Electronic Supplementary Information for "Adhesion of a Tape Loop"

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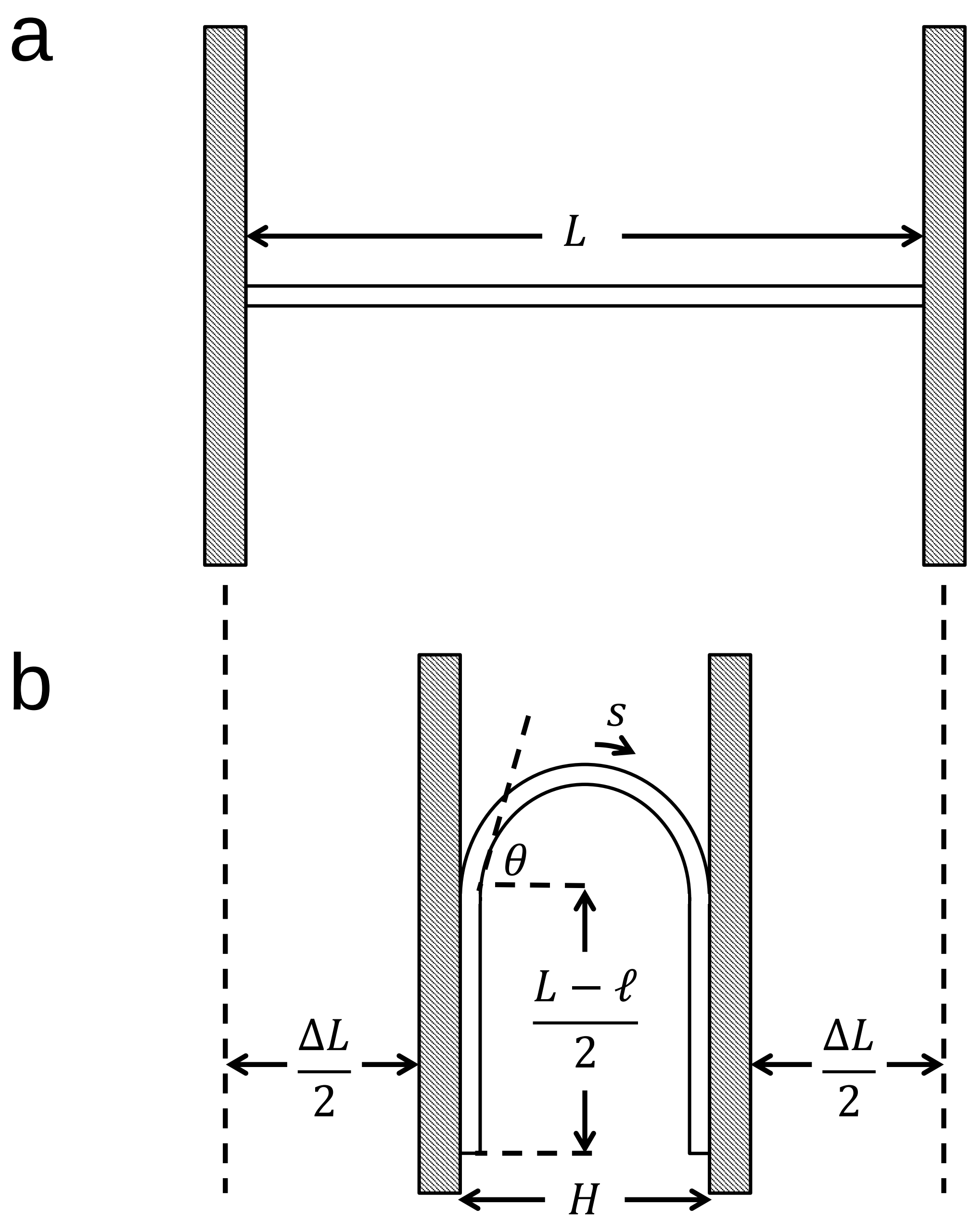

FIG. 1. Geometry of a tape loop. a. before compression of the walls, the film of length L lies flat. b. shows the film after some compression of the walls.

