Electronic Supplementary Material (ESI) for Physical Chemistry Chemical Physics.

Supplementary Information for *Harnessing Deep Reinforcement Learning to Construct Time-Dependent Optimal Fields for Quantum Control Dynamics*

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The following flowchart provides additional details for our implementation of the DDQN algorithm for quantum optimal control.

Algorithm 1 DDQN for Quantum Optimal Control

1: Discretize the range of admissible electric fields uniformly into D levels. In our work, a value of $D = 15$ was used.
2: Initialize all neural network parameters Θ (e.g., zero-initialize biases and Xavier-uniform initialization for weights). The neural network $q_\Theta(s, a)$ has $D = 15$ outputs, one for each discretized action.
3: Set the initial wavefunction $\psi(x, t = 0) = \psi_O(x)$.
4: Initialize state $s_0 = [P_0^0 = 1, P_0^1 = 0, ..., P_0^K = 0, g_0 = 0]$, where K is a design parameter. In our work, a value of $K = 4$ was used.
5: **for** $t = 0, \cdots,$ **do**
6: Sample a_t according to the epsilon greedy scheme defined as follows:
Step 1. $\xi \sim \mathcal{U}(0, 1)$
Step 2. if $\xi > \epsilon$, then $a_t = \text{argmax}_a\, q_\Theta(s_t, a)$; if $\xi < \epsilon$, then $a_t \sim [0, 1, ..., D = 15]$.
7: Normalize a_t to be within $[-\bar{E}, \bar{E}]$.
8: Set $E(t) = a_t$.
9: Starting with $\psi(x, t)$ and the external electric field, $E(t)$, perform the time-evolution of the Schrödinger equation for one step using the Crank–Nicolson scheme to obtain $\psi(x, t + 1)$.
10: Project $\psi(x, t + 1)$ to the first $K + 1$ eigenstates (denoted as P_{t+1}^k for the kth projection).
11: Calculate the gradient of the fidelity with respect to the external electric field $E(t)$: $g_{t+1} = \frac{\partial P_{t+1}^\kappa}{\partial E(t)}$. This can be done either analytically or using automatic differentiation packages.
12: Obtain $s_{t+1} = [P_{t+1}^0, P_{t+1}^1, ..., P_{t+1}^K, g_{t+1}]$.
13: Obtain the reward, $r_{t+1} = P_{t+1}^\kappa$.
14: Store $(s_t, a_t, r_{t+1}, s_{t+1})$ into the replay buffer $\mathcal{D}$.
15: Sample mini-batch $\mathcal{B} = \{(s, a, r, s')\}$ from $\mathcal{D}$.
16: Perform one step of a gradient descent on the objective function (Eq. 16) to update Θ.
17: **if** $\text{mod}(t, 100) = 0$ **then**
18: Update the target neural network weight $\Theta^- \leftarrow \Theta$.
19: **if** $P_t^\kappa > \bar{P}$ **then**
20: Break

The following flowchart provides additional details for our implementation of the SAC algorithm for quantum optimal control.

Algorithm 2 SAC for Quantum Optimal Control

1: Initialize all neural network parameters ϕ, θ^1, and θ^2 (e.g. zero-initialize biases and Xavier-uniform initialization for weights).
2: Set the initial wavefunction $\psi(x, t = 0) = \psi_O(x)$.
3: Initialize state $s_0 = [P_0^0 = 1, P_0^1 = 0, ..., P_0^K = 0, g_0 = 0]$, where K is a design parameter. In our work, a value of $K = 4$ was used.
4: **for** $t = 0, \cdots,$ **do**
5: Sample a_t from the policy neural network as follows:
Step 1. $\xi_t \sim \mathcal{N}(0, 1)$
Step 2. $a_t = \tanh(\xi_t \cdot \sigma_\phi(s_t) + \mu_\phi(s_t))$
6: Normalize a_t to be within $[-\bar{E}, \bar{E}]$.
7: Set $E(t) = a_t$.
8: Starting with $\psi(x, t)$ and the external electric field, $E(t)$, perform the time-evolution of the Schrödinger equation for one step using the Crank–Nicolson scheme to obtain $\psi(x, t + 1)$.
9: Project $\psi(x, t + 1)$ to the first $K + 1$ eigenstates (denoted as P_{t+1}^k for the kth projection).
10: Calculate the gradient of the fidelity with respect to the external electric field $E(t)$: $g_{t+1} = \frac{\partial P_{t+1}^\kappa}{\partial E(t)}$. This can be done either analytically or using automatic differentiation packages.
11: Obtain $s_{t+1} = [P_{t+1}^0, P_{t+1}^1, ..., P_{t+1}^K, g_{t+1}]$.
12: Obtain the reward, $r_{t+1} = P_{t+1}^\kappa$.
13: Store $(s_t, a_t, r_{t+1}, s_{t+1})$ into the replay buffer $\mathcal{D}$.
14: Sample mini-batch $\mathcal{B} = \{(s, a, r, s')\}$ from $\mathcal{D}$.
15: Perform one step of a gradient descent on the objective function (Eq. 20) to update θ^1 and θ^2.
16: Perform one step of a gradient ascent on the objective function (Eq. 22) to update ϕ.
17: Update the target neural network weight θ^{1-}, θ^{2-} via Eq. 24.
18: **if** $P_t^\kappa > \bar{P}$ **then**
19: Break

