Electronic Supplementary Material (ESI) for Nanoscale.

Supplemental Material

1 Appendix A: The SW transform

To transform the 4x4 Hamiltonian to a 2x2 we used a SW transform. This will inevitably rotate our Pauli basis, here we will calculate the effect of that rotation. The procedure we used to calculate the SW transform is outlined in Ref.[1]. The generator for the SW transform can be written as

$$e^S = \mathbb{I} + S + \frac{1}{2!}S^2 + \dots, \tag{1}$$

$$S = S^{(1)} + S^{(2)} + S^{(3)} + \dots, \tag{2}$$

where S is completely non-block diagonal and is anti-Hermitian. For the purpose of this calculation we only require S up to second order, the terms in the expansion of S are

$$S^{(1)}_{ml} = -\frac{H'_{mm'}}{E_m - E_l}, \tag{3}$$

$$S^{(2)}_{ml} = \frac{1}{E_m - E_l}\left[\sum_{m'} \frac{H'_{mm'}H'_{m'l}}{E_{m'} - E_l} - \sum_{l'} \frac{H'_{ml'}H'_{l'l}}{E_m - E_{l'}}\right]. \tag{4}$$

m indicates indices in the upper block and l in the lower block. For the TI Hamiltonian $E_m - E_l = -2\mathcal{M}$ and

$$H' = \begin{pmatrix} m_z & m_- & \mathcal{B}k_z & \mathcal{A}k_- \\ m_+ & -m_z & \mathcal{A}k_+ & -\mathcal{B}k_z \\ \mathcal{B}k_z & \mathcal{A}k_- & m_z & m_- \\ \mathcal{A}k_+ & -\mathcal{B}k_z & m_+ & -m_z \end{pmatrix}. \tag{5}$$

There are two questions we must ask with this transform: (1) does the rotation change the STT terms calculated? (2) Is our claim the the SO-field goes to 0 when the magnetisation is turned off still valid?
First we will look at how the SW transform rotates the 4×4 spin operators. Here we only care about a block diagonal portion of the transformed operators as our density matrix will be purely block diagonal due to the SW transform. The upper block diagonal of the transformed spin operators are

$$\left[e^{-S}(\mathbb{I}_{2\times2} \otimes \sigma_x)e^S\right]_{[1,2;1,2]} \approx \Bigg(\sigma_x - \frac{\mathcal{A}^2k_\parallel^2 + \mathcal{B}^2k_z^2}{4\mathcal{M}^2}\sigma_x + \frac{\mathcal{A}\mathcal{B}k_xk_z}{2\mathcal{M}^2}\sigma_z + \frac{\mathcal{A}^2k_xk_y}{2\mathcal{M}^2}\sigma_y + \frac{\mathcal{A}^2(k_xk_ym_x - k_y^2m_x) + \mathcal{A}\mathcal{B}k_xk_zm_z - \mathcal{B}^2k_z^2m_y}{2\mathcal{M}^3}\mathbb{I}_{2\times2}\Bigg), \tag{6}$$

$$\left[e^{-S}(\mathbb{I}_{2\times2} \otimes \sigma_y)e^S\right]_{[1,2;1,2]} \approx \Bigg(\sigma_y - \frac{\mathcal{A}^2k_\parallel^2 + \mathcal{B}^2k_z^2}{4\mathcal{M}^2}\sigma_y + \frac{\mathcal{A}\mathcal{B}k_yk_z}{2\mathcal{M}^2}\sigma_z + \frac{\mathcal{A}^2k_xk_y}{2\mathcal{M}^2}\sigma_x + \frac{\mathcal{A}^2(k_xk_ym_x - k_x^2m_y) + \mathcal{A}\mathcal{B}k_yk_zm_z - \mathcal{B}^2k_z^2m_y}{2\mathcal{M}^3}\mathbb{I}_{2\times2}\Bigg), \tag{7}$$

$$\left[e^{-S}(\mathbb{I}_{2\times2} \otimes \sigma_z)e^S\right]_{[1,2;1,2]} \approx \Bigg(\sigma_z - \frac{\mathcal{A}^2k_\parallel^2 + \mathcal{B}^2k_z^2}{4\mathcal{M}^2}\sigma_z + \frac{\mathcal{A}\mathcal{B}k_xk_z}{2\mathcal{M}^2}\sigma_x + \frac{\mathcal{A}\mathcal{B}k_yk_z}{2\mathcal{M}^2}\sigma_y + \frac{-\mathcal{A}^2(k_x^2m_z + k_y^2m_z) + \mathcal{A}\mathcal{B}(k_xk_zm_x + k_yk_zm_y)}{2\mathcal{M}^3}\mathbb{I}_{2\times2}\Bigg). \tag{8}$$

The second, third and fourth terms in all of these expansions can be neglected as they will simply give a correction to the calculated spin polarisations that is of a higher order in the perturbation parameters. For the values of k at which these corrections will be significant the SW transform will break down. The terms that are $\propto \mathbb{I}_{2\times2}$ show that the scalar response in our 2×2 effective Pauli basis can give a spin response. The homogeneous scalar response will not generate any spin polarisation as it is linear in $\boldsymbol{k}$, so the spin expectation value will integrate to 0. The inhomogeneous scalar response to lowest order in the SO field is $n_{0,\nabla} \approx 0$. So, there will be some corrections from the scalar response to the spin polarisation, though they will be negligible. Hence the answer to the first question is, the rotated spin matrices will not affect our results.

