Supplementary Information for

A Thin-Film Model for Droplet Spreading on Soft Solid Substrates

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Scaled governing equations

Before dropping the higher order terms, the scaled governing equations in the liquid are

$$\epsilon^2 Re\Big[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_z\frac{\partial v_x}{\partial z}\Big] = -\frac{\partial p_l}{\partial x} + \Big[\epsilon^2\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\Big], \tag{1}$$

$$\epsilon^4 Re\Big[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_z\frac{\partial v_z}{\partial z}\Big] = -\frac{\partial p_l}{\partial z} + \epsilon^2\Big[\epsilon^2\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\Big], \tag{2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0. \tag{3}$$

where $Re = \rho_l U R_0/\eta_l$ is the liquid Reynolds number.

In the solid, we have

$$\epsilon^2 Re_m\frac{\partial^2 u_x}{\partial^2 t} = -\frac{\partial p_s}{\partial x} + G\Big[\frac{\partial^2 u_x}{\partial z^2} + \epsilon^2\frac{\partial^2 u_x}{\partial x^2}\Big] + m\frac{\partial}{\partial t}\ \Big[\frac{\partial^2 u_x}{\partial z^2} + \epsilon^2\frac{\partial^2 u_x}{\partial x^2}\Big] = 0, \tag{4}$$

$$\epsilon^4 Re_m\frac{\partial^2 u_z}{\partial^2 t} = -\frac{\partial p_s}{\partial z} + \epsilon^2 G\Big[\frac{\partial^2 u_z}{\partial z^2} + \epsilon^2\frac{\partial^2 u_z}{\partial x^2}\Big] + \epsilon^2 m\frac{\partial}{\partial t}\ \Big[\frac{\partial^2 u_z}{\partial z^2} + \epsilon^2\frac{\partial^2 u_z}{\partial x^2}\Big] = 0, \tag{5}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \tag{6}$$

where $Re_m = (\rho_s/\rho_l)Re$ is a modified Reynolds number, $G = ER_0/\sigma\epsilon^3$ is a dimensionless parameter quantifying the ratio of elastic forces to liquid-solid interfacial-tension forces and $m = \eta_s/\eta_l$ is the viscosity ratio between the solid and liquid phases.

At $z = -H$ the boundary conditions are

$$u_x = u_z = 0, \tag{7}$$

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$$v_x = v_z = 0. \tag{8}$$

At the liquid-air interface $z = \zeta(x,t)$, the kinematic, normal stress, and tangential stress boundary conditions are respectively given by

$$\frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} = v_z, \tag{9}$$

$$\frac{1}{Z}\Big[-\frac{\partial \zeta}{\partial x}\Big(-p_l + 2\epsilon^2 \frac{\partial v_x}{\partial x}\Big) + \Big(\frac{\partial v_x}{\partial z} + \epsilon^2 \frac{\partial v_z}{\partial x}\Big)\Big] = -\frac{C_l^{-1}}{Z^4}\frac{\partial^2 \zeta}{\partial x^2}\frac{\partial \zeta}{\partial x} - \frac{1}{Z}\Pi \frac{\partial \zeta}{\partial x}, \tag{10}$$

$$\frac{1}{Z}\Big[-\frac{\partial \zeta}{\partial x}\Big(\epsilon^2 \frac{\partial v_x}{\partial z} + \epsilon^4 \frac{\partial v_z}{\partial x}\Big) + \Big(-p_l + 2\epsilon^2 \frac{\partial v_z}{\partial x}\Big) + \Big] = \frac{C_l^{-1}}{Z^3}\frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{Z}\Pi \tag{11}$$

where $Z = \sqrt{(1 + \epsilon\, \partial\zeta/\partial x)^2}$ and $C_l^{-1} = \epsilon^3 \sigma / \eta_l U$ is a rescaled capillary number based on the liquid-air interfacial tension. Note that due to our choice of characteristic velocity, the magnitude of C_l^{-1} is equal to unity.

At the liquid-solid interface $z = \xi(x,t)$, the continuity-of-velocity, normal stress, and tangential stress boundary conditions are respectively given by

