Electronic Supplementary Information

1 Derivation of Equation (38a,b)

Before diving into the derivation, we state the Gauss and Weingarten formulae

$$\frac{\partial \mathbf{g}_\alpha}{\partial \xi_\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{g}_\gamma + b_{\alpha\beta}\mathbf{n} \text{ (Gauss formula)} \tag{S1a}$$

$$\mathbf{n}_{,\alpha} = -b^\gamma{}_\alpha \mathbf{g}_\gamma = -b_{\alpha\gamma}\mathbf{g}^\gamma \text{ (Weingarten formula)} \tag{S1b}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbols. The surface gradient of a surface vector field $\mathbf{q} = q^\alpha \mathbf{g}_\alpha$ is

$$\begin{aligned}\nabla_s \mathbf{q} &= \frac{\partial q^\alpha \mathbf{g}_\alpha}{\partial \xi^\gamma} \otimes \mathbf{g}^\gamma = \frac{\partial q^\alpha}{\partial \xi^\gamma}\mathbf{g}_\alpha \otimes \mathbf{g}^\gamma + q^\alpha \frac{\partial \mathbf{g}_\alpha}{\partial \xi^\gamma} \otimes \mathbf{g}^\gamma = \frac{\partial q^\alpha}{\partial \xi^\gamma}\mathbf{g}_\alpha \otimes \mathbf{g}^\gamma + q^\alpha \left(\Gamma^\omega_{\alpha\gamma}\mathbf{g}_\omega + b_{\alpha\gamma}\mathbf{n}\right) \otimes \mathbf{g}^\gamma \\ &= \left\{\frac{\partial q^\alpha}{\partial \xi^\gamma} + \Gamma^\alpha_{\omega\gamma} q^\omega\right\}\mathbf{g}_\alpha \otimes \mathbf{g}^\gamma + b_{\alpha\gamma} q^\alpha \mathbf{n} \otimes \mathbf{g}^\gamma\end{aligned} \tag{S2}$$

where we have used (S1a). The term inside the curly bracket is, by definition, the surface covariant derivative of q^α,

$$q^\alpha{}_{|\gamma} \equiv \frac{\partial q^\alpha}{\partial \xi^\gamma} + \Gamma^\alpha_{\omega\gamma} q^\omega \tag{S3}$$

The surface divergence of $\mathbf{q}$ is

$$\begin{aligned}\nabla_s \cdot \mathbf{q} &= \frac{\partial q^\alpha \mathbf{g}_\alpha}{\partial \xi^\gamma} \cdot \mathbf{g}^\gamma = \frac{\partial q^\alpha}{\partial \xi^\gamma}\mathbf{g}_\alpha \cdot \mathbf{g}^\gamma + q^\alpha \frac{\partial \mathbf{g}_\alpha}{\partial \xi^\gamma} \cdot \mathbf{g}^\gamma = \frac{\partial q^\alpha}{\partial \xi^\alpha} + q^\alpha \left(\Gamma^\omega_{\alpha\gamma}\mathbf{g}_\omega + b_{\alpha\gamma}\mathbf{n}\right) \cdot \mathbf{g}^\gamma \\ &= \frac{\partial q^\alpha}{\partial \xi^\alpha} + \Gamma^\alpha_{\omega\alpha} q^\omega = q^\alpha{}_{|\alpha}\end{aligned} \tag{S4}$$

For a 2D tensor field $\mathbf{Q} = Q^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$, the surface divergence is:

$$\begin{aligned}\nabla_s \cdot \mathbf{Q} &= \frac{\partial Q^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta}{\partial \xi^\gamma} \cdot \mathbf{g}^\gamma = \frac{\partial Q^{\alpha\beta}}{\partial \xi^\gamma}\mathbf{g}_\alpha \delta^\gamma_\beta + Q^{\alpha\beta}\frac{\partial \mathbf{g}_\alpha}{\partial \xi^\gamma}\delta^\gamma_\beta + Q^{\alpha\beta}\mathbf{g}_\alpha \otimes \frac{\partial \mathbf{g}_\beta}{\partial \xi^\gamma} \cdot \mathbf{g}^\gamma \\ &= \frac{\partial Q^{\alpha\beta}}{\partial \xi^\beta}\mathbf{g}_\alpha + Q^{\alpha\beta}\left(\Gamma^\omega_{\alpha\beta}\mathbf{g}_\omega + b_{\alpha\beta}\mathbf{n}\right) + Q^{\alpha\beta}\mathbf{g}_\alpha \otimes \left(\Gamma^\omega_{\beta\gamma}\mathbf{g}_\omega + b_{\beta\gamma}\mathbf{n}\right) \cdot \mathbf{g}^\gamma \\ &= \left(\frac{\partial Q^{\alpha\beta}}{\partial \xi^\beta} + \Gamma^\alpha_{\gamma\beta} Q^{\gamma\beta} + \Gamma^\gamma_{\beta\gamma} Q^{\alpha\beta}\right)\mathbf{g}_\alpha + Q^{\alpha\beta} b_{\alpha\beta}\mathbf{n} = Q^{\alpha\beta}{}_{|\beta} + Q^{\alpha\beta} b_{\alpha\beta}\mathbf{n}\end{aligned} \tag{S5a}$$

where

$$Q^{\alpha\beta}{}_{|\beta} \equiv \frac{\partial Q^{\alpha\beta}}{\partial \xi^{\beta}} + \Gamma^{\alpha}_{\gamma\beta} Q^{\gamma\beta} + \Gamma^{\gamma}_{\beta\gamma} Q^{\alpha\beta} \tag{S5b}$$

The first term in the surface equilibrium equation can be written as

$$\boldsymbol{\sigma}_s - \mathbf{b}\cdot\mathbf{m}_s = \sigma_s^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta - b^{\alpha}{}_{\gamma}\mathbf{g}_\alpha \otimes \mathbf{g}^{\gamma}\cdot m_s^{\omega\beta}\mathbf{g}_\omega \otimes \mathbf{g}_\beta = \left(\sigma_s^{\alpha\beta} - b^{\alpha}{}_{\gamma} m_s^{\gamma\beta}\right)\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \tag{S6}$$

Using (S5a,b), the surface divergence of this term is

$$\begin{aligned}\nabla_s\cdot\left(\boldsymbol{\sigma}_s - \mathbf{b}\cdot\mathbf{m}_s\right) &= \left(\sigma_s^{\alpha\beta} - b^{\alpha}{}_{\gamma} m_s^{\gamma\beta}\right)_{|\beta}\mathbf{g}_\alpha + \left(\sigma_s^{\alpha\beta} - b^{\alpha}{}_{\gamma} m_s^{\gamma\beta}\right) b_{\alpha\beta}\mathbf{n} \\ &= \left(\sigma_s^{\alpha\beta} - b^{\beta}{}_{\gamma} m_s^{\alpha\gamma}\right)_{|\alpha}\mathbf{g}_\beta + \left(\sigma_s^{\alpha\beta} - b^{\beta}{}_{\gamma} m_s^{\alpha\gamma}\right) b_{\alpha\beta}\mathbf{n}\end{aligned} \tag{S7}$$

where we have used the fact that $\boldsymbol{\sigma}_s$, $\mathbf{m}_s$ and $\mathbf{b}$ are symmetric tensors. The second term in the surface equilibrium equation can be expressed as

$$\mathbf{n}\otimes\left(\mathbf{D}\cdot\mathbf{m}_s\right) = \mathbf{n}\otimes\left[\left(\mathbf{1}-\mathbf{n}\otimes\mathbf{n}\right)\cdot\nabla_s\cdot\mathbf{m}_s\right] = \mathbf{n}\otimes\left[\left(\mathbf{1}-\mathbf{n}\otimes\mathbf{n}\right)\cdot\left(m_s^{\alpha\beta}{}_{|\beta}\mathbf{g}_\alpha + m_s^{\alpha\beta} b_{\alpha\beta}\mathbf{n}\right)\right] = m_s^{\alpha\beta}{}_{|\beta}\mathbf{n}\otimes\mathbf{g}_\alpha . \tag{S8}$$

