Electronic Supplementary Information to **Predicting the Characteristics of Defect Transitions on Curved Surfaces** submitted to Soft Matter

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1 Complete covariant formalism

We follow the covariant formalism developed by Giomi and Bowick [1], Bowick and Giomi [2] in the following. Let $\mathbb{P}$ be a smooth two-dimensional surface of a crystal lattice in $\mathbb{R}^3$. The elastic free energy of the crystal may be expressed in the form:

$$F = F_{\mathrm{el}} + F_c + F_0, \tag{1}$$

where F_0 is the free energy of the defect-free monolayer, F_c is the core energy of defects and F_{el} is the elastic energy associated with defect interaction. As we will change neither the shape of the surface nor the number of defects, any minimization is governed by F_{el}, which can be written as

$$F_{\mathrm{el}} = \frac{Y}{2} \int \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, G_{2L}(\mathbf{x}, \mathbf{y}) q_T(\mathbf{x}) q_T(\mathbf{y}), \tag{2}$$

where Y is the Young's modulus for the planar crystal and $G_{2L}(\mathbf{x}, \mathbf{y})$ is the Green's function for the covariant biharmonic operator on the manifold. The quantity $q_T(\mathbf{x})$ represents the effective topological charge density; in the presence of discrete $q = +1$ disclinations at $\mathbf{x}_\alpha$ it takes the form

$$q_T(\mathbf{x}) = \sum_\alpha \frac{\pi}{3} \delta(\mathbf{x}, \mathbf{x}_\alpha) - K_G(\mathbf{x}), \tag{3}$$

where $\delta(\mathbf{x}, \mathbf{x}_\alpha) = g^{-1/2} \delta(x_1 - x_{\alpha_1}) \delta(x_2 - x_{\alpha_2})$ is the Dirac delta function on the manifold parametrized by $\mathbf{x} = (x_1, x_2) (= (r, \phi)$ in polar coordinates). The second term $K_G(\mathbf{x})$ is the Gaussian curvature of the surface. On a topological disk with total charge Q=+6, the minimal number of disclinations is 6. Any disclination located at the boundary will not contribute to F_{el}, because G_{2L} vanishes there. In this work, we compare energies of configurations with all 6 disclinations at the boundary to those with one disclination located at an arbitrary $\mathbf{x}_D$. Therefore, we consider the simpler $q_T(\mathbf{x}, \mathbf{x}_D) = \frac{\pi}{3} \delta(\mathbf{x}, \mathbf{x}_D) - K_G(\mathbf{x})$. We begin by observing that the Airy stress function χ solves the following inhomogeneous biharmonic equation:

$$\Delta^2 \chi(\mathbf{x}, \mathbf{x}_D) = Y q_T(\mathbf{x}, \mathbf{x}_D), \tag{4}$$

with no stress boundary conditions

$$\chi(\mathbf{x},\mathbf{x}_D)=0,\quad \mathbf{x}\in\partial\mathbb{P};\quad \nu_i\nabla^i\chi(\mathbf{x},\mathbf{x}_D)=0,\quad \mathbf{x}\in\partial\mathbb{P}. \tag{5}$$

The solution of (4) will then be

$$\chi(\mathbf{x},\mathbf{x}_D)=\int d^2yG_L(\mathbf{x},\mathbf{y})\Gamma(\mathbf{y},\mathbf{y}_D), \tag{6}$$

where $G_L(\mathbf{x},\mathbf{y})$ is the Green's function of the covariant Laplace operator on $\mathbb{P}$ with Dirichlet boundary conditions

$$\Delta G_L(\mathbf{x},.)=\delta(\mathbf{x},.),\quad \mathbf{x}\in\mathbb{P};\quad G_L(\mathbf{x},.)=0,\quad \mathbf{x}\in\partial\mathbb{P}, \tag{7}$$

and $\Gamma(\mathbf{x},\mathbf{x}_D)=\Delta\chi(\mathbf{x},\mathbf{x}_D)$ is the solution of the Poisson problem:

$$\Delta\Gamma(\mathbf{x},\mathbf{x}_D)=Yq_T(\mathbf{x},\mathbf{x}_D), \tag{8}$$

which can be expressed formally as:

$$\Gamma(\mathbf{x},\mathbf{x}_D)=Y\int q_T(\mathbf{y},\mathbf{y}_D)G_L(\mathbf{x},\mathbf{y})d^2y=-\Gamma_D(\mathbf{x},\mathbf{x}_D)-\Gamma_S(\mathbf{x})+U(\mathbf{x},\mathbf{x}_D), \tag{9}$$

where

$$\Gamma_D(\mathbf{x},\mathbf{x}_D)=-\frac{Y\pi}{3}G_L(\mathbf{x},\mathbf{x}_D),\quad \Gamma_S(\mathbf{x})=Y\int K_G(\mathbf{y})G_L(\mathbf{x},\mathbf{y})d^2y, \tag{10}$$

and $U(\mathbf{x},\mathbf{x}_D)$ is a harmonic function on $\mathbb{P}$ that enforces the Neumann boundary conditions. The first term of (9) represents the bare contribution of disclinations while the second term captures the screening effect of Gaussian curvature. In this paper we restrict ourselves to allowing only one disclination to migrate from the boundary to the apex of the manifold. The Green's function satisfying (7) can be computed explicitly by conformally mapping the surface $\mathbb{P}$ onto the unit disk of the complex plane where the Green's function is known:

$$G_L(\mathbf{x},\mathbf{y})=\frac{1}{2\pi}\log\left|\frac{z(\mathbf{x})-z(\mathbf{y})}{1-z(\mathbf{x})\overline{z(\mathbf{y})}}\right|, \tag{11}$$

where $z(\mathbf{x})=\varrho e^{i\phi}$, a point in the unit disk, is the image of a point on the surface $\mathbb{P}$ under the conformal mapping. The Green's function vanishes when the disclination is located at the boundary. For a surface $X(r,\phi)$ with first fundamental form $E=\partial X/\partial r\cdot\partial X/\partial r$, $F=\partial X/\partial r\cdot\partial X/\partial\phi$ and $G=\partial X/\partial\phi\cdot\partial X/\partial\phi$, the metric of the surface will be

$$ds^2=Edr^2+2Fdrd\phi+Gd\phi^2 \tag{12}$$

whereas the unit disk has the metric

$$ds^2=w(z)\left(\varrho^2dr^2+\varrho^2d\phi^2\right) \tag{13}$$

where $w(z)$ is a positive conformal weight. The remaining task is now to find the conformal factor $w(z)$ and the conformal radius $\varrho(r)$ by equating the two metrics; these can be explicitly obtained for many rotationally symmetric surfaces but in general, may not be analytically computable.

Taking the two image points on the unit disk as $z(r, \phi) = \varrho_x(r)e^{i\phi}$ and $\zeta(r', \phi') = \varrho_y(r')e^{i\phi'}$, the contribution due to the background Gaussian curvature is split into two parts $\Gamma_S(\mathbf{x}) = \Gamma_{S,1}(\mathbf{x}) - \Gamma_{S,2}(\mathbf{x})$, where

$$\Gamma_{S,1}(r, \phi) = \frac{Y}{2\pi} \int d\phi'\, dr'\, \sqrt{g} K(r') \log|z - \zeta|, \tag{14a}$$

$$\Gamma_{S,2}(r, \phi) = \frac{Y}{2\pi} \int d\phi'\, dr'\, \sqrt{g} K(r') \log|1 - z\overline{\zeta}|, \tag{14b}$$

are evaluated analytically for the specific surfaces considered in this paper. The computation of the harmonic function on the manifold $\mathbb{P}$ is more involved and its contribution to the energy density is given by,

$$U(\mathbf{x}, \mathbf{x}_D) = -Y \int d^2y H(\mathbf{x}, \mathbf{y}) q_T(\mathbf{y}, \mathbf{y}_D), \tag{15}$$

where $H(\mathbf{x}, \mathbf{y})$ is harmonic kernel associated with the Green's function of the weighted biharmonic operator arising from the conformal mapping of the manifold $\mathbb{P}$ onto the unit disk in the complex plane. The harmonic kernel can be written in integral form as [3]