I. SURFACE ENERGIES

The net change in surface energy upon compression of a thin film between two walls can be determined by considering the total surface energy before and after a deformation.

Figure 1 shows the film before (a) and after (b) deformation. Initially the total of the interfacial energy is given by:

$$U_0 = 2L\gamma_{sa} + 2L'\gamma_{wa}, \tag{1}$$

where the sheet length is L, the wall length is L', and the subscripts s, w, a refer to sheet, wall, and air respectively. After compression some amount of sheet/wall contact is created, which comes at a cost γ_{sw}. If a length ℓ of sheet remains free from wall contact after compression, the total energy is given as:

$$U_1 = (L + \ell)\gamma_{sa} + 2(L' - (L - \ell)/2)\gamma_{wa} + (L - \ell)\gamma_{sw}. \tag{2}$$

The difference in energy is

$$\begin{aligned} U_1 - U_0 =& (\ell - L)\gamma_{sa} + (\ell - L)\gamma_{wa} - (\ell - L)\gamma_{sw} \\ U_1 - U_0 =& (\ell - L)(\gamma_{sa} + \gamma_{wa} - \gamma_{sw}) \\ =& (\ell - L)\Delta\gamma \end{aligned} \tag{3}$$

To ensure interfacial energy is reduced upon compression, $\Delta\gamma$ must be positive.

II. T-PEEL TEST

The sticky elastica variation shown in figure 1b is a variant of a geometry known in the literature as the T-Peel test.[1, 2] In a T-Peel test, the geometry shown in Figure 2, a load per unit width of film, P, is applied to the free end of two sheets which have been adhered to one another ($P = F/b$, F the applied force and b the out of page dimension in Fig. 2). As the sheets are pulled apart, they reach a steady state geometry and meet at an angle of 90° (assuming identical sheets). Analysis of the test is simplified by noting that the geometry reduces to two simultaneous 90° peel tests. In a peel test the energy release rate, G_c, is given by:

$$G_c = P(1 - \cos\theta), \tag{4}$$

where b is the film's width, θ is the angle between the film and the substrate and the film has been assumed to be inextensible.[3, 4] The relation arises through the geometry relating the work done by moving the load a distance (say dx) to the distance moved by the crack at the interface (say dc). In 90° peel $dx = dc$ if the film is inextensible.

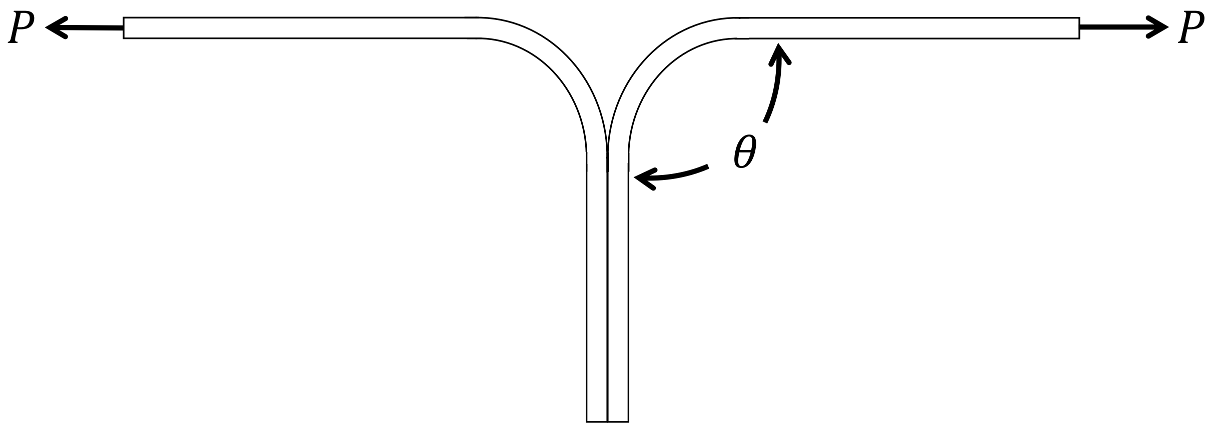

FIG. 2. Geometry of a T-Peel test.