Hence, its surface divergence is

$$\begin{aligned}\nabla_s\cdot\left[\mathbf{n}\otimes\left(\mathbf{D}\cdot\mathbf{m}_s\right)\right] &= \nabla_s\cdot\left(m_s^{\alpha\beta}{}_{|\beta}\mathbf{n}\otimes\mathbf{g}_\alpha\right) \equiv \frac{\partial\left[m_s^{\alpha\beta}{}_{|\beta}\mathbf{n}\otimes\mathbf{g}_\alpha\right]}{\partial\xi^{\gamma}}\cdot\mathbf{g}^{\gamma} \\ &= \frac{\partial\left(m_s^{\alpha\beta}{}_{|\beta}\right)}{\partial\xi^{\gamma}}\mathbf{n}\delta_\alpha^\gamma + m_s^{\alpha\beta}{}_{|\beta}\frac{\partial\mathbf{n}}{\partial\xi^{\gamma}}\delta_\alpha^\gamma + m_s^{\alpha\beta}{}_{|\beta}\mathbf{n}\otimes\frac{\partial\mathbf{g}_\alpha}{\partial\xi^{\gamma}}\cdot\mathbf{g}^{\gamma} \\ &= \left(m_s^{\alpha\beta}{}_{|\beta}\right)_{,\gamma}\mathbf{n}\delta_\alpha^\gamma + m_s^{\alpha\beta}{}_{|\beta}\left(-b^{\omega}{}_{\gamma}\mathbf{g}_\omega\right)\delta_\alpha^\gamma + m_s^{\alpha\beta}{}_{|\beta}\mathbf{n}\otimes\left(\Gamma^{\omega}_{\alpha\gamma}\mathbf{g}_\omega + b_{\alpha\gamma}\mathbf{n}\right)\cdot\mathbf{g}^{\gamma} \\ &= \left[\left(m_s^{\alpha\beta}{}_{|\beta}\right)_{,\alpha} + m_s^{\alpha\beta}{}_{|\beta}\Gamma^{\gamma}_{\alpha\gamma}\right]\mathbf{n} - b^{\beta}{}_{\gamma} m_s^{\gamma\alpha}{}_{|\alpha}\mathbf{g}_\beta \\ &= \left(m_s^{\alpha\beta}{}_{|\beta}\right)_{|\alpha}\mathbf{n} - b^{\beta}{}_{\gamma} m_s^{\gamma\alpha}{}_{|\alpha}\mathbf{g}_\beta = m_s^{\alpha\beta}{}_{|\alpha\beta}\mathbf{n} - b^{\beta}{}_{\gamma} m_s^{\gamma\alpha}{}_{|\alpha}\mathbf{g}_\beta\end{aligned} \tag{S9}$$

where we have used the symmetry of $\mathbf{m}_s$ and (S1b). The LHS of the surface equilibrium equation is

$$\left[\boldsymbol{\sigma}\right]_-^+\cdot\mathbf{n} = \left[\sigma\right]_-^{+33}\mathbf{n} + \left[\sigma\right]_-^{+\beta 3}\mathbf{g}_\beta , \tag{S10}$$

Combining (S7)-(S9) and separating the components in the tangent space from the normal direction, we obtain the equilibrium equation in component form:

$$\left(\sigma_s^{\alpha\beta} - b^{\beta}{}_{\gamma} m_s^{\alpha\gamma}\right) b_{\alpha\beta} + m_s^{\alpha\beta}{}_{|\alpha\beta} + \left[\sigma\right]_-^{+33} = 0 \tag{S11a}$$

$$\left(\sigma_s^{\alpha\beta} - b^{\beta}{}_{\gamma} m_s^{\alpha\gamma}\right)_{|\alpha} - b^{\beta}{}_{\gamma} m_s^{\gamma\alpha}{}_{|\alpha} + \left[\sigma\right]_-^{+\beta 3} = 0 \tag{S11b}$$

2. Derivation of Equation (42a,b)

In plane strain problem, we parameterize any curve by its arc length *s*; the arc length parameterization introduces a unit tangent vector, i.e., $\mathbf{g}_1 = \mathbf{s}$. $\mathbf{g}_2 = \mathbf{v}$ is a constant unit vector out-of-plane. Therefore, we have

$$\mathbf{g}_{1,1} = h\mathbf{n}, \ \mathbf{g}_{1,2} = 0, \ \mathbf{g}_{2,1} = 0, \ \mathbf{g}_{2,2} = 0 \tag{S12}$$

where h is the in-plane curvature. By Gauss Formula, it is evident that all Christoffel symbols are zero indicating that covariant differentiation $Q^{\alpha\beta}{}_{|\beta}$ is equal to $Q^{\alpha\beta}{}_{,\beta}$. Further, since all the quantities of $Q^{\alpha\beta}$ are irrelevant with the out-of-plane direction and $Q^{12} = Q^{21} = 0$, the only non-trivial term is $Q^{11}{}_{,1}$. Therefore, equation (38a,b) reduces to (40a,b).