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TABLE I: Parameters Used to Generate $V_i(x)$ in Eq. (25): Easy Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
1	9.3036	1.0804	0.9115	3.7232	-1.8055	1.9567	6.1351	-1.5839	0.9614
2	6.1973	1.6050	1.5435	7.7532	-1.2968	1.0775	2.7186	2.1432	0.9093
3	6.2382	0.5516	0.8725	2.3524	2.9547	0.5960	4.2352	2.9302	0.8222
4	3.7158	1.5999	1.0306	2.7889	-1.9428	1.6553	1.0304	-2.2768	1.5743
5	2.5956	-1.6894	1.0225	8.9720	-0.3319	0.8518	9.0664	-2.9862	0.8448
6	6.6764	0.9489	0.7300	8.2687	2.9736	0.6253	1.7029	-0.0444	1.6568
7	5.6923	-2.0758	1.6776	2.3914	0.0438	1.8830	5.6496	2.5316	1.4176
8	2.7063	2.9833	1.3633	7.0393	-2.8191	1.4105	6.1721	0.0912	0.8273
9	4.7035	1.7585	1.4883	4.5406	0.6686	0.6595	9.9855	-2.1200	1.6940
10	7.3758	-1.2422	1.2959	1.0432	2.6808	1.6832	6.3843	1.8400	1.0195
11	7.0059	0.1567	0.6128	3.5442	2.7091	0.9582	8.2721	-1.5878	0.5066
12	6.3149	2.3968	0.9940	6.3036	-0.4790	1.0882	1.0382	-2.9053	0.6861
13	5.4897	2.0607	1.9400	1.9084	-1.0836	1.9530	5.0735	1.1993	1.7835
14	1.3281	-2.6195	0.8165	6.1420	2.9766	0.6804	7.8161	0.2575	1.8006
15	5.9688	-0.6938	0.8708	6.3402	1.9563	0.9708	1.3069	0.8325	1.3658
16	4.8967	-2.9032	1.7589	8.6125	1.3233	1.8680	3.5204	-1.9655	0.8039
17	1.6869	-1.0654	1.6352	8.5996	-2.6645	1.1960	9.1878	0.8449	1.4695
18	3.2660	-2.3053	0.6017	6.8465	2.6619	0.6300	7.0437	0.4373	1.0656
19	2.0613	0.1907	0.7776	9.0306	-2.8098	1.9712	5.6349	1.2253	1.2546
20	5.1206	-0.1246	0.8793	2.2858	-0.5718	0.8442	7.1881	-2.5451	0.6492
21	6.0540	0.2075	0.5725	8.4690	2.5923	1.2814	1.0407	-1.0699	1.9181
22	3.2274	-1.2871	0.8138	1.6908	-0.8279	1.2690	4.5128	2.5597	1.7856
23	1.4313	2.8517	0.8859	6.0184	2.1528	1.4281	6.7655	-0.5091	0.8657
24	2.9593	-0.9413	1.5522	6.2920	1.5760	0.5060	6.0062	-0.5842	1.8204
25	1.3633	-1.4272	0.7729	5.1766	-1.4233	0.5803	5.5539	0.6186	0.7619
26	3.0920	1.2721	0.7116	2.3540	-2.2210	0.7294	2.3861	-0.8778	0.8088
27	1.3839	-1.8555	1.4661	5.5170	0.1127	1.7419	4.0935	-2.8277	0.6261
28	7.0794	1.7830	1.4746	6.4402	-2.4888	1.3502	2.4558	-0.9405	1.5243
29	4.3115	2.3386	0.5714	8.6595	1.5045	1.8398	7.6144	-0.5969	1.0908
30	5.7769	0.1602	0.7594	7.1063	-2.5017	1.3637	1.1421	0.5491	1.3784
31	5.7343	-0.9260	1.6580	9.1926	-0.2238	1.5737	9.2441	-2.9664	0.5751
32	4.4796	0.7389	0.7375	9.9935	0.8643	1.8297	9.9438	-1.7265	0.9412
33	4.1467	-1.4965	0.8337	2.7765	0.1568	1.6882	3.5603	2.0248	1.1536
34	3.7266	-2.8619	1.0885	2.2321	-2.5155	1.4033	4.6533	-0.5352	0.5150
35	4.7209	2.2114	1.7988	5.3658	2.7526	0.8216	9.3410	-0.4536	1.4881
36	7.1026	-2.4932	0.8286	4.3076	0.0875	1.4233	3.1310	0.0534	0.9966
37	3.3333	1.4002	1.5112	1.8995	-2.3888	0.6460	2.5499	-1.7663	1.6554
38	8.1008	1.9466	0.9819	9.8706	-0.9727	1.9972	2.4984	-0.8385	0.6737
39	6.8223	-1.8441	1.9317	5.2436	1.7652	0.8317	2.4450	0.6322	1.8297
40	9.6917	-2.8532	1.6402	7.3750	0.3582	0.6435	4.0074	-1.0245	1.6730
41	3.5026	-2.6402	1.0578	2.6322	-0.7010	1.1168	3.5907	1.4870	0.8789
42	3.9637	-0.2348	0.7307	3.1759	-2.8398	0.7693	1.3824	-1.9495	1.7833
43	7.1582	-1.8206	1.1303	5.6659	0.7683	1.5996	5.8726	2.9707	1.7395
44	7.2511	0.4214	0.6729	8.3076	-2.5023	1.1949	8.6636	2.8060	1.2102
45	2.9309	-2.2046	1.8764	4.5506	1.2641	0.8811	3.0547	-1.7582	1.7766
46	1.2863	1.2347	1.5615	5.3873	0.3279	0.5622	5.6203	-1.7265	0.8440
47	4.0181	-2.3070	0.5768	4.3283	-0.3703	1.1416	3.9657	2.0339	0.8518
48	7.2996	-2.6071	0.6770	2.1916	1.0255	1.0896	7.5407	-0.2355	1.1510
49	5.8485	2.8179	0.6678	7.2510	1.7599	1.2163	7.9369	0.1079	0.7386
50	9.7206	0.6067	1.1735	2.5076	2.7472	1.5813	8.4921	-1.6112	0.7157
51	4.4233	-1.9421	1.4215	3.8384	1.8358	0.9344	1.7001	0.8762	1.5458
52	2.4192	1.8088	1.1607	6.6605	-1.2565	1.2642	4.7046	2.4593	1.4946
53	4.8921	1.4587	1.4142	5.9281	-1.5389	0.7096	3.2330	0.9161	1.6544
54	6.6153	1.4309	0.8151	5.1806	-1.8373	1.8865	1.8782	-2.3881	1.8358
55	4.4252	1.5169	0.5057	7.1204	0.9610	1.0263	8.5698	-1.1408	0.8818
56	3.0252	-1.8099	1.0641	7.5467	1.2546	1.0875	4.7360	-1.4774	0.9760
57	8.9485	0.1435	1.9610	5.3057	-1.6309	0.5691	2.1971	1.0400	0.6586
58	8.5453	-2.2930	1.1674	4.4711	2.5327	1.4863	6.7281	0.5332	1.3368
59	2.8677	-2.4387	1.3685	1.0793	-2.5443	1.8065	3.7891	1.3016	1.2880
60	1.6951	2.8416	1.6677	6.0711	1.9378	1.8018	6.7028	-1.6753	1.3087
61	6.2736	1.9147	1.4129	8.1291	0.1058	0.5052	5.0195	2.0995	0.7985