$$H(z, \zeta) = -\int_{|\zeta|}^{1} \frac{dt}{\pi t} \int_0^t ds\, \sqrt{g} k \left[\frac{\varrho^2(s)}{t^2} \zeta\overline{z}\right], \tag{16a}$$

$$k(z\overline{\zeta}) = \sum_{n \geq 0} \frac{(z\overline{\zeta})^n}{c_n(1)} + \sum_{n<0} \frac{(\overline{z}\zeta)^{|n|}}{c_{|n|}(1)}, \tag{16b}$$

$$c_n(t) = 2\int_0^t ds\, \sqrt{g} \varrho^{2n}(s). \tag{16c}$$

After making appropriate substitutions, one obtains

$$\begin{aligned}
H(z, \zeta) &= -\int_{|\zeta|}^{1} \frac{dt}{\pi t} \int_0^t ds\, \sqrt{g} \left[\frac{1}{c_0(1)} + \sum_{n \geq 1} \frac{1}{c_n(1)} \frac{\varrho^{2n}(s)}{t^{2n}} \varrho_x^n \varrho_y^n \left(e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}\right)\right] \\
&= -\int_{|\zeta|}^{1} \frac{dt}{\pi t} \int_0^t ds\, \sqrt{g} \left[\frac{1}{c_0(1)} + \sum_{n \geq 1} \frac{2}{c_n(1)} \frac{\varrho^{2n}(s)}{t^{2n}} \varrho_x^n \varrho_y^n \cos n(\phi - \phi')\right] \\
&= -\int_{\varrho_y}^{1} \frac{dt}{\pi t} \left[\frac{1}{2} \frac{c_0(t)}{c_0(1)} + \sum_{n \geq 1} \frac{c_n(t)}{c_n(1)} \frac{2}{t^{2n}} \varrho_x^n \varrho_y^n \cos n(\phi - \phi')\right] \\
&= -\frac{1}{2\pi} f_0(\varrho_y) - \sum_{n \geq 1} \frac{1}{\pi} \varrho_x^n \varrho_y^n \cos n(\phi - \phi') f_n(\varrho_y),
\end{aligned} \tag{17}$$

where

$$f_n(\varrho_y) = \int_{\varrho_y}^{1} \frac{c_n(t)}{c_n(1) t^{2n+1}} dt. \tag{18}$$

Note that the variable s lies on the manifold $\mathbb{P}$, whereas t lies on the unit disk in $\mathbb{R}^2$. Therefore, explicitly

$$\begin{aligned}
U(r, \phi, r_D, \phi_D) = -Y \int_0^{2\pi} d\phi' \int_0^{r_b} dr'\, \sqrt{g}(r')\, q_T(r', \phi', r_D, \phi_D) \\
\left[\left(-\frac{1}{2\pi} f_0(\varrho_y) - \sum_{n \geq 1} \frac{1}{\pi} \varrho_x^n \varrho_y^n \cos n(\phi - \phi') f_n(\varrho_y)\right)\right].
\end{aligned} \tag{19}$$

Here, f_0 is the azimuthally symmetric contribution while the higher order modes capture asymmetries arising due to intermediate singularity positions. ϱ is the effective topological charge density as defined according to (3) and Y is the material Young's modulus. We note that the isotropic contribution, i.e., due to f_0 can be shown to be the same as reported in our previous work [4] and is equivalent to $U = \frac{1}{A}\int \Gamma dA$ for rotationally symmetric surfaces with symmetric defect placement.