$$\frac{\partial u_x}{\partial t}|_{z=0} = v_x|_{z=\xi}, \tag{12}$$

$$\frac{\partial u_z}{\partial t}|_{z=0} = v_z|_{z=\xi}, \tag{13}$$

$$\begin{aligned} &\frac{1}{\Xi}\Big[-\frac{\partial \xi}{\partial x}\Big(-p_l + 2\epsilon^2 \frac{\partial v_x}{\partial x}\Big) + \Big(\frac{\partial v_x}{\partial z} + \epsilon^2 \frac{\partial v_z}{\partial x}\Big)\Big] - \frac{1}{\Xi}\Big[-\frac{\partial \xi}{\partial x}\Big(-p_s + 2G\epsilon^2 \frac{\partial u_x}{\partial x} \\ &+ 2m\epsilon^2 \frac{\partial}{\partial t}\frac{\partial u_x}{\partial x}\Big) + G\Big(\frac{\partial u_x}{\partial z} + \epsilon^2 \frac{\partial u_z}{\partial x}\Big) + m\frac{\partial}{\partial t}\Big(\frac{\partial u_x}{\partial z} + \epsilon^2 \frac{\partial u_z}{\partial x}\Big)\Big] = \frac{C_s^{-1}}{\Xi^4}\frac{\partial^2 \xi}{\partial x^2}\frac{\partial \xi}{\partial x} \end{aligned} \tag{14}$$

$$\begin{aligned} &\frac{1}{\Xi}\Big[-\frac{\partial \xi}{\partial x}\Big(\epsilon^2 \frac{\partial v_x}{\partial z} + \epsilon^4 \frac{\partial v_z}{\partial x}\Big) + \Big(-p_l + 2\epsilon^2 \frac{\partial v_x}{\partial x}\Big) + \Big] - \frac{1}{\Xi}\Big[-\frac{\partial \xi}{\partial x}\Big\{G\Big(\epsilon^2 \frac{\partial u_x}{\partial z} + \epsilon^4 \frac{\partial u_z}{\partial x}\Big) \\ &+ m\epsilon^2 \frac{\partial}{\partial t}\Big(\frac{\partial u_x}{\partial z} + \epsilon^2 \frac{\partial u_z}{\partial x}\Big)\Big\} + \Big(-p_s + 2G\epsilon^2 \frac{\partial u_x}{\partial z} + 2m\epsilon \frac{\partial}{\partial t}\frac{\partial u_x}{\partial z}\Big)\Big] = -\frac{C_s^{-1}}{\Xi^3}\frac{\partial^2 \xi}{\partial x^2} \end{aligned} \tag{15}$$

where $\Xi = \sqrt{(1 + \epsilon\, \partial\xi/\partial x)^2}$ and $C_s^{-1} = \epsilon^3 \gamma / \eta_l U$ is the rescaled capillary number based on the liquid-solid interfacial tension. Substituting the characteristic velocity $U = \sigma\epsilon^3/\eta_l$ leads to $C_s^{-1} = \gamma/\sigma$, which is a ratio of the liquid-solid and liquid-air interfacial tensions.

To obtain the leading-order equations, we assume $\epsilon \to 0$ in equations (1)-(15). The leading-order equations in the liquid are

$$-\frac{\partial p_l}{\partial x} + \frac{\partial^2 v_x}{\partial z^2} = 0, \tag{16}$$

$$\frac{\partial p_l}{\partial z} = 0, \tag{17}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0. \tag{18}$$

In the solid, we have

$$-\frac{\partial p_s}{\partial x} + G\frac{\partial^2 u_x}{\partial z^2} + m\frac{\partial}{\partial t}\frac{\partial^2 u_x}{\partial z^2} = 0, \tag{19}$$

$$\frac{\partial p_s}{\partial z} = 0, \tag{20}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0. \tag{21}$$

At $z = -H$ the boundary conditions are

$$u_x = u_z = 0, \tag{22}$$

$$v_x = v_z = 0. \tag{23}$$

At the liquid-air interface $z = \zeta(x,t)$, the kinematic, normal stress, and tangential stress boundary conditions are respectively given by

$$\frac{\partial \zeta}{\partial t} + v_x\frac{\partial \zeta}{\partial x} = v_z, \tag{24}$$

$$p_l = p_{cap,l} - \Pi, \tag{25}$$

$$\frac{\partial v_x}{\partial z} = 0, \tag{26}$$

where $p_{cap,l} = -C_l^{-1}(\partial^2\zeta/\partial x^2)$ is the capillary pressure in the liquid.

At the liquid-solid interface $z = \xi(x,t)$, the continuity-of-velocity, normal stress, and tangential stress boundary conditions are respectively given by

$$\frac{\partial u_x}{\partial t}|_{z=0} = v_x|_{z=\xi}, \tag{27}$$

$$\frac{\partial u_z}{\partial t}|_{z=0} = v_z|_{z=\xi}, \tag{28}$$

$$p_s = p_{cap,l} + p_{cap,s}, \tag{29}$$

$$\frac{\partial v_x}{\partial z} - G\frac{\partial u_x}{\partial z} - m\frac{\partial}{\partial t}\frac{\partial u_x}{\partial z} = 0, \tag{30}$$

where $p_{cap,s} = -C_s^{-1}(\partial^2\xi/\partial x^2)$ is a capillary-like pressure in the solid.