cdot(b^2-(f-y)^2))}{(d^2+y^2)^2\cdot(d^2+y^2+(b^2-(f-y)^2))^2}\right)$$

It is very hard to integrate this equation. A reasonable assumption can be made here that the nanosphere is so close to the nanorod so that the higher order of x can be igored all:

The result is listed below:

$$\int dx \frac{1}{4}\left(\frac{3\arctan(\frac{\sqrt{b^2-f^2+2fx}}{d})}{d^5}+\frac{\sqrt{b^2-f^2+2fx}\cdot(5d^2+3\cdot(b^2-f^2+2fx))}{d^4\cdot(d^2+b^2-f^2+2fx)^2}\right)=$$

$$\frac{1}{4d^5}\left\{\begin{array}{l}
\dfrac{(b^2-f^2+2fx)^{1.5}Hypergeometric2F1\left[1.5,1,2.5,-\dfrac{b^2-f^2+2fx}{d^2}\right]}{fd}+ \\
\dfrac{0.667\cdot(b^2-f^2+2fx)^{1.5}Hypergeometric2F1\left[1.5,2,2.5,-\dfrac{b^2-f^2+2fx}{d^2}\right]}{fd}+ \\
3x\cdot\arctan(\dfrac{\sqrt{b^2-f^2+2fx}}{d})-\dfrac{3}{fd\sqrt{b^2-f^2+2fx}}\cdot(-0.333b^4+ \\
(0.5f^2-0.5b^2)(b^2-f^2+2fx)Hypergeometric2F1\left[0.5,1,1.5,-\dfrac{b^2-f^2+2fx}{d^2}\right]+ \\
0.167\cdot(b^2-f^2+2fx)^2Hypergeometric2F1\left[1.5,1,2.5,-\dfrac{b^2-f^2+2fx}{d^2}\right]+ \\
0.5b^2\cdot(b^2-f^2+2fx)-0.167\cdot(b^2-f^2+2fx)^2-0.5f^2\cdot(b^2-f^2+2fx)+ \\
0.667\cdot b^2f^2-0.333b^2fx-0.333f^4+0.333f^3x+0.667f^2x^2)
\end{array}\right\}$$

Then put the upper limit, which is the furthest point from the observed point P ($x=f+b$) and lower limit, which is the nearest point ($x=f-b$), into x, the complete integration as a function of d can be written as:

$$\frac{1}{4d^5}\left\{\begin{aligned}&\frac{(b^2-f^2+2f(f+b))^{1.5}Hypergeometric2F1\left[1.5,1,2.5,-\frac{b^2-f^2+2f(f+b)}{d^2}\right]}{fd}+\\&\frac{0.667\cdot(b^2-f^2+2f(f+b))^{1.5}Hypergeometric2F1\left[1.5,2,2.5,-\frac{b^2-f^2+2f(f+b)}{d^2}\right]}{fd}+\\&3(f+b)\cdot\arctan(\frac{\sqrt{b^2-f^2+2f(f+b)}}{d})-\frac{3}{fd\sqrt{b^2-f^2+2f(f+b)}}\cdot(-0.333b^4+\\&(0.5f^2-0.5b^2)(b^2-f^2+2f(f+b))Hypergeometric2F1\left[0.5,1,1.5,-\frac{b^2-f^2+2f(f+b)}{d^2}\right]+\\&0.167\cdot(b^2-f^2+2f(f+b))^2Hypergeometric2F1\left[1.5,1,2.5,-\frac{b^2-f^2+2f(f+b)}{d^2}\right]+\\&0.5b^2\cdot(b^2-f^2+2f(f+b))-0.167\cdot(b^2-f^2+2f(f+b))^2-0.5f^2\cdot(b^2-f^2+2f(f+b))+\\&0.667\cdot b^2f^2-0.333b^2f(f+b)-0.333f^4+0.333f^3(f+b)+0.667f^2(f+b)^2)\end{aligned}\right\}-$$