2 Breaking rotational symmetry

As stated in the main text, we apply a shape perturbation to a general surface of revolution, generating an ellipse from a circular boundary by stretching/contracting perpendicular axes by ϵ. Thus,

$$x = (1+\epsilon)r\cos\phi, \quad y = (1-\epsilon)r\sin\phi, \quad z = f(r) \tag{20}$$

with $\epsilon \ll 1$. The metric tensor in the small-slope limit is

$$g_{ij} = \begin{bmatrix} 1+\epsilon(2+\epsilon)\cos 2\phi & -2r\epsilon\sin 2\phi \\ -2r\epsilon\sin 2\phi & r^2\left(1+\epsilon(\epsilon-2)\cos 2\phi\right) \end{bmatrix}, \tag{21}$$

where $\sqrt{g} = r(1-\epsilon^2)$ and eccentricity $e = \sqrt{1-\frac{(1-\epsilon)^2}{(1+\epsilon)^2}} = 2\sqrt{\epsilon} + \mathcal{O}(\epsilon^{3/2})$, while the Gaussian curvature is constant to $\mathcal{O}(\epsilon)$ and is given by

$$K_G(r,\phi) = \kappa^2 + \mathcal{O}\left(\epsilon^2\right), \tag{22}$$

where $\kappa = f''(0)$ is the apical curvature. The elliptical boundary of a section cut parallel to the xy-plane is given by

$$r(\phi) = \frac{r_b(1-\epsilon^2)}{\sqrt{1+\epsilon(\epsilon-2)\cos 2\phi}} \approx r_b(1+\epsilon\cos 2\phi). \tag{23}$$

Using a regular perturbation expansion for $\chi = \chi_0 + \epsilon\chi_1$ (and consequently the stresses σ) while also expanding the Laplace-Beltrami operator for small ϵ ($\nabla_\perp^2 \approx \nabla_0^2 + \epsilon\nabla_1^2$), we solve the following biharmonic equation for χ order-wise up to $O(\epsilon)$:

$$\frac{1}{Y}(\nabla_0^2+\epsilon\nabla_1^2)(\nabla_0^2+\epsilon\nabla_1^2)(\chi_0+\epsilon\chi_1) = \frac{\pi}{3}\frac{\delta(r-r_D)\delta(\phi-\phi_D)}{\sqrt{g}} - K_G(r,\phi). \tag{24}$$

The Airy stress χ relates to the stress components via the usual relations in polar coordinates:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2}, \quad \sigma_{r\phi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\phi}\right), \quad \sigma_{\phi\phi} = \frac{\partial^2\chi}{\partial r^2}. \tag{25}$$

For $\epsilon \ll 1$, the normal vector components can be expanded as: $n_r = 1 + O(\epsilon^2)$ and $n_\phi = 2\epsilon\sin 2\phi + O(\epsilon^2)$. Consistently expanding both the stress components and their arguments ($r(\phi) = r_b + \epsilon r_b\cos 2\phi$), one obtains up to $O(\epsilon)$:

$$\sigma_{rr_0}(r_b) + \epsilon\left(r_b\cos 2\phi\frac{\partial\sigma_{rr_0}}{\partial r}\bigg|_{r_b} + 2\sigma_{r\phi_0}(r_b)\sin 2\phi + \sigma_{rr_1}(r_b)\right) = 0 \tag{26a}$$

$$\sigma_{r\phi_0}(r_b) + \epsilon\left(r_b\cos 2\phi\frac{\partial\sigma_{r\phi_0}}{\partial r}\bigg|_{r_b} + 2\sigma_{\phi\phi_0}(r_b)\sin 2\phi + \sigma_{r\phi_1}(r_b)\right) = 0 \tag{26b}$$

For a stress-free boundary the elastic energy is written simply in terms of the trace of the stress tensor [1], i.e., $\Gamma(r,\phi) \approx \Gamma_0(r,\phi) + \epsilon\Gamma_1(r,\phi) = \sigma_{rr_0} + \sigma_{\phi\phi_0} + \epsilon(\sigma_{rr_1} + \sigma_{\phi\phi_1})$. Therefore, the elastic energy formally reads:

$$\begin{aligned} F_{\text{el}} &= \frac{1}{2Y}\int_0^{2\pi}\int_0^{r_b} \sqrt{g}\,\Gamma(r,\phi)^2 dr d\phi \\ &= \frac{1}{2Y}\int_0^{2\pi}\int_0^{r_b} r\left(\Gamma_0(r,\phi)^2 + 2\epsilon\Gamma_0(r,\phi)\Gamma_1(r,\phi)\right) dr d\phi + \mathcal{O}(\epsilon^2), \end{aligned} \tag{27}$$

which is the expression given in the main text. This energy integral is executed analytically exploiting the orthogonality of trigonometric functions.