TABLE I: Parameters Used to Generate $V_i(x)$ in Eq. (25): Easy Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
62	3.4378	2.9940	1.1665	7.7577	0.7433	0.6254	5.6505	2.8914	0.9455
63	3.9090	0.1021	0.8642	5.6245	-1.0251	1.0176	7.7444	2.3217	1.0104
64	3.7331	1.3750	0.9743	6.6271	-2.9552	1.1743	5.8043	-0.2270	1.4897
65	4.1593	0.3496	1.8671	1.2553	-2.1459	1.1027	8.1465	1.8848	1.9830
66	7.2219	-1.7248	0.5768	9.6483	-0.1575	1.2788	9.5479	2.4552	1.3272
67	5.4036	1.9002	1.2159	9.3026	-1.7536	1.6927	3.2579	1.6433	1.4625
68	6.4902	-2.7775	1.8004	3.7459	2.0318	0.5433	4.8181	0.6255	1.3349
69	2.4484	-0.6979	1.5248	3.4420	-1.0587	0.6867	5.6303	1.1661	0.6158
70	4.1851	1.1597	1.6972	3.7314	0.8798	0.5198	6.6380	-1.7682	1.2632
71	3.9836	-2.4539	1.7250	5.2089	2.1683	0.9525	2.9964	-0.2387	1.3366
72	4.6151	0.6173	1.1839	6.3072	-1.9569	0.9824	3.7237	2.3272	1.7644
73	5.4099	-2.5388	0.6799	3.1713	0.3235	0.5937	4.7405	-0.6526	1.0566
74	6.8969	0.1455	0.6148	9.2946	2.4225	0.6491	5.8919	0.9477	1.1188
75	2.5636	2.1851	0.6235	1.9599	0.3096	1.8803	2.5761	-1.6669	1.9127
76	8.0164	0.5756	0.6305	5.8796	-1.4945	1.1272	5.1241	-2.7014	1.5270
77	2.0334	1.2474	1.5001	7.5519	-1.1751	0.5998	8.7317	1.0788	1.6291
78	2.4166	1.2614	0.7616	2.9076	-0.2150	0.8821	4.1714	-2.7166	0.9735
79	3.8679	1.0012	0.8343	8.2696	-1.5781	0.8657	4.8476	0.9992	1.3454
80	1.5919	1.6950	0.7205	6.6922	-1.6651	1.2086	5.2675	1.4176	1.4383
81	9.9410	-1.4901	1.8231	9.7590	1.9909	1.1121	2.0709	-0.5894	1.4758
82	9.6974	1.1923	1.0079	7.6476	-1.0729	0.6809	1.1037	-2.9369	1.9130
83	6.4175	2.4396	0.7392	2.4057	1.4256	1.4805	9.8122	-0.8345	1.9836
84	6.2775	2.8542	1.4572	1.0049	-2.0252	1.4179	5.0342	-0.2597	1.0061
85	8.2195	-0.4628	0.6165	7.1582	-2.3134	0.5135	4.0501	-1.9735	1.6131
86	3.8479	-0.1457	1.6908	8.2349	1.2857	0.6908	8.0066	-1.5611	1.3685
87	2.1239	1.9638	0.7306	5.2785	1.6305	1.8712	6.9735	-1.9225	1.6539
88	3.9341	-0.8307	1.6674	6.9046	2.3581	0.7933	3.8728	-0.0446	1.2000
89	2.0350	2.4650	1.3435	3.8597	-2.6741	0.7000	3.6889	-0.2411	0.9287