$$\frac{1}{4d^5}\left\{\begin{aligned}&\frac{(b^2-f^2+2f(f-b))^{1.5}Hypergeometric2F1\left[1.5,1,2.5,-\frac{b^2-f^2+2f(f-b)}{d^2}\right]}{fd}+\\&\frac{0.667\cdot(b^2-f^2+2f(f-b))^{1.5}Hypergeometric2F1\left[1.5,2,2.5,-\frac{b^2-f^2+2fx}{d^2}(f-b)\right]}{fd}+\\&3(f-b)\cdot\arctan(\frac{\sqrt{b^2-f^2+2f(f-b)}}{d})-\frac{3}{fd\sqrt{b^2-f^2+2f(f-b)}}\cdot(-0.333b^4+\\&(0.5f^2-0.5b^2)(b^2-f^2+2f(f-b))Hypergeometric2F1\left[0.5,1,1.5,-\frac{b^2-f^2+2f(f-b)}{d^2}\right]+\\&0.167\cdot(b^2-f^2+2f(f-b))^2Hypergeometric2F1\left[1.5,1,2.5,-\frac{b^2-f^2+2f(f-b)}{d^2}\right]+\\&0.5b^2\cdot(b^2-f^2+2f(f-b))-0.167\cdot(b^2-f^2+2f(f-b))^2-0.5f^2\cdot(b^2-f^2+2f(f-b))+\\&0.667\cdot b^2f^2-0.333b^2f(f-b)-0.333f^4+0.333f^3(f-b)+0.667f^2(f-b)^2)\end{aligned}\right\}$$

Then put d as the variable x (d=x) and integrate the above equation so that the total vdw force contribution of a nanorod to the observed point P can be calculated. The result is:

$$
\begin{aligned}
F(x) = \frac{1}{4fx^5}\Big\{ & \frac{0.2b^4}{(b^2+2bf-3f^2)^{0.5}} + \frac{0.2b^3 f}{(b^2+2bf-3f^2)^{0.5}} - \frac{b^2 f^2}{(b^2+2bf-3f^2)^{0.5}} + \frac{0.6bf^3}{(b^2+2bf-3f^2)^{0.5}} \\
& + \frac{6.661*10^{-17} f^4}{(b^2+2bf-3f^2)^{0.5}} - 0.3b^2(b^2+2bf-3f^2)^{0.5} + 0.3f^2(b^2+2bf-3f^2)^{0.5} + 0.1(b^2+2bf-3f^2)^{1.5} \\
& - \frac{0.2b^4}{(b^2+2bf+f^2)^{0.5}} - \frac{0.2b^3 f}{(b^2+2bf+f^2)^{0.5}} + \frac{0.6b^2 f^2}{(b^2+2bf+f^2)^{0.5}} + \frac{bf^3}{(b^2+2bf+f^2)^{0.5}} + \frac{0.4f^4}{(b^2+2bf+f^2)^{0.5}} \\
& + 0.3b^2(b^2+2bf+f^2)^{0.5} - 0.3f^2(b^2+2bf+f^2)^{0.5} - 0.1(b^2+2bf+f^2)^{1.5} \\
& + 0.332(b^2+2bf-3f^2)^{1.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{1.5,1\},\{3.5\},-\frac{b^2+2bf-3f^2}{x^2}\right] \\
& - 0.332(b^2+2bf+f^2)^{1.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{1.5,1\},\{3.5\},-\frac{b^2+2bf+f^2}{x^2}\right] \\
& + 0.443(b^2+2bf-3f^2)^{1.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{1.5,2\},\{3.5\},-\frac{b^2+2bf-3f^2}{x^2}\right] \\
& - 0.443(b^2+2bf+f^2)^{1.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{1.5,2\},\{3.5\},-\frac{b^2+2bf+f^2}{x^2}\right] \\
& + 0.884b^2(b^2+2bf-3f^2)^{0.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{0.5,1,2.5\},\{1.5,3.5\},-\frac{b^2+2bf-3f^2}{x^2}\right] \\
& - 0.884f^2(b^2+2bf-3f^2)^{0.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{0.5,1,2.5\},\{1.5,3.5\},-\frac{b^2+2bf-3f^2}{x^2}\right] \\
& - 0.884b^2(b^2+2bf+f^2)^{0.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{0.5,1,2.5\},\{1.5,3.5\},-\frac{b^2+2bf+f^2}{x^2}\right] \\
& + 0.884f^2(b^2+2bf+f^2)^{0.5}\, HypergeometricPFQ\,\mathrm{Re}\,gularized\left[\{0.5,1,2.5\},\{1.5,3.5\},-\frac{b^2+2bf+f^2}{x^2}\right] \Big\}
\end{aligned}
$$