3. Derivation of Equation (29a)

Consider the surface Helmholtz free energy without surface bending in equation (20), i.e.,

$$A\left(I_1^s, J_s\right) = a_1\left(I_1^s - 2\right) + a_2\left(J_s - 1\right) + \frac{a_3}{2}\left(J_s - 1\right)^2. \tag{S13}$$

The surface stress, by equation (19c), is then

$$\boldsymbol{\sigma}_s = \frac{2a_1}{J_s}\mathbf{B}_s + \left[a_2 + a_3\left(J_s - 1\right)\right]\mathbf{1}_s. \tag{S14}$$

Within the scope of small strain linear elasticity,

$$\left[\mathbf{B}_s\right] = \begin{bmatrix} 1 + 2\varepsilon_{11} & 2\varepsilon_{12} \\ 2\varepsilon_{12} & 1 + 2\varepsilon_{11} \end{bmatrix}, \ J_s = 1 + \varepsilon_{\gamma\gamma} \tag{S15}$$

where $\varepsilon_{\alpha\beta}$ is the components of the surface strain tensor. Substituting (S15) into (S14) yields

$$\begin{aligned} \left[\boldsymbol{\sigma}_s\right] &= \frac{2a_1}{1+\varepsilon_{\gamma\gamma}}\begin{bmatrix} 1 + 2\varepsilon_{11} & 2\varepsilon_{12} \\ 2\varepsilon_{12} & 1 + 2\varepsilon_{22} \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix} + a_3\begin{bmatrix} \varepsilon_{\gamma\gamma} & 0 \\ 0 & \varepsilon_{\gamma\gamma} \end{bmatrix} \\ &= 2a_1\begin{bmatrix} 1 + \varepsilon_{11} - \varepsilon_{22} & 2\varepsilon_{12} \\ 2\varepsilon_{12} & 1 - \varepsilon_{11} + \varepsilon_{22} \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix} + a_3\begin{bmatrix} \varepsilon_{\gamma\gamma} & 0 \\ 0 & \varepsilon_{\gamma\gamma} \end{bmatrix} \\ &= \left(2a_1 + a_2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2a_1\begin{bmatrix} \varepsilon_{11} - \varepsilon_{22} & 2\varepsilon_{12} \\ 2\varepsilon_{12} & -\varepsilon_{11} + \varepsilon_{22} \end{bmatrix} + a_3\begin{bmatrix} \varepsilon_{\gamma\gamma} & 0 \\ 0 & \varepsilon_{\gamma\gamma} \end{bmatrix} \\ &= \left(2a_1 + a_2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4a_1\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} + \left(a_3 - 2a_1\right)\begin{bmatrix} \varepsilon_{\gamma\gamma} & 0 \\ 0 & \varepsilon_{\gamma\gamma} \end{bmatrix} \end{aligned} \tag{S16a}$$

Write (S16a) in the indicial form

$$\sigma_{\alpha\beta} = 4a_1\varepsilon_{\alpha\beta} + \left(a_3 - 2a_1\right)\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + \left(2a_1 + a_2\right)\delta_{\alpha\beta}. \tag{S16b}$$

As illustrated in the manuscript that surface residual stress is $\sigma_0 = 2a_1 + a_2$, surface shear modulus is $G_s = 2a_1$ and surface bulk modulus is $K_s = a_3$, we recover the Shuttleworth equation,

$$\sigma_{\alpha\beta} = 2G_s\varepsilon_{\alpha\beta} + \left(K_s - G_s\right)\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + \sigma_0\delta_{\alpha\beta} \tag{S17}$$

4. Derivation of Equations (45) and (46a,b)

The bulk deformation gradient tensor is

$$\mathbf{F} = \frac{dr}{dR}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \tag{S18}$$