TABLE II: Parameters Used to Generate $V_i(x)$ in Eq. (25): Medium-Difficulty Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
1	1.5660	-1.3063	0.9929	6.5647	-1.4440	1.3640	7.6502	1.6400	1.0433
2	9.0520	-0.7776	1.8310	1.6773	-2.6660	0.9222	8.3738	2.7440	1.1730
3	9.4891	-1.1117	1.3000	6.7122	1.2068	0.5676	1.5276	2.9063	1.4141
4	6.6415	2.8636	1.0103	5.2695	-0.8102	1.1272	5.6741	0.6273	1.8887
5	1.8186	-0.5905	1.9793	4.5343	-2.5140	0.5344	4.5050	0.2757	1.0753
6	1.1626	0.0254	1.9283	7.4005	-0.0656	0.8550	8.8064	2.7794	1.2542
7	2.9425	0.9115	1.2025	1.3760	1.5572	0.8429	4.4198	-1.2809	0.5729
8	1.1363	-1.8092	0.8208	1.9024	2.1567	1.0084	1.0808	-1.0731	1.8229
9	6.2786	-2.6647	1.9490	5.9028	0.8730	0.7469	2.1732	-1.4185	1.4673
10	4.5997	-1.6526	0.8224	1.1554	-2.8955	1.3814	5.3225	1.0368	1.0997
11	4.3446	2.9663	0.6372	1.2981	-0.0684	1.4203	2.2333	0.3747	0.9565
12	7.4570	2.3265	1.3124	2.6612	0.5781	1.4456	5.9520	-0.3549	0.5892
13	3.1878	1.8563	0.7063	6.1949	-0.6966	0.5427	4.9758	1.1222	0.8186
14	4.0570	-0.6537	0.6210	4.2526	2.1283	1.7583	1.2598	2.0124	1.6234
15	9.6355	-1.2368	1.2092	3.4994	-1.7388	0.5297	9.4361	0.5755	0.6181
16	5.2833	0.9396	0.8734	1.4746	-1.2652	0.9217	4.2737	-2.9895	1.5630
17	2.2485	-2.1842	1.7022	4.3232	1.1965	1.3682	2.4323	-1.7575	0.5731
18	7.9011	0.6055	1.5109	3.2766	1.0343	1.6159	8.0179	-1.6880	0.5604
19	7.5718	-1.0043	1.8663	9.4968	-2.4863	0.5793	9.2987	0.7694	1.7036
20	1.6784	-1.0905	0.5480	2.4965	1.5908	1.1347	1.6635	-2.6647	0.9114
21	3.5155	-2.8521	1.9844	6.7255	-0.5960	1.9578	5.2197	1.7196	0.6765
22	7.9644	-0.7625	0.9829	6.9049	-2.7219	0.5493	3.3695	0.9028	0.7029
23	6.4127	0.0631	0.9953	1.1230	1.2132	0.5108	6.6853	2.8090	0.8163
24	9.4556	1.1590	1.0825	5.5006	-1.0796	0.9768	5.4240	-1.9554	0.7423
25	6.9397	2.2194	0.7443	1.1787	-1.8920	1.0216	5.7932	-0.2414	0.9938
26	6.0568	1.7474	1.3457	4.5042	-0.1185	0.6455	9.9123	-2.6603	1.6002
27	8.8700	-0.5561	0.8792	7.9503	1.5783	0.5409	5.5809	0.7567	1.9136
28	3.2459	-2.2294	1.9909	3.7042	0.9754	1.4475	5.9027	-1.5088	1.7484