What changes in the T-peel geometry is that the load at two positions must move a distance dx in order to open the same distance of crack, dc. The energy release rate in a T-Peel test is therefore only slightly different:

$$G_c = 2P. \tag{5}$$

III. STICKY ELASTICA

In this section, we adapt the "Sticky Elastica" model of Wagner and Vella [5] to the geometry shown in Figure 1 (a), and derive the governing equation and boundary condition applicable to our system.

We consider a configuration, as shown in Figure 1 (b), where a thin inextensible sheet of total length L and width b is compressed by moving the confining walls closer by a total distance ΔL. The angle made by the tangent of the sheet at any given arc-length coordinate s with the horizontal is denoted by $\theta(s)$. The two end points of the free part of the sheet where it contacts the confining plates are given by $s = \pm l/2$, and the entire configuration is assumed to be symmetric under reflection in a plane parallel to the confining walls and passing through the mid point of the sheet at $s = 0$. Exploiting this symmetry of the system, we consider the following functional comprising the total energy of one symmetric half of the free part of the sheet, and the inextensibility constraint enforced by a constant Lagrange multiplier α,

$$U = \int_0^{l/2} ds \left[\frac{1}{2}B\theta'^2 + \Delta\gamma\right] - \alpha\left(\frac{L-\Delta L}{2} - \int_0^{l/2} ds \cos\theta\right), \tag{6}$$

Here B is the bending rigidity of the sheet, and the prime represent the total derivative w.r.t. the arc-length parameter s, i.e. $()' \equiv \frac{d}{ds}$. The first term in the first integral above represents the bending energy density of the sheet, and the second term represents the adhesive penalty per unit length due to peeling. The constant α is a Lagrange multiplier enforcing the inextensibility constraint on the sheet. The boundary conditions are $\theta(0) = 0$ and $\theta(l/2) = -\pi/2$. We also note, that for clarity in the derivation we opt to absorb the constant term $-L\Delta\gamma$ into the energy here. We leave this term in the main manuscript to explicitly show how the total energy does scale with the size of the sheet. However, the shape of the film is determined locally through the boundary conditions at the point of contact regardless of the overall sheet size.

Before proceeding further, we will rewrite the total energy U in a slightly different way so as to subsume the inextensibility constraint and the energy density under a single integral,

$$U = \int_0^{l/2} ds \left[\frac{1}{2}B\theta'^2 + \Delta\gamma - \alpha\left(x'(s) - \cos\theta\right)\right]. \tag{7}$$

Here $x(s)$ is the horizontal coordinate of point s of the sheet. Note that the constraint written above is exactly equivalent to what is present in equation (6), as can be verified by noting that $\int_0^{l/2} ds\, x'(s) = x(l/2) - x(0) = (L - \Delta L)/2$.

To keep the upcoming computations clean, we will now denote the integrand in (7) by $\mathcal{L}(s;\theta,\theta')$ and write the integral as,

$$U = \int_0^{s_0} ds\, \mathcal{L}(s;\theta,\theta'), \tag{8}$$

where the *dependent* variable θ (and θ') appearing in the argument of $\mathcal{L}$ are considered to be functions of the *independent* variable s appearing before the semicolon. Also, s_0 is equal to $l/2$. Following [6], we will compute the first order variation of the functional (8) under shifts in the *dependent* as well as the *independent* variables. As we will see towards the end, the former will give us the governing equation for the elastica, and the later will deliver the non-trivial boundary condition of adhesion.