To answer the second question, we can already see by looking at the transformations of the spin operators that if the magnetisation is turned off the scalar component vanishes. Now, this means that our effective 2×2 Hamiltonian with the magnetization turned off must have to have a component $\propto \boldsymbol{\sigma}$ in order to have any spin-orbit field. Now since without the magnetization the Hamiltonian is completely scalar we can conclude that the SO field does indeed vanish. To further my point since the conduction band states have a total angular momentum of $\frac{1}{2}$ you should not be able to get any spin orbit effects without breaking some symmetry.

2 Appendix B: Spin density

To obtain the spin density from the density matrix we simply trace the angular averaged spin dependent part of the density matrix with the appropriate spin operator

$$\langle s_i \rangle = \mathrm{Tr}\left[\frac{1}{2}\sigma_i\left(\frac{1}{2}\boldsymbol{\sigma}\cdot\overline{\boldsymbol{s}}\right)\right]. \tag{9}$$

For a general magnetisation with all gradients present we get a general form for the spin density is

$$\begin{aligned}\langle s\rangle_i =&(\boldsymbol{E}\cdot\nabla)m_i + \nabla(\boldsymbol{m}\cdot\boldsymbol{E}) + (\nabla\cdot\boldsymbol{m})E_i + (\boldsymbol{m}\times(\boldsymbol{E}\cdot\nabla)\boldsymbol{m})_i + (\boldsymbol{m}\times\nabla(\boldsymbol{m}\cdot\boldsymbol{E}))_i\\ &+ (\boldsymbol{m}\times(\nabla\cdot\boldsymbol{m})\boldsymbol{E})_i + m_i(\boldsymbol{E}\cdot\nabla)(\boldsymbol{m}\cdot\boldsymbol{m}) + m_i(\boldsymbol{m}\cdot\nabla)(\boldsymbol{E}\cdot\boldsymbol{m}) + m_i(\boldsymbol{E}\cdot\boldsymbol{m})(\nabla\cdot\boldsymbol{m})\\ &+ m_j m_k(\boldsymbol{E}\cdot\nabla)(\boldsymbol{m}\cdot\boldsymbol{m}) + m_j m_k(\boldsymbol{m}\cdot\nabla)(\boldsymbol{E}\cdot\boldsymbol{m}) + m_j m_k(\boldsymbol{E}\cdot\boldsymbol{m})(\nabla\cdot\boldsymbol{m}),\end{aligned} \tag{10}$$

where each of these terms has a tensor coefficient. These are not shown here as each term is very large and complex.

3 Appendix C: The scattering term in the Born approximation

In the Born approximation the scattering term takes the form

$$J(\hat{f}) = \frac{1}{\hbar^2}\int_0^\infty dt' e^{-\eta t'}\left[\hat{U}, e^{-i\hat{H}t'/\hbar}[\hat{U},\hat{f}]e^{i\hat{H}t'/\hbar}\right]_{\boldsymbol{kk}}. \tag{11}$$

The commutator in the integral is

$$\frac{1}{\hbar^2}\left[\hat{U}, e^{-i\hat{H}t'/\hbar}[\hat{U},\hat{f}]e^{i\hat{H}t'/\hbar}\right]_{\boldsymbol{kk}} = \frac{1}{\hbar^2}\sum_{\boldsymbol{k}'}|U_{\boldsymbol{kk}'}|^2\left[e^{-i\hat{H}_{\boldsymbol{k}'}t'/\hbar}\left(f_{\boldsymbol{k}} - f_{\boldsymbol{k}'}\right)e^{iH_{\boldsymbol{k}}t'/\hbar} + e^{-iH_{\boldsymbol{k}}t'/\hbar}\left(f_{\boldsymbol{k}} - f_{\boldsymbol{k}'}\right)e^{iH_{\boldsymbol{k}'}t'/\hbar}\right], \tag{12}$$