TABLE II: Parameters Used to Generate $V_i(x)$ in Eq. (25): Medium-Difficulty Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
29	7.6423	-0.8712	1.9833	7.8699	2.9376	0.9534	3.1908	0.7016	1.9622
30	1.1648	-1.5094	1.7779	9.3642	-0.2046	1.8455	6.9380	-2.7249	0.5493
31	4.7508	-1.6744	0.7272	7.1324	2.3866	1.6258	4.5761	-0.1250	0.5866
32	2.6492	-2.8151	0.6062	2.8647	-1.9699	1.5711	3.6374	0.8444	1.0720
33	4.2548	-0.1634	0.5810	5.6362	-2.9492	1.6035	2.9859	1.9884	0.5536
34	6.6828	-2.2252	0.6318	7.2787	0.0023	1.0033	2.1618	2.4760	0.8759
35	4.3613	-2.2923	0.5293	2.7087	0.4274	1.7729	1.2814	0.4940	1.2687
36	6.8237	-2.9649	1.2484	5.4076	0.8431	1.0321	3.3437	-0.6131	0.8001
37	4.5128	-1.4230	1.6230	6.2051	1.3185	1.7208	4.0209	-0.5015	1.4203
38	1.6888	-0.1167	1.2795	1.7070	-2.1489	1.2396	2.2699	2.0065	0.5149
39	5.2941	-2.1366	1.4963	6.4010	-0.4264	1.2670	7.8994	1.6220	0.6244
40	7.2251	2.1241	0.5975	7.5291	0.2382	1.5446	6.2771	-2.5681	1.6703
41	3.7966	2.0375	1.1850	8.4117	-0.2310	1.8650	4.9568	-2.2725	0.5845
42	5.5641	-2.2040	1.2653	6.0797	2.1133	0.9339	4.4241	-0.0931	0.6978
43	1.9241	0.0781	1.5610	2.4616	0.9557	0.6218	2.8350	-2.6411	0.9607
44	4.8234	-2.2383	0.6324	5.5595	-0.7333	1.3733	6.5895	1.4600	1.0832
45	9.1277	0.3550	1.8007	3.4325	1.0859	0.5989	6.5263	-1.3119	0.6428
46	1.5104	2.6799	0.6359	5.3385	2.8990	1.2557	8.2864	-0.5239	1.7131
47	1.9026	0.7315	1.2411	5.6296	-2.4921	0.5667	2.8665	-0.2671	0.7393

TABLE III: Parameters Used to Generate $V_i(x)$ in Eq. (25): Hard Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
1	8.2816	-2.1193	1.0199	6.7014	0.5296	0.7498	1.2063	0.2037	1.8793
2	5.1814	-1.5645	1.8455	3.5358	2.6235	0.8806	1.3909	1.7815	1.6579
3	3.3273	2.7414	0.5943	8.8774	-2.3141	1.7991	6.6585	0.8824	0.6925
4	7.8867	-2.6579	0.9971	6.0352	0.0916	1.4307	4.4009	1.4232	0.5815
5	3.6424	-0.1761	1.0289	5.9449	-1.7876	0.9930	6.3157	1.8231	0.8117
6	1.6833	-1.3933	0.7840	3.0237	2.4202	1.6908	2.1463	-2.6288	0.7005
7	7.2552	2.8138	0.6912	6.0171	0.1756	1.6723	4.3055	-1.5096	1.9326
8	2.8617	2.1703	1.5564	1.6184	-2.3564	0.5486	2.0912	-1.8763	1.4922
9	6.1322	0.2946	1.5948	5.7457	2.5901	0.5318	2.9993	-1.3442	1.5301
10	5.9820	-2.8719	0.9933	9.3567	1.0597	0.8750	7.0698	-1.2716	1.2492
11	4.4205	1.3927	1.8821	6.3277	-2.2107	1.0041	2.9844	1.4067	1.2370
12	5.9461	2.2123	0.9011	4.9833	0.1914	1.2945	6.9383	-1.6440	0.6435
13	4.4132	2.4116	1.4879	6.3946	-2.1082	0.5409	7.1580	0.0918	1.5176
14	7.4811	2.6910	1.2631	1.1250	-0.2449	0.6754	5.4022	-0.0505	0.6657
15	1.4429	-0.2677	0.7800	4.4748	-1.8086	0.7879	4.7161	2.6969	1.6247
16	7.8798	0.5409	1.5543	6.3298	-2.0898	0.8003	6.5420	2.7033	0.5015
17	9.6547	-1.9695	1.5006	2.1642	2.5954	1.8681	7.7866	0.6642	0.6134
18	2.7473	0.9665	1.4903	6.2410	-1.8464	0.9666	4.0225	1.1489	0.8628
19	5.7602	-2.0095	1.1418	2.0368	1.7766	1.2094	3.4051	1.2694	1.3624
20	4.2084	-2.0202	0.9958	3.0750	2.2724	0.8752	2.6125	0.4727	1.3522
21	7.4737	-1.4524	1.9195	5.3749	1.1968	0.5991	1.0746	-2.2405	1.4690
22	5.5934	1.2508	0.6350	4.8573	-1.1497	1.2211	1.9574	-2.5050	0.7403
23	4.8423	-2.2110	0.5955	4.4393	0.3531	1.2544	1.3403	2.3773	0.8904
24	6.2065	-1.5580	0.5622	8.7841	0.6400	1.9298	3.3550	1.3510	0.5249
25	3.4984	-0.8288	0.6191	1.1127	-0.3852	0.5577	3.7373	1.9680	1.0311
26	2.5749	1.5493	0.7996	3.9424	-1.5138	1.2394	1.3665	2.3736	1.2318
27	4.8224	1.5486	1.1739	2.7461	1.6985	1.7013	7.9561	-2.0134	1.4584
28	1.7603	-2.6204	1.0482	3.2191	-1.7642	1.4795	4.8190	1.8854	1.4893
29	3.5854	0.0280	0.5428	7.5740	-1.7946	1.4075	9.5081	2.7653	1.7147
30	4.4932	-2.0002	0.7313	2.6545	1.1623	1.5464	2.4120	1.1244	1.2295
31	9.1222	-1.1986	1.8729	5.4549	0.0002	0.7701	5.9994	-2.8120	0.6138
32	9.3549	0.8636	1.6842	4.7187	1.9148	1.2889	9.5174	-1.5019	0.7354
33	5.4471	-1.6075	0.5266	6.9224	2.5371	1.7041	3.7665	-0.4206	1.0921
34	9.8936	-1.0513	1.1039	1.5219	0.7133	1.4771	8.3088	1.4655	0.6580
35	5.4002	-2.7771	1.4660	2.2815	1.4586	1.7590	2.8380	0.8667	1.0530
36	3.3980	1.3309	0.7639	5.8609	0.6972	1.2816	9.5972	-2.8283	1.8826
37	4.1185	-1.6386	1.8586	8.0690	1.0567	0.6016	7.0534	-1.5644	1.8174