Incompressibility implies

$$\det\left(\mathbf{F}\right) = \frac{dr}{dR}\frac{r}{R} = 1. \tag{S19}$$

Solving equation (S19) with the boundary condition $r\big|_{R=R_A} = r_A$ gives

$$r\left(R\right) = \sqrt{R^2 + r_A^2 - R_A^2}. \tag{S20}$$

Equation (S20) implies that the inner and outer radii before and after deformation satisfies

$$r_B^2 - R_B^2 = r_A^2 - R_A^2. \tag{S21}$$

The Cauchy (true) stress tensor for neo-Hookean material is related to the deformation gradient by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{S22}$$

or in indicial form (with respect to the basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$)

$$\sigma_{rr} = -p + \mu\left(\frac{dr}{dR}\right)^2, \quad \sigma_{\theta\theta} = -p + \mu\left(\frac{r}{R}\right)^2, \tag{S23a,b}$$

where p is the Lagrange multiplier to enforce incompressibility, and $\mathbf{I}$ is the 3D identity tensor. Radial equilibrium in the bulk yields

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{S24}$$

Substituting (S23a, b) into (S24) gives

$$\frac{d\sigma_{rr}}{dr}+\mu\frac{R^2}{r^3}-\mu\frac{r}{R^2}=0\,. \tag{S25}$$

To solve (S25) we replace R by $\sqrt{r^2-r_A^2+R_A^2}$, and the solution is

$$\sigma_{rr}=\frac{\mu}{2r^2}\left[\ln\left(\frac{r^2-r_A^2+R_A^2}{r^2}\right)r^2-r_A^2+R_A^2\right]+C_1, \tag{S26}$$

where C_1 is an unknown constant. Using (S23a,b), σ_{rr} at the outer and inner radii of the deformed shell are:

$$\sigma_{rr}\big|_{r=r_B}=\frac{\mu}{2}\left[\ln\left(\frac{R_B^2}{r_B^2}\right)-1+\frac{R_B^2}{r_B^2}\right]+C_1, \tag{S27a}$$

$$\sigma_{rr}\big|_{r=r_A}=\frac{\mu}{2}\left[\ln\left(\frac{R_A^2}{r_A^2}\right)-1+\frac{R_A^2}{r_A^2}\right]+C_1\,. \tag{S27b}$$

$\mathbf{s}$, $\mathbf{n}$ are the unit tangent and normal vectors for the inner surface, and $\mathbf{s}=\mathbf{e}_\theta$, $\mathbf{n}=-\mathbf{e}_r$. The surface deformation gradient tensor on the inner surface is

$$\mathbf{F}_s=\lambda_A\mathbf{s}\otimes\mathbf{s}=\lambda_A\mathbf{e}_\theta\otimes\mathbf{e}_\theta\,, \tag{S28}$$

where λ_A is the surface stretch ratio of the inner surface, i.e., $\lambda_A\equiv r_A/R_A$. The in-plane curvature at $r=r_A$ is $h=1/r_A$,

$$I_3^s\big|_{r=r_A}=\lambda_A^2/r_A\,. \tag{S29}$$

Similarly, the surface stretch ratio at outer surface ($r=r_B$) is $\lambda_B=r_B/R_B$ and the trace of the relative curvature tensor $I_3^s\big|_{r=r_B}=\lambda_B^2/r_B$.

The in-plane surface stress and in-plane surface bending moment are given by the constitutive model; for the case of plane strain, they are

$$\begin{aligned}&\sigma_s^{11}\big|_{r=r_A}=\sigma_0+E_s^*\left(\lambda_A-1\right),\quad \sigma_s^{11}\big|_{r=r_B}=\sigma_0+E_s^*\left(\lambda_B-1\right)\\ &m_s^{11}\big|_{r=r_A}=-\lambda_A D_s\left(\frac{\lambda_A^2}{r_A}-\frac{1}{R_A}\right),\quad m_s^{11}\big|_{r=r_B}=-\lambda_B D_s\left(\frac{\lambda_B^2}{r_B}-\frac{1}{R_B}\right)\end{aligned}\,. \tag{S30a-d}$$