We subject functional (8) to a set of transformations in the *independent* and the *dependent* variable as $s \to \bar{s}$ and $\theta \to \bar{\theta}$. The transformed functional is then written as,

$$\bar{U} = \int_0^{\bar{s}_0} d\bar{s}\, \mathcal{L}(\bar{s}; \bar{\theta}, \bar{\theta}')\,, \tag{9}$$

where the transformed variable $\bar{\theta}$ is now to be considered a function of $\bar{s}$ appearing before the semicolon. Note that this transformation has also shifted the domain of integration to $\bar{s}_0$. We assume that the transformations are related by infinitesimal shifts of the form,

$$\bar{s} = s + \delta s(s)\,, \qquad \bar{\theta}(\bar{s}) = \theta(s) + \delta\theta(s)\,, \tag{10}$$

The inifinitesimal shifts above are written as functions of the untransformed variable s. These could very well be considered to be the functions of the transformed variable $\bar{s}$ since it can be shown by a short computation that up to first order $\delta\theta(\bar{s}) = \delta\theta(s)$. The δ operator here measures the total change in θ due to changes in the independent variable s, as well as changes in θ at a fixed material point. Since the two sides of $(10)_2$ are defined at different points in the space of the independent variable, the operator δ does not commute with the total derivative $()'$. Therefore, we define an operator $\tilde{\delta}$ which only measures the changes in the dependent variable θ at a fixed material label s.

$$\bar{\theta}(s) = \theta(s) + \tilde{\delta}\theta(s)\,. \tag{11}$$

Using relations (10) and (11) the two variational operators can be related as,

$$\delta\theta = \tilde{\delta}\theta + \theta'\delta s\,. \tag{12}$$

With all these definitions at hand, we write the difference between (9) and (8) as follows,

$$\bar{U} - U = \Delta U = \int_0^{\bar{s}_0} d\bar{s}\, \mathcal{L}(\bar{s}; \bar{\theta}, \bar{\theta}') - \int_0^{s_0} ds\, \mathcal{L}(s; \theta, \theta') \tag{13}$$

In order to combine the two integrals above, we need to ensure that the domains of integration of the two integrals coincide. We achieve that by transforming the shifted volume form in the first integral back to original domain and obtain the following,

$$\Delta U = \int_0^{s_0} ds\, \left[\frac{d\bar{s}}{ds}\mathcal{L}(\bar{s}; \bar{\theta}, \bar{\theta}') - \mathcal{L}(s; \theta, \theta')\right]\,, \tag{14}$$

From $(10)_1$, we note that $\frac{d\bar{s}}{ds} = 1 + \frac{d\delta s}{ds}$, and the first Lagrangian function above is Taylor expanded upto first order terms as $\mathcal{L}(s+\delta s; \theta+\delta\theta, \theta'+\delta\theta') = \mathcal{L}(s; \theta, \theta') + \frac{\partial \mathcal{L}}{\partial s}\delta s + \frac{\partial \mathcal{L}}{\partial \theta}\delta\theta + \frac{\partial \mathcal{L}}{\partial \theta'}\delta\theta'$.

Substituting these two expressions in (14), and retaining only first order terms we obtain,

$$\delta U = \int_0^{s_0} ds\, \left[\mathcal{L}\frac{d\delta s}{ds} + \frac{\partial \mathcal{L}}{\partial s}\delta s + \frac{\partial \mathcal{L}}{\partial \theta}\delta\theta + \frac{\partial \mathcal{L}}{\partial \theta'}\delta\theta' \right] . \tag{15}$$

We now replace the operator δ in the integrand above with $\tilde{\delta}$ using (12), and after some manipulations obtain the following,

$$\delta U = \int_0^{s_0} ds\, \frac{d}{ds}\left(\mathcal{L}\delta s\right) + \int_0^{s_0} ds\, \left[\frac{\partial \mathcal{L}}{\partial \theta}\tilde{\delta}\theta + \frac{\partial \mathcal{L}}{\partial \theta'}\tilde{\delta}\theta' \right] . \tag{16}$$