TABLE III: Parameters Used to Generate $V_i(x)$ in Eq. (25): Hard Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
38	4.2264	1.7879	1.6850	5.9482	0.7868	1.2606	7.6421	-1.5229	0.5562
39	4.1089	0.7331	1.4374	4.9525	-2.4937	1.1310	2.0786	2.7293	1.4333
40	5.1232	-1.7374	0.5415	4.8107	-0.6652	0.8953	8.4803	2.3807	1.8626
41	3.0263	1.9799	0.6410	9.1278	-1.2544	1.5799	5.0454	2.2078	1.4209
42	5.7242	-2.0769	1.0384	8.6493	1.8436	1.9733	1.7794	-1.3116	0.8094
43	7.6718	-2.8275	0.5051	8.9521	0.3698	0.6310	6.8428	-1.5547	0.5749
44	6.8475	-2.5763	0.7513	5.8829	-1.7268	1.7858	8.4889	-0.2883	0.6448
45	1.7265	-0.5718	1.3053	9.2621	1.0669	0.9521	9.8621	-2.2093	1.3396
46	1.4715	-1.3257	1.5620	6.2677	-0.2336	1.6122	6.1504	2.9446	0.8102
47	2.0840	-1.6590	1.6013	7.5027	1.7905	1.7477	5.1019	-2.4875	1.1553
48	1.9663	2.4072	0.5424	2.1583	0.8413	1.4761	2.3018	-1.5442	0.5514
49	4.4935	2.7705	1.9591	3.5382	-1.1667	0.7628	1.0550	-2.5559	1.1588
50	4.7047	-1.2842	1.9025	2.9142	2.6767	0.5088	1.4918	2.3052	1.1742
51	4.3471	2.7616	1.6944	3.4067	2.7587	1.0544	8.4059	-1.3030	1.9427
52	9.5293	-2.9164	1.4920	1.8908	2.7490	1.5984	7.2694	-0.2009	0.5179
53	4.2661	-1.1838	1.2232	5.3475	1.2225	0.5431	2.5293	-2.8637	1.8201
54	2.1398	0.2595	0.6738	8.6600	-2.0585	0.7910	8.9363	0.8628	1.5642
55	5.7655	-2.1568	1.1041	7.1058	0.2296	0.8607	3.4644	-2.6986	0.5107
56	9.4772	0.7746	0.8961	3.8203	-0.7831	0.9058	9.8632	-2.2004	0.9029
57	9.2489	0.5161	0.8308	9.6082	-1.9975	0.8008	1.7376	0.6763	1.5485
58	1.0325	-1.3057	1.8666	5.1324	-1.3609	0.5334	5.8530	0.8736	0.8233
59	2.0818	-2.8698	1.4437	1.0719	0.1337	0.7098	2.0858	2.2049	0.5535
60	2.1260	-2.3569	0.7219	6.6571	-1.0820	1.8052	5.5285	1.7485	0.6404
61	6.3722	-2.3937	1.9213	1.3019	0.2109	1.0144	6.0865	2.7482	1.6807
62	5.4730	-1.2247	0.7562	2.1533	0.2890	0.9749	6.1996	2.3189	1.1565
63	2.1546	2.2984	1.9834	6.4564	1.3847	1.6790	8.8235	-2.8496	1.6758
64	2.4988	1.4363	1.1315	2.7000	-2.9451	0.9084	1.8162	-0.9956	0.5828
65	4.0271	-0.5665	1.1196	8.1640	2.2279	1.6273	7.0686	-2.0853	0.8282
66	3.6036	-2.9339	0.7614	9.1740	-1.8376	1.7903	8.1407	0.6172	0.8410
67	5.1539	1.9928	1.5862	2.4466	1.8453	1.3600	6.8310	-1.6741	1.0831