The equilibrium equations become

$$\frac{1}{r_B}\left[\sigma_0 + E_s^*\left(\lambda_B - 1\right) + \lambda_A D_s\left(\frac{\lambda_B^2}{r_B} - \frac{1}{R_B}\right)\frac{1}{r_B}\right] + \sigma_{rr}\big|_{r=r_B} = 0\,, \tag{S31a}$$

$$\frac{1}{r_A}\left[\sigma_0 + E_s^*\left(\lambda_A - 1\right) + \lambda_A D_s\left(\frac{\lambda_A^2}{r_A} - \frac{1}{R_A}\right)\frac{1}{r_A}\right] + \left(-T - \sigma_{rr}\big|_{r=r_A}\right) = 0\,, \tag{S31b}$$

where we have used the fact that the jump in stress across the outer and inner surface are $\sigma_{rr}\big|_{r=r_B}$ and $-T-\sigma_{rr}\big|_{r=r_A}$, respectively.

Using (S27a,b) and (S31a,b), the relationship between the applied pressure T and surface stretch ratios λ_A, λ_B is

$$T = \frac{\mu}{2}\left[2\ln\frac{\lambda_A}{\lambda_B} + \frac{1}{\lambda_B^2} - \frac{1}{\lambda_A^2}\right] + \frac{1}{\lambda_A R_A}\left[\sigma_0 + E_s^*\left(\lambda_A - 1\right) + D_s\frac{\lambda_A - 1}{R_A^2}\right] + \frac{1}{\lambda_B R_B}\left[\sigma_0 + E_s^*\left(\lambda_B - 1\right) + D_s\frac{\lambda_B - 1}{R_B^2}\right]. \tag{S32}$$

By normalization and using equation (S32)

$$\begin{aligned}\overline{T} = {} & \frac{1}{2}\left[2\ln\frac{\lambda_A}{\sqrt{1+\left(\lambda_A^2-1\right)f^2}} + \frac{1}{1+\left(\lambda_A^2-1\right)f^2} - \frac{1}{\lambda_A^2}\right] + \overline{\sigma}_0\left[\frac{1}{\lambda_A} + \frac{f}{\sqrt{1+\left(\lambda_A^2-1\right)f^2}}\right] \\ & + \overline{E}_s^*\left[\frac{\lambda_A - 1}{\lambda_A} + \frac{\left(\sqrt{1+\left(\lambda_A^2-1\right)f^2}-1\right)f}{\sqrt{1+\left(\lambda_A^2-1\right)f^2}}\right] + D_s^3\left[\frac{\lambda_A - 1}{\lambda_A} + \frac{\left(\sqrt{1+\left(\lambda_A^2-1\right)f^2}-1\right)f^3}{\sqrt{1+\left(\lambda_A^2-1\right)f^2}}\right].\end{aligned} \tag{S33}$$

For given normalized pressure $\overline{T}$, initial geometry f, and surface properties $\overline{\sigma}_0$, $\overline{E}_s^*$, $\overline{D}_s$, we solve for λ_A numerically. Once λ_A has been determined, the normalized constant $\overline{C}_1 = C_1/\mu$ can be obtained by equilibrium at the inner surface, i.e.,

$$\overline{C}_1 = \frac{\overline{\sigma}_0}{\lambda_A} + \frac{\left(\overline{E}_s^* + \overline{D}_s^3\right)\left(\lambda_A - 1\right)}{\lambda_A} - \frac{1}{2}\left(2\ln\frac{1}{\lambda_A} - 1 + \frac{1}{\lambda_A^2}\right) - \overline{T}\,. \tag{S34}$$

The normalized stress $\overline{\sigma}_{rr}$ and $\overline{\sigma}_{\theta\theta}$ are given by

$$\overline{\sigma}_{rr} = \frac{1}{2}\left[\ln\left(\frac{\overline{r}^2 - \lambda_A^2 + 1}{\overline{r}^2}\right) - \frac{\lambda_A^2}{\overline{r}^2} + \frac{1}{\overline{r}^2}\right] + \overline{C}_1\,, \tag{S35a}$$

$$\overline{\sigma}_{\theta\theta} = \frac{1}{2}\left[\ln\left(\frac{\overline{r}^2 - \lambda_A^2 + 1}{\overline{r}^2}\right) - \frac{\lambda_A^2}{\overline{r}^2} + \frac{1}{\overline{r}^2}\right] - \frac{\overline{r}^2 - \lambda_A^2 + 1}{\overline{r}^2} + \frac{\overline{r}^2}{\overline{r}^2 - \lambda_A^2 + 1} + \overline{C}_1\,. \tag{S35b}$$