While Azadi and Grason [5] focused on the azimuthally symmetric center or boundary placement of the defect, following work going back to Michell [6], we use a Fourier series expansion of the Airy stress function $\chi(r,\phi) = a_0(r) + \sum_{n=1}^{\infty}(a_n(r)\cos n\phi + b_n(r)\sin n\phi)$, to solve this system of equations for a general position (r_D, ϕ_D) of the disclination. The stress components expressed in terms of this expansion read

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2} \\ &= \frac{1}{r}\left[a_0^{(1)}(r) + \sum_{n=1}^{\infty}\left\{\left(a_n^{(1)}(r) - n^2\frac{a_n(r)}{r}\right)\cos n\phi + \left(b_n^{(1)}(r) - n^2\frac{b_n(r)}{r}\right)\sin n\phi\right\}\right], \end{aligned} \tag{28a}$$

$$\sigma_{r\phi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\phi}\right) = \frac{1}{r}\sum_{n=1}^{\infty} n\left[\left(a_n^{(1)}(r) - \frac{a_n(r)}{r}\right)\sin n\phi - \left(b_n^{(1)}(r) - \frac{b_n(r)}{r}\right)\cos n\phi\right], \tag{28b}$$

$$\sigma_{\phi\phi} = \frac{\partial^2\chi}{\partial r^2} = a_0^{(2)}(r) + \sum_{n=1}^{\infty}\left(a_n^{(2)}(r)\cos n\phi + b_n^{(2)}(r)\sin n\phi\right). \tag{28c}$$

2.1 Zeroth order solution

At $O(1)$, we employ a Fourier series expansion of the Airy stress function such that $\chi_0(r,\phi) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos n(\phi - \phi_D)$. The solution will be azimuthally symmetric only when the singularity is placed at the center or at the boundary of the surface [5]. This results in:

$$\frac{1}{Y}\nabla_0^4\chi_0(r,\phi) = -\kappa^2 + \frac{\pi}{3}\frac{\delta(r - r_D)}{\pi r}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n(\phi - \phi_D))\right] \tag{29}$$

where $\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$ and subject to the radial stress component boundary condition ($\sigma_{rr_0}(r_b) = 0$, $\sigma_{r\phi_0}(r_b) = 0$). The Fourier expansion of the stress components in terms of these modes is as follows:

$$\sigma_{rr_0} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2} = \frac{1}{r}\left[a_0^{(1)}(r) + \sum_{n=1}^{\infty}\left(a_n^{(1)}(r) - n^2\frac{a_n(r)}{r}\right)\cos n(\phi - \phi_D)\right], \tag{30a}$$

$$\sigma_{r\phi_0} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\phi}\right) = \frac{1}{r}\sum_{n=1}^{\infty} n\left(a_n^{(1)}(r) - \frac{a_n(r)}{r}\right)\sin n(\phi - \phi_D), \tag{30b}$$

$$\sigma_{\phi\phi_0} = \frac{\partial^2\chi}{\partial r^2} = a_0^{(2)}(r) + \sum_{n=1}^{\infty} a_n^{(2)}(r)\cos n(\phi - \phi_D). \tag{30c}$$

The system of ODEs one thus obtains is:

$$\frac{1}{Y}\left[a_0^{(4)}(r) + \frac{2}{r}a_0^{(3)}(r) - \frac{1}{r^2}a_0^{(2)}(r) + \frac{1}{r^3}a_0^{(1)}(r)\right] = -\kappa^2 + \frac{\delta(r - r_D)}{6r} \tag{31a}$$

$$a_n^{(4)}(r) + \frac{2}{r}a_n^{(3)}(r) - \frac{(1 + 2n^2)}{r^2}a_n^{(2)}(r) + \frac{1 + 2n^2}{r^3}a_n^{(1)}(r) + \frac{n^2(n^2 - 4)}{r^4}a_n(r) = \frac{Y\delta(r - r_D)}{3r}, \tag{31b}$$