68	8.7046	-1.1615	0.8971	2.5054	1.4661	0.8058	7.6049	1.9021	1.4900
69	1.8561	-0.7193	0.9593	8.1023	2.6204	1.0010	6.2196	-0.4524	1.3408
70	6.4681	1.2091	0.7528	8.2375	-1.5532	1.1740	1.6867	2.6364	1.8701
71	4.6208	-0.1368	0.6623	5.1645	-1.9633	0.8757	6.1268	2.6751	1.4515
72	5.6483	-1.8520	0.5141	9.9392	0.7432	0.8246	5.1268	-1.2155	0.9557
73	2.5582	-0.7343	0.5020	6.7320	-0.9724	1.7011	8.9536	2.8935	1.6255
74	7.2251	-0.8650	0.9775	9.7726	1.5816	0.5151	3.0981	-0.0434	0.5499
75	4.8717	-0.3872	0.8303	5.1181	-1.2858	0.6880	9.5108	2.3987	1.8061
76	4.1838	0.1828	0.6254	7.1509	2.8789	0.6587	4.1780	0.9941	1.3150
77	2.9511	1.4320	0.7079	2.0410	-2.9011	0.6478	3.1778	-1.2419	1.9478
78	2.4620	-0.9607	1.2428	2.6216	0.3062	1.4653	4.9387	2.7777	0.6934
79	5.0806	-0.5982	1.8283	3.6762	-2.4112	0.5426	4.5511	1.6453	0.7029
80	7.1516	2.5016	0.6923	7.0501	-0.5263	1.1607	3.4819	1.2571	1.3692
81	8.3684	0.3759	0.7125	3.2789	0.2618	1.3273	9.9653	2.4277	0.5439
82	1.5983	-0.9650	1.4526	9.3109	1.7511	1.5290	8.6115	-2.4942	1.5600
83	3.0121	-0.7867	1.4992	1.8088	-1.8880	0.7265	4.7087	2.0804	0.6377
84	6.6993	0.1302	0.9130	2.1459	2.4689	0.9228	6.6370	-2.4709	0.7159
85	1.4833	2.5727	0.5325	4.0820	-0.5997	0.8338	3.7522	2.9336	1.7215
86	1.3930	0.8661	1.8458	3.0595	-1.9742	0.7476	2.6673	0.6249	0.6625
87	3.2004	0.8857	0.6223	1.6938	1.0350	1.4687	4.7395	-1.6824	0.6248
88	6.5959	0.0159	0.7395	7.6240	-2.7712	1.2428	1.7651	1.7739	1.8439
89	8.9826	2.6927	1.7437	2.9650	0.4214	1.1616	9.4562	-2.1770	1.5063
90	3.9563	0.0701	0.8023	7.2738	-0.6175	1.6456	9.6413	2.0284	0.5868
91	8.7297	-1.7076	0.9047	5.3930	0.5485	1.0067	7.9874	2.9735	1.2596
92	9.4498	-2.7305	1.6840	3.8903	1.7918	0.6498	7.6214	0.6553	1.2250
93	6.8012	-1.4254	1.4972	4.0386	-2.4468	0.6949	9.9904	2.5291	1.8477
94	4.5616	-2.8352	0.8275	3.1808	1.0521	1.0428	2.3519	-0.1972	1.0056
95	7.9903	-0.8483	1.4337	8.2682	1.3903	0.7400	4.7396	-1.9671	0.5481
96	4.6299	-0.7138	1.5596	7.1878	-2.2441	1.2914	9.8300	2.6389	1.7905
97	6.6269	-2.4697	1.3678	4.5189	1.8688	1.6587	3.1615	0.9712	1.6054
98	3.9441	-2.5777	1.2089	9.2734	2.2478	1.8576	5.5345	-0.8962	0.7347