4. Derivations of Equations (51a-c) and (52a,b)

The bulk deformation gradient tensor is[2]

$$\mathbf{F}=\sqrt{\frac{\eta}{2X_1}}\mathbf{e}_r\otimes\mathbf{E}_1+\sqrt{\frac{2X_1}{\eta}}\mathbf{e}_\theta\otimes\mathbf{E}_2 . \tag{S36}$$

By equations (S22) and (S24), it gives

$$\sigma_{rr}=\frac{\mu\eta^2}{2r^2}+\frac{\mu r^2}{2\eta^2}+C_2,\quad \sigma_{\theta\theta}=-\frac{\mu\eta^2}{2r^2}+\frac{3\mu r^2}{2\eta^2}+C_2, \tag{S37}$$

where C_2 is an unknow to be determined.

The surface deformation gradient is not easy to obtain intuitively compared to our first example. Therefore, we compute it following the standard recipe in Section 2. We first parametrize the inner surface in the reference configuration by $\mathbf{R}=(c_A, X_2)$ which is mapped into $\mathbf{r}=\left(\sqrt{2\eta c_A}\cos\frac{X_2}{\eta},\sqrt{2\eta c_A}\sin\frac{X_2}{\eta}\right)$ in the current configuration. The curvilinear coordinate is X_2. Hence,

$$\mathbf{G}_1=(0,1)=\mathbf{E}_2,\ \mathbf{G}^1=(0,1)=\mathbf{E}_2, \tag{S38a,b}$$

$$\mathbf{g}_1=\left(-\sqrt{\frac{2c_A}{\eta}}\sin\frac{X_2}{\eta},\sqrt{\frac{2c_A}{\eta}}\cos\frac{X_2}{\eta}\right)=\sqrt{\frac{2c_A}{\eta}}\mathbf{s},\ \mathbf{g}^1=\sqrt{\frac{\eta}{2c_A}}\mathbf{s} . \tag{S38c,d}$$

The surface deformation gradient tensor at the inner surface is

$$\mathbf{F}_s=\mathbf{g}_1\otimes\mathbf{G}^1=\sqrt{\frac{2c_A}{\eta}}\mathbf{s}\otimes\mathbf{E}_2 . \tag{S39}$$

Therefore, the surface stretch ratio $\lambda_A=\sqrt{\frac{2c_A}{\eta}}$ at $r=r_A$. The non-vanishing component second fundamental form b_{11} in the basis $\mathbf{g}^1\otimes\mathbf{g}^1$ is

$$b_{11}=\mathbf{g}_{1,1}\cdot\mathbf{n}=\left(-\frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}}\cos\frac{X_2}{\eta},-\frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}}\sin\frac{X_2}{\eta}\right)\cdot\left(-\cos\frac{X_2}{\eta},-\sin\frac{X_2}{\eta}\right)=\frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}} . \tag{S40}$$

Express it in the basis $\mathbf{s}\otimes\mathbf{s}$,

$$\mathbf{b} = \frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}}\mathbf{g}^1 \otimes \mathbf{g}^1 = \sqrt{\frac{1}{2\eta c_A}}\mathbf{s} \otimes \mathbf{s} = \frac{1}{r_A}\mathbf{s} \otimes \mathbf{s}\,. \tag{S41}$$

Next, we compute the relative curvature tensor,

$$\boldsymbol{\kappa} \equiv -\mathbf{F}_s^{\mathrm{T}} \cdot \mathbf{b} \cdot \mathbf{F}_s = \frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}}\mathbf{E}_2 \otimes \mathbf{E}_2\,. \tag{S42}$$

And $I_3^s = \frac{1}{\eta}\sqrt{\frac{2c_A}{\eta}}$. Similarly, at $r = r_B$, the stretch ratio of the surface is $\lambda_B = \sqrt{\frac{2c_B}{\eta}}$, and $I_3^s = \frac{1}{\eta}\sqrt{\frac{2c_B}{\eta}}$.