where $^{()}$ indicates differentiation with respect to r, along with the following system of boundary conditions

$$\frac{1}{r_b}\left[a_0^{(1)}(r_b)+\sum_{n=1}^{\infty}\left(a_n^{(1)}(r_b)-n^2\frac{a_n(r_b)}{r_b}\right)\cos n(\phi-\phi_D)\right]=0 \tag{32a}$$

$$\frac{1}{r_b}\left[\sum_{n=1}^{\infty}n\left(a_n^{(1)}(r_b)-\frac{a_n(r_b)}{r_b}\right)\sin n(\phi-\phi_D)\right]=0 \tag{32b}$$

The leading order (small-slope) Fourier amplitudes for a rotationally symmetric surface with central curvature κ in the presence of an arbitrarily positioned disclination defect is given by:

$$\begin{aligned} a_0(r) =& -\frac{Y\kappa^2r^2}{64}(r^2-2r_b^2)+\frac{Y}{24}H(r-r_D)\left(r_D^2-r^2+(r_D^2+r^2)\log\left(\frac{r}{r_D}\right)\right)\\ &+\frac{Y}{48}r^2\left(1-\frac{r_D^2}{r_b^2}+2\log\left(\frac{r_D}{r_b}\right)\right), \end{aligned} \tag{33}$$

$$\begin{aligned} a_n(r) =&\frac{Y}{24}\frac{H(r-r_D)}{n(n^2-1)}\left[\left(\frac{r_D}{r}\right)^n\left((n+1)r^2-(n-1)r_D^2\right)+\left(\frac{r}{r_D}\right)^n\left((n-1)r^2-(n+1)r_D^2\right)\right]\\ &-\frac{Y}{24}\frac{(r/r_D)^n}{n(n^2-1)}\Bigg[(n-1)r^2-(n+1)r_D^2\\ &+\left(\frac{r_D}{r_b}\right)^{2n}\left(n(n-1)\left(\frac{rr_D}{r_b}\right)^2-(n^2-1)(r^2+r_D^2)+n(n+1)r_b^2\right)\Bigg], \end{aligned} \tag{34}$$

where H is the Heaviside function and $a_0(r)$ is the azimuthally symmetric solution reported in [5]. All the $n \neq 0$ modes go to zero when the singularity is decorated at the boundary or at the center of the surface but not otherwise.

2.2 First order solution

At $O(\epsilon)$, analogous to the zeroth order but more generally, we employ a Fourier series decomposition of the Airy stress function such that, $\chi_1(r,\phi)=c_0(r)+\sum_{n=1}^{\infty}\left(c_n(r)\cos n\phi+d_n(r)\sin n\phi\right)$. This results in

$$\frac{1}{Y}\left[\nabla_0^4\chi_1(r,\phi)+2\nabla_1^2(\nabla_0^2\chi_0(r,\phi))\right]=0 \tag{35}$$

where $\nabla_1^2 = 2\cos^2\phi\frac{\partial^2}{\partial r^2}+2\sin\phi\left(\frac{2\cos\phi}{r^2}\frac{\partial}{\partial\phi}+\frac{\sin\phi}{r}\frac{\partial}{\partial r}\right)+2\sin\phi\left(\frac{\sin\phi}{r^2}\frac{\partial^2}{\partial\phi^2}-\frac{2\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}\right)$ and subject to the following boundary conditions:

$$r_b\cos 2\phi\left.\frac{\partial\sigma_{rr_0}}{\partial r}\right|_{r_b}+\cancelto{0}{2\sigma_{r\phi_0}(r_b)}\sin 2\phi+\sigma_{rr_1}(r_b)=0, \tag{36a}$$

$$r_b\cos 2\phi\left.\frac{\partial\sigma_{r\phi_0}}{\partial r}\right|_{r_b}+2\sigma_{\phi\phi_0}(r_b)\sin 2\phi+\sigma_{r\phi_1}(r_b)=0. \tag{36b}$$