In order to simplify the result, we assume that f is very small which means the nanosphere is just slightly off the axis of the nanorod so that the aspect ratio can be defined as $s=b/d$, which is a constant in this step. Integrate again, the equation above can be simplified into:

$$F(x)=-\frac{1}{40\,fx^5}\left\{\begin{array}{l}
(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
0.667\cdot(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]-\\
(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
0.667\cdot(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]+\\
\frac{1}{\sqrt{b^2-3f^2+2bf}}\cdot(-b^4-b^3f+-0.333f^4+\\
(1.5f^2-1.5b^2)(b^2-3f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]+\\
0.5\cdot(b^2-3f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
5\cdot b^2f^2+1.5b^2\cdot(b^2-3f^2+2bf)-0.5\cdot(b^2-3f^2+2bf)^2-\\
1.5f^2\cdot(b^2-3f^2+2bf)-3bf^3-3.33\times10^{-16}f^4)+\\
\frac{1}{\sqrt{b^2+f^2+2bf}}\cdot(b^4+b^3f+\\
(1.5b^2-1.5f^2)(b^2+f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]-\\
0.5\cdot(b^2+f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
3\cdot b^2f^2-1.5b^2\cdot(b^2+f^2+2bf)+0.5\cdot(b^2+f^2+2bf)^2-\\
1.5f^2\cdot(b^2+f^2+2bf)-5bf^3-2f^4)
\end{array}\right\}$$

So the *vdW* force between a rod and an observed point from a nanosphere is (substitute the x with the upper limit $t+L/2$ and lower limit $t-L/2$ of x, where t is the separation between point P and the middle plane of the rod):

$$-\frac{1}{40f(t+\frac{L}{2})^5}\left\{\begin{array}{l}
(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
0.667\cdot(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]-\\
(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
0.667\cdot(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]+\\
\frac{1}{\sqrt{b^2-3f^2+2bf}}\cdot(-b^4-b^3f+-0.333f^4-1.5f^2\cdot(b^2-3f^2+2bf)-3bf^3+\\
(1.5f^2-1.5b^2)(b^2-3f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]+\\
0.5\cdot(b^2-3f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
5\cdot b^2f^2+1.5b^2\cdot(b^2-3f^2+2bf)-0.5\cdot(b^2-3f^2+2bf)^2-3.33\times10^{-16}f^4)+\\
\frac{1}{\sqrt{b^2+f^2+2bf}}\cdot(b^4+b^3f-1.5f^2\cdot(b^2+f^2+2bf)-5bf^3-2f^4+\\
(1.5b^2-1.5f^2)(b^2+f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]-\\
0.5\cdot(b^2+f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
3\cdot b^2f^2-1.5b^2\cdot(b^2+f^2+2bf)+0.5\cdot(b^2+f^2+2bf)^2)
\end{array}\right\}+$$

$$\frac{1}{40f(t-\frac{L}{2})^5}\left\{\begin{array}{l}
(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
0.667\cdot(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]-\\
(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
0.667\cdot(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]+\\
\frac{1}{\sqrt{b^2-3f^2+2bf}}\cdot(-b^4-b^3f+-0.333f^4-1.5f^2\cdot(b^2-3f^2+2bf)-3bf^3+\\
(1.5f^2-1.5b^2)(b^2-3f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]+\\
0.5\cdot(b^2-3f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
5\cdot b^2f^2+1.5b^2\cdot(b^2-3f^2+2bf)-0.5\cdot(b^2-3f^2+2bf)^2-3.33\times10^{-16}f^4)+\\
\frac{1}{\sqrt{b^2+f^2+2bf}}\cdot(b^4+b^3f-1.5f^2\cdot(b^2+f^2+2bf)-5bf^3-2f^4+\\
(1.5b^2-1.5f^2)(b^2+f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]-\\
0.5\cdot(b^2+f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
3\cdot b^2f^2-1.5b^2\cdot(b^2+f^2+2bf)+0.5\cdot(b^2+f^2+2bf)^2)
\end{array}\right\}$$