Note that the additional effect of varying the independent variable s is now clearly visible in the first term above. If we were only varying the dependent variable θ, as is done in standard fixed domain calculus of variation problems, then the first integral in (16) would not appear, and the two variational operators $\tilde{\delta}$ and δ would coincide. On integrating by parts the second term in the second integral on the right side of (16) we obtain the following,

$$\delta U = \int_0^{s_0} ds\, \frac{d}{ds}\left[\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \theta'}\theta'\right)\delta s + \frac{\partial \mathcal{L}}{\partial \theta'}\delta\theta \right] + \int_0^{s_0} ds\, \left[\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)\right] \left(\delta\theta - \theta'\delta s\right) , \tag{17}$$

$$= \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \theta'}\theta'\right)\delta s\Big|_0^{s_0} + \frac{\partial \mathcal{L}}{\partial \theta'}\delta\theta\Big|_0^{s_0} + \int_0^{s_0} ds\, \left[\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)\right] \left(\delta\theta - \theta'\delta s\right) . \tag{18}$$

Now recognising the fact that $\delta s(0) = 0$, and $\delta\theta(0) = \delta\theta(s_0) = 0$, we arrive at the following,

$$\delta U = \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \theta'}\theta'\right)\delta s_0 + \int_0^{s_0} ds\, \left[\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)\right] \left(\delta\theta - \theta'\delta s\right) . \tag{19}$$

For U to be extremal, we must have $\delta U = 0$, which requires the following relations to hold for arbitrary variations in s and θ,

$$\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \theta'}\theta' = 0\,, \qquad \text{at } s = s_0\,, \tag{20}$$

$$\left[\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)\right] = 0\,, \qquad \text{for } 0 < s < s_0\,. \tag{21}$$

On substituting the expression for $\mathcal{L}$ from (7), we immediately see that (20) delivers the adhesion boundary condition [7], and (21) gives us the governing equation of the Euler elastica.

$$\tfrac{1}{2}B(\theta')^2 = \Delta\gamma\,, \tag{22}$$

$$\theta'' + (\alpha/B)\sin\theta = 0\,. \tag{23}$$

IV. SOLUTIONS OF EQN. 23

Integrating Eqn. 23 once gives an expression for the curvature:

$$\theta' = \sqrt{2(P/B)\cos(\theta) + c_1} \tag{24}$$

with c_1 an integration constant. The boundary condition of Eqn. 22 allows c_1 to be determined, and the differential equation can then be solved:

$$\theta(s) = 2am\left(\sqrt{\frac{P+\Delta\gamma}{2B}}s, \sqrt{\frac{2}{1+\Delta\gamma/P}}\right), \tag{25}$$

where am is the Jacobi amplitude function. The point of contact, ℓ, can also be analytically determined:

$$\frac{\ell}{2} = F\left(-\frac{\pi}{4}, \sqrt{\frac{2}{1+\Delta\gamma/P}}\right)\sqrt{\frac{2B}{P+\Delta\gamma}}, \tag{26}$$

where F is the eliptic integral of the first kind.

V. ENERGY RELEASE RATE

The inextensible film has a mechanical energy given by:

$$U_M = 2b\int_0^{\ell/2} \frac{B}{2}(\theta')^2 \partial s, \tag{27}$$

as the fixed grips allow no contribution from work during a virtual crack displacement. Using Eqn. 24 the energy can be rewritten as:

$$U_M = 2b\int_0^{\ell/2} (P\cos(\theta) + \Delta\gamma)\partial s. \tag{28}$$

The first term of the integral results in a constant due to the inextensibility constraint, and the second integral is trivial. Hence, the total mechanical energy can be written:

$$U_M = bP(L - \Delta L) + b\ell\Delta\gamma. \tag{29}$$

The energy relase rate is given by the areal derivative of the mechanical energy. In this case the element of area is given by $\partial A = -b\partial\ell$, where the negative sign arises because ∂A is positive as the crack closes. Hence,

$$G = -\frac{\partial U_M}{b\partial\ell} = -\Delta\gamma. \tag{30}$$

As G is constant, the areal derivative of G is equal to zero.

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