Inserting the Fourier expansion, (35) reduces to the following system of ODEs that now have a forcing term from the previous order:

$$c_0^{(4)}(r)+\frac{2}{r}c_0^{(3)}(r)-\frac{1}{r^2}c_0^{(2)}(r)+\frac{1}{r^3}c_0^{(1)}(r)=-2\left(\frac{2r^2{a_0}^{(3)}(r)-ra_0^2(r)+a_0^1(r)}{r^3}+{a_0}^{(4)}(r)\right)$$
$$-\frac{\cos(2\phi_D)\left(3a_2'(r)+r\left(r\left(4{a_2}^{(3)}(r)+r{a_2}^{(4)}(r)\right)-3a_2''(r)\right)\right)}{r^3} \tag{37a}$$

$$c_n^{(4)}(r)+\frac{2}{r}c_n^{(3)}(r)-\frac{(1+2n^2)}{r^2}c_n^{(2)}(r)+\frac{1+2n^2}{r^3}c_n^{(1)}(r)+\frac{n^2(n^2-4)}{r^4}c_n(r)=$$
$$-\frac{2\cos(n\phi_D)}{r^4}\Big(r\left(\left(2n^2+1\right)a_n'(r)+r\left(\left(-2n^2-1\right)a_n''(r)+r\left(2{a_n}^{(3)}(r)+r{a_n}^{(4)}(r)\right)\right)\right)$$
$$+\left(n^2-4\right)n^2a_n(r)\Big)-\frac{\cos((n-2)\phi_D)}{r^4}\Big(r\Big((2(n-2)(n-1)n+3)a_{n-2}'(r)$$
$$+r\left(r\left(r{a_{n-2}}^{(4)}(r)-2(n-2){a_{n-2}}^{(3)}(r)\right)-3a_{n-2}''(r)\right)\Big)-(n-2)^2n(n+2)a_{n-2}(r)\Big)$$
$$-\frac{\cos((n+2)\phi_D)}{r^4}\Big(r\Big((3-2n(n+1)(n+2))a_{n+2}'(r)$$
$$+r\left(r\left(2(n+2){a_{n+2}}^{(3)}(r)+r{a_{n+2}}^{(4)}(r)\right)-3a_{n+2}''(r)\right)\Big)+(2-n)n(n+2)^2a_{n+2}(r)\Big) \tag{37b}$$

$$d_n^{(4)}(r)+\frac{2}{r}d_n^{(3)}(r)-\frac{(1+2n^2)}{r^2}d_n^{(2)}(r)+\frac{1+2n^2}{r^3}d_n^{(1)}(r)+\frac{n^2(n^2-4)}{r^4}d_n(r)=$$
$$-\frac{2\sin(n\phi_D)}{r^4}\Big(r\left(\left(2n^2+1\right)a_n'(r)+r\left(\left(-2n^2-1\right)a_n''(r)+r\left(2{a_n}^{(3)}(r)+r{a_n}^{(4)}(r)\right)\right)\right)$$
$$+\left(n^2-4\right)n^2a_n(r)\Big)-\frac{\sin((n-2)\phi_D)}{r^4}\Big(r\Big((2(n-2)(n-1)n+3)a_{n-2}'(r)$$
$$+r\left(r\left(r{a_{n-2}}^{(4)}(r)-2(n-2){a_{n-2}}^{(3)}(r)\right)-3a_{n-2}''(r)\right)\Big)-(n-2)^2n(n+2)a_{n-2}(r)\Big)$$
$$-\frac{\sin((n+2)\phi_D)}{r^4}\Big(r\Big((3-2n(n+1)(n+2))a_{n+2}'(r)$$
$$+r\left(r\left(2(n+2){a_{n+2}}^{(3)}(r)+r{a_{n+2}}^{(4)}(r)\right)-3a_{n+2}''(r)\right)\Big)+(2-n)n(n+2)^2a_{n+2}(r)\Big) \tag{37c}$$

where $n=1,2,3,4...$ These together with (36) are solved analytically to obtain the $\mathcal{O}(\epsilon)$ Fourier coefficients whose lengthy expressions are available upon request.

References

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