The equilibrium at the surfaces are

$$-\frac{\mu\eta}{4c_A} - \frac{\mu c_A}{\eta} - C_2 = -\frac{1}{\sqrt{2\eta c_A}}\left[\sigma_0 + E_s^*\left(\sqrt{\frac{2c_A}{\eta}} - 1\right) + \frac{1}{\eta^2}\sqrt{\frac{2c_A}{\eta}}D_s\right], \tag{S43a}$$

$$\frac{\mu\eta}{4c_B} + \frac{\mu c_B}{\eta} + C_2 = -\frac{1}{\sqrt{2\eta c_B}}\left[\sigma_0 + E_s^*\left(\sqrt{\frac{2c_B}{\eta}} - 1\right) + \frac{1}{\eta^2}\sqrt{\frac{2c_B}{\eta}}D_s\right]. \tag{S43b}$$

Therefore, the system of equations determines η and C_2 for given c_A and c_B. It must be solved numerically. Once η and C_2 are determined we compute the applied moment at the end of the plate; it is

$$M = \int_{r_A}^{r_B} r\tau_{\theta\theta}\,\mathrm{d}r + r_A\,\sigma_s^{11}\Big|_{r=r_A} - m_s^{11}\Big|_{r=r_A} + r_B\,\sigma_s^{11}\Big|_{r=r_B} - m_s^{11}\Big|_{r=r_B} = -\int_{r_A}^{r_B} r\tau_{rr}\,\mathrm{d}r\,, \tag{S44}$$

where we have used the boundary conditions and integrate the radical equilibrium by part. The applied moment is equal to

$$M = -\frac{\mu\eta^2}{2}\ln\frac{r_B}{r_A} - \frac{\mu\left(r_B^4 - r_A^4\right)}{8\eta^2} - \frac{C_2\left(r_B^2 - r_A^2\right)}{2}. \tag{S45}$$

Using normalization and combining equations (S43a,b) lead to

$$\frac{\bar{c}_B - \bar{c}_A}{\bar{\eta}} - \frac{\bar{\eta}}{4}\left(\frac{1}{\bar{c}_A} - \frac{1}{\bar{c}_B}\right) = \left(\frac{\bar{E}_s^*}{\sqrt{2\bar{\eta}}} - \frac{\bar{\sigma}_0}{\sqrt{2\bar{\eta}}}\right)\left(\frac{1}{\sqrt{\bar{c}_B}} + \frac{1}{\sqrt{\bar{c}_A}}\right) - \frac{2\bar{E}_s^*}{\bar{\eta}} - \frac{2\bar{D}_s^3}{\bar{\eta}^3}\,. \tag{S46}$$

Equation (S46) allows us to solve for $\bar{\eta}$ numerically. Then we find $\bar{C}_2$ by

$$\bar{C}_2 = -\frac{1}{\sqrt{2\bar{\eta}\bar{c}_B}}\left[\bar{\sigma}_0 + \bar{E}_s^*\left(\sqrt{\frac{2\bar{c}_B}{\bar{\eta}}} - 1\right) + \frac{\bar{D}_s^3}{\bar{\eta}^2}\sqrt{\frac{2\bar{c}_B}{\bar{\eta}}}\right] - \frac{\bar{\eta}}{4\bar{c}_B} - \frac{\bar{c}_B}{\bar{\eta}}\,. \tag{S47}$$

The normalized radical and circumferential stresses are

$$\bar{\sigma}_{rr} = \frac{\bar{\eta}^2}{2\bar{r}^2} + \frac{\bar{r}^2}{2\bar{\eta}^2} + \bar{C}_2, \quad \bar{\sigma}_{\theta\theta} = -\frac{\bar{\eta}^2}{2\bar{r}^2} + \frac{3\bar{r}^2}{2\bar{\eta}^2} + \bar{C}_2. \tag{S48a,b}$$

The normalized applied moment and normalized average in-plane curvature are

$$\bar{M} \equiv \frac{M}{\mu\left(c_B - c_A\right)^2} = -\frac{\bar{\eta}^2}{4}\ln\frac{\bar{c}_B}{\bar{c}_A} - \frac{\left(\bar{c}_B + \bar{c}_A\right)}{2} - \bar{\eta}\bar{C}_2, \tag{S49a}$$

$$\bar{h} = \frac{1}{\sqrt{\bar{\eta}\left(\bar{c}_B + \bar{c}_A\right)}}. \tag{S49b}$$

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