Then integrate the above equation for t is a variable(t=x):

$$
\int \left(\begin{aligned}
&-\frac{1}{40f(x+\frac{L}{2})^5}(R_2^2-e^2+2ex-x^2)\left\{ \begin{aligned}
&(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
&0.667\cdot(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]-\\
&(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
&0.667\cdot(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]+\\
&\frac{1}{\sqrt{b^2-3f^2+2bf}}\cdot(-b^4-b^3f-0.333f^4-1.5f^2\cdot(b^2-3f^2+2bf)-3bf^3+\\
&(1.5f^2-1.5b^2)(b^2-3f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]+\\
&0.5\cdot(b^2-3f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
&5\cdot b^2f^2+1.5b^2\cdot(b^2-3f^2+2bf)-0.5\cdot(b^2-3f^2+2bf)^2-3.33\times10^{-16}f^4)+\\
&\frac{1}{\sqrt{b^2+f^2+2bf}}\cdot(b^4+b^3f-1.5f^2\cdot(b^2+f^2+2bf)-5bf^3-2f^4+\\
&(1.5b^2-1.5f^2)(b^2+f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]-\\
&0.5\cdot(b^2+f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
&3\cdot b^2f^2-1.5b^2\cdot(b^2+f^2+2bf)+0.5\cdot(b^2+f^2+2bf)^2)
\end{aligned} \right\}+\\
&\frac{1}{40f(x-\frac{L}{2})^5}(R_2^2-e^2+2ex-x^2)\left\{ \begin{aligned}
&(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
&0.667\cdot(b^2+f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]-\\
&(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
&0.667\cdot(b^2-3f^2+2bf)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]+\\
&\frac{1}{\sqrt{b^2-3f^2+2bf}}\cdot(-b^4-b^3f+-0.333f^4-1.5f^2\cdot(b^2-3f^2+2bf)-3bf^3+\\
&(1.5f^2-1.5b^2)(b^2-3f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]+\\
&0.5\cdot(b^2-3f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]+\\
&5\cdot b^2f^2+1.5b^2\cdot(b^2-3f^2+2bf)-0.5\cdot(b^2-3f^2+2bf)^2-3.33\times10^{-16}f^4)+\\
&\frac{1}{\sqrt{b^2+f^2+2bf}}\cdot(b^4+b^3f-1.5f^2\cdot(b^2+f^2+2bf)-5bf^3-2f^4+\\
&(1.5b^2-1.5f^2)(b^2+f^2+2bf)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]-\\
&0.5\cdot(b^2+f^2+2bf)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]-\\
&3\cdot b^2f^2-1.5b^2\cdot(b^2+f^2+2bf)+0.5\cdot(b^2+f^2+2bf)^2)
\end{aligned} \right\}
\end{aligned} \right) dx
$$

The result is as follows, where e is the center-center distance between the rod and the sphere:
:

$$F(x)=\frac{1}{60f(L^2-4x^2)^4}(-L^6+192x^4(r^2-2x^2)+4L^4(3r^2+4x^2)+48L^2(6r^2x^2-7x^4)$$
$$-12e^2(L^4+24L^2x^2+16x^4)+128e(5L^2x^3+4x^5))$$
$$\{-(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]$$
$$+(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]$$
$$+\frac{1}{(b^2+2bf-3f^2)^{0.5}}(\ -b^4-b^3f+5b^2f^2-3bf^3-3.33*10^{-16}f^4+1.5b^2(b^2+2bf-3f^2)$$
$$-1.5f^2(b^2+2bf-3f^2)-0.5(b^2+2bf-3f^2)^2$$
$$+(b^2+2bf-3f^2)(-1.5b^2+1.5f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]$$
$$+0.5(b^2+2bf-3f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )$$
$$+\frac{1}{(b^2+2bf+f^2)^{0.5}}(\ b^4+b^3f-3b^2f^2-5bf^3-2f^4-1.5b^2(b^2+2bf+f^2)$$
$$+1.5f^2(b^2+2bf+f^2)+0.5(b^2+2bf+f^2)^2$$
$$+(1.5b^2-1.5f^2)(b^2+2bf+f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]$$
$$-0.5(b^2+2bf+f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )$$
$$-0.667(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]$$
$$+0.667(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]\ \}$$