TABLE III: Parameters Used to Generate $V_i(x)$ in Eq. (25): Hard Cases

No.	A_1	μ_1	σ_1	A_2	μ_2	σ_2	A_3	μ_3	σ_3
99	3.7274	0.6220	0.5327	9.1753	-2.8890	0.8391	9.2494	-0.4424	1.0783
100	4.1383	-2.2091	1.1575	4.1719	2.7295	0.6302	4.5931	0.3218	1.8223
101	8.2240	2.8733	0.9135	9.9231	-0.6132	1.8102	1.3169	1.0101	1.4100
102	7.2715	-0.1632	1.6766	8.7007	2.6034	1.3525	8.4744	-2.5892	1.2666
103	8.8755	-0.1633	1.0483	8.1679	-1.6821	0.8307	9.7241	1.6052	0.6275
104	5.4233	-2.2546	0.7030	4.3720	0.4791	1.4007	1.0287	0.7564	0.9122
105	5.5848	-1.4812	0.5162	6.6517	0.3365	1.9049	4.5626	2.2038	1.2550
106	5.1538	2.5838	0.6104	5.5418	0.1165	0.9739	3.1711	-2.2978	0.5928
107	3.1548	-2.8979	0.6347	2.6385	2.0198	0.9809	2.1423	-0.5175	0.6935
108	5.2354	1.6727	0.6322	4.0221	-1.6303	1.8373	2.5549	-0.6864	0.8216
109	6.1744	-0.6669	1.8219	1.1942	0.2193	0.6906	6.1402	2.6353	0.7498
110	5.1748	2.3842	0.9339	2.4676	-2.8757	0.8957	4.9877	-1.0155	1.6273
111	1.6870	-2.6736	0.6749	9.9882	2.5056	1.9944	8.3755	-1.8586	1.5441
112	8.4272	-1.0934	0.5917	8.4238	1.4885	1.8299	4.1383	2.1836	1.5200
113	5.7780	2.7470	0.7719	5.2417	0.1375	1.0350	4.1591	-2.3777	1.4658
114	4.6446	-1.9080	0.6757	9.6686	-0.5401	1.6976	8.1537	1.5965	0.7756
115	2.1178	2.4450	1.0227	1.1944	0.0942	1.7260	2.3692	-2.8152	1.0515
116	6.3438	2.6719	0.5501	9.7263	0.9036	1.3927	9.7939	-2.0028	0.8691
117	9.5646	-2.1013	1.4225	4.2482	0.9914	0.7597	5.3832	2.0472	1.6945
118	9.1819	1.5961	0.8472	9.7527	-2.1815	1.6004	2.8948	-0.3388	1.0330
119	9.7874	-1.1513	1.4731	7.8458	1.9705	1.5295	4.4204	2.6413	0.5098
120	7.2643	-0.4926	1.4613	9.0888	2.9386	1.1531	3.1489	-1.6006	1.8654
121	2.5115	0.5777	0.7864	7.4316	0.0420	0.5315	9.9610	2.5129	0.9361
122	6.8083	-1.8564	1.0408	6.4967	0.7006	1.7103	5.7143	2.4770	0.7616
123	1.4989	-2.4973	1.2690	8.8772	2.7718	1.9999	7.8664	-1.9039	1.9579
124	3.7574	2.6516	1.4233	5.1350	-1.3188	1.5183	1.2871	2.6218	0.6522
125	6.5894	0.4106	0.7594	9.2004	-0.7681	0.5154	9.6167	1.8480	0.6481
126	8.3481	-2.2147	1.5003	6.1248	0.4901	1.8755	5.9472	1.9016	0.9043
127	4.2700	-2.1290	1.7569	9.2509	0.4600	0.7581	6.7046	-2.1651	1.1496
128	8.7104	-0.5467	1.5972	7.5711	1.8155	0.5361	2.8214	-1.8294	0.6110
129	9.2887	1.8264	1.9749	9.1152	0.4032	0.6212	8.5498	2.6730	0.7933
130	7.2264	2.3581	0.6467	1.3477	2.9387	0.7262	8.4382	-0.1562	1.2817
131	8.2046	-2.3125	0.9283	9.0045	-0.8564	1.4529	8.5438	0.7061	0.5571
132	5.1734	1.5196	1.8292	7.7806	-2.1567	1.4780	3.6555	1.1909	0.6520
133	2.9777	-2.1352	0.5584	5.2835	-1.7227	1.5021	8.7030	1.8059	1.7796
134	5.3289	2.3128	1.3625	5.4820	0.9231	0.8983	8.4722	-1.1162	0.6648
135	1.0430	0.1433	1.2468	5.5765	-0.5802	1.4053	5.8064	2.5888	0.7931
136	5.3016	0.6261	0.6479	6.6516	-1.6758	0.5588	4.4731	-0.2122	1.6408
137	3.1147	-1.1055	1.9691	2.4602	-1.7319	1.1056	4.6030	1.7945	0.9786
138	7.1655	2.6932	1.2401	3.5814	-1.5966	0.8399	7.1992	-0.4673	1.8789
139	3.0913	-1.3975	0.7152	9.3701	1.9691	1.2174	7.8853	-1.1431	1.4670
140	8.9395	-1.9191	0.7954	3.9602	1.6498	1.3042	8.1778	0.6354	1.6214
141	1.4264	-2.6602	0.6250	2.3584	2.7478	0.6945	3.0304	0.1956	0.5341
142	2.8629	1.6221	1.9565	7.8495	1.2630	1.2230	8.7454	-1.1698	0.5591
143	3.7679	0.8903	0.6368	4.2377	1.3062	0.8584	9.3536	-1.9658	1.4990
144	5.5254	-1.0505	0.8718	8.8490	0.3051	1.8084	8.2281	2.4460	0.9920
145	8.3340	0.4934	0.5470	2.0766	-1.0151	0.6629	7.4305	-1.9316	0.8658
146	6.1370	-1.8623	1.8087	3.2270	-1.6780	0.7930	7.2679	1.0457	0.6967
147	7.8539	1.7788	1.1682	2.3280	0.7749	1.5521	9.9330	-1.8495	1.2819
148	5.2449	-2.5025	0.9644	4.6862	-0.4990	0.5627	7.3720	2.6714	1.9385
149	4.3583	-0.3634	1.2298	2.9558	2.1955	0.6524	2.5860	2.9291	0.6320