Then the vdW force between the nanorode and the nanosphere is equal to $vdW=F(e+R_2)-F(e-R_2)$:

$$vdw=\left\{\begin{aligned}
&\frac{1}{60f(L^2-4(e+R_2)^2)^4}\left[\begin{aligned}
&(-L^6+192(e+R_2)^4(R_2^2-2(e+R_2)^2)+4L^4(3R_2^2+4(e+R_2)^2)+\\
&48L^2(6R_2^2(e+R_2)^2-7(e+R_2)^4)-12e^2(L^4+24L^2(e+R_2)^2+\\
&16(e+R_2)^4)+128e(5L^2(e+R_2)^3+4(e+R_2)^5))\\
&\{-(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\\
&+(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\\
&+\frac{1}{(b^2+2bf-3f^2)^{0.5}}(\ -b^4-b^3f+5b^2f^2-3bf^3-3.33*10^{-16}f^4+\\
&1.5b^2(b^2+2bf-3f^2)\\
&-1.5f^2(b^2+2bf-3f^2)-0.5(b^2+2bf-3f^2)^2\\
&+(b^2+2bf-3f^2)(-1.5b^2+1.5f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]\\
&+0.5(b^2+2bf-3f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )\\
&+\frac{1}{(b^2+2bf+f^2)^{0.5}}(\ b^4+b^3f-3b^2f^2-5bf^3-2f^4-1.5b^2(b^2+2bf+f^2)\\
&+1.5f^2(b^2+2bf+f^2)+0.5(b^2+2bf+f^2)^2\\
&+(1.5b^2-1.5f^2)(b^2+2bf+f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]\\
&-0.5(b^2+2bf+f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )\\
&-0.667(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]\\
&+0.667(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]\ \}
\end{aligned}\right]-\\
&\frac{1}{60f(L^2-4(e-R_2)^2)^4}\left[\begin{aligned}
&(-L^6+192(e-R_2)^4(R_2^2-2(e-R_2)^2)+4L^4(3R_2^2+4(e-R_2)^2)+\\
&48L^2(6R_2^2(e-R_2)^2-7(e-R_2)^4)-12e^2(L^4+24L^2(e-R_2)^2+\\
&16(e-R_2)^4)+128e(5L^2(e-R_2)^3+4(e-R_2)^5))\\
&\{-(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\\
&+(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\\
&+\frac{1}{(b^2+2bf-3f^2)^{0.5}}(\ -b^4-b^3f+5b^2f^2-3bf^3-3.33*10^{-16}f^4+\\
&1.5b^2(b^2+2bf-3f^2)\\
&-1.5f^2(b^2+2bf-3f^2)-0.5(b^2+2bf-3f^2)^2\\
&+(b^2+2bf-3f^2)(-1.5b^2+1.5f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]\\
&+0.5(b^2+2bf-3f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )\\
&+\frac{1}{(b^2+2bf+f^2)^{0.5}}(\ b^4+b^3f-3b^2f^2-5bf^3-2f^4-1.5b^2(b^2+2bf+f^2)\\
&+1.5f^2(b^2+2bf+f^2)+0.5(b^2+2bf+f^2)^2\\
&+(1.5b^2-1.5f^2)(b^2+2bf+f^2)Hypergeometric2F1\left[0.5,1,1.5,-\frac{1}{s^2}\right]\\
&-0.5(b^2+2bf+f^2)^2\,Hypergeometric2F1\left[1.5,1,2.5,-\frac{1}{s^2}\right]\ )\\
&-0.667(b^2+2bf-3f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]\\
&+0.667(b^2+2bf+f^2)^{1.5}\,Hypergeometric2F1\left[1.5,2,2.5,-\frac{1}{s^2}\right]\ \}
\end{aligned}\right]
\end{aligned}\right\}$$