$\hat{f}$ can be split into a scalar and spin dependent parts $\hat{f}_k = n_k\mathbb{I} + \frac{1}{2}\,\boldsymbol{s}\cdot\boldsymbol{\sigma}$ so the commutator will become

$$\begin{aligned}&\frac{1}{\hbar^2}\sum_{\boldsymbol{k}'}|U_{\boldsymbol{kk}'}|^2\left[e^{-i\hat{H}_{\boldsymbol{k}'}t'/\hbar}e^{iH_{\boldsymbol{k}}t'/\hbar} + e^{-iH_{\boldsymbol{k}}t'}e^{iH_{\boldsymbol{k}'}t'/\hbar}\right](n_{\boldsymbol{k}} - n_{\boldsymbol{k}'})\\ &\qquad\frac{1}{2}|U_{\boldsymbol{kk}'}|^2\left[e^{-i\hat{H}_{\boldsymbol{k}'}t'/\hbar}\boldsymbol{\sigma}e^{iH_{\boldsymbol{k}}t'/\hbar} + e^{-iH_{\boldsymbol{k}}t'/\hbar}\boldsymbol{\sigma}e^{iH_{\boldsymbol{k}'}t'/\hbar}\right]\cdot(\boldsymbol{s}_{\boldsymbol{k}} - \boldsymbol{s}_{\boldsymbol{k}'}).\end{aligned} \tag{13}$$

Carrying out the time integral while only including terms to first order in the SO-field gives the following disorder averaged scattering term:

$$\begin{aligned}\hat{J}(\hat{f}) = &\frac{2\pi n_i}{\hbar}\sum_{\boldsymbol{k}'}|U_{\boldsymbol{kk}'}|^2\left[\delta(\epsilon_0 - \epsilon_0') + \frac{\hbar}{2}\boldsymbol{\sigma}\cdot(\boldsymbol{\Omega}_k - \boldsymbol{\Omega}_{k'})\frac{\partial}{\partial\epsilon_0}\delta(\epsilon_0 - \epsilon_0')\right](n_{\boldsymbol{k}} - n_{\boldsymbol{k}'}) +\\ &\frac{1}{2}|U_{\boldsymbol{kk}'}|^2\left[\boldsymbol{\sigma}\delta(\epsilon_0 - \epsilon_0') + \frac{\hbar}{4}\left(\{\boldsymbol{\sigma},\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_k\} - \{\boldsymbol{\sigma},\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_{k'}\}\right)\frac{\partial}{\partial\epsilon_0}\delta(\epsilon_0 - \epsilon_0')\right]\cdot(\boldsymbol{s}_{\boldsymbol{k}} - \boldsymbol{s}_{\boldsymbol{k}'}).\end{aligned} \tag{14}$$

Here the delta functions have been expanded about ϵ_0 using a Taylor expansion. This term can be broken up into two different terms based on their dependence on $\boldsymbol{\Omega}$

$$\hat{J}(\hat{f}) = \hat{J}_0(\hat{f}) + \hat{J}_\Omega(\hat{f}). \tag{15}$$

Note: Here Ω represents all spin dependent parts of the Hamiltonian.

4 Appendix D: General solution to the spin dependent part of the kinetic equation

The kinetic equation for the spin density matrix can be written as

$$\frac{\partial S_{Ek}}{\partial t} + \frac{i}{\hbar}[H, S_{Ek}] + \hat{J}(S_{Ek}) = D_{Ek}\,, \tag{16}$$

where $S_{Ek} = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{s}_{Ek}$. Now let $S_{Ek} = \overline{S_{Ek}} + T_{Ek}$, where $\overline{S_{Ek}}$ is the angular average of S_{Ek} and represents spin polarisation, and T_{Ek} is the anisotropic component and captures the spin currents. For our perturbative solution $\hat{J}$ will just be $\hat{J}_0$, other scattering terms will be included in the driving term. Splitting H and S into their isotropic and anisotropic components gives

$$\frac{\partial T_{Ek}}{\partial t} + \frac{\partial \overline{S_{Ek}}}{\partial t} + \frac{i}{\hbar}[H + \overline{H} - \overline{H}, T_{Ek} + \overline{S_{Ek}}] + \frac{T_{Ek} + \overline{S_{Ek}} - \overline{S_{Ek}}}{\tau_0} = D_{Ek}\,. \tag{17}$$

This equation can be split into a pair of isotropic and anisotropic equations

$$\frac{\partial \overline{S_{Ek}}}{\partial t} + \frac{i}{\hbar}[\overline{H}, \overline{S_{Ek}}] + \frac{i}{\hbar}\overline{[H, T_{Ek}]} = \overline{D_{Ek}}\,, \tag{18}$$

$$\frac{\partial T_{Ek}}{\partial t} + \frac{i}{\hbar}[H, T_{Ek}] + \frac{T_{Ek}}{\tau_0} = D_{Ek} - \overline{D_{Ek}} - \frac{i}{\hbar}[H - \overline{H}, \overline{S_{Ek}}] + \frac{i}{\hbar}\overline{[H, T_{Ek}]}\,. \tag{19}$$

To solve these coupled equations we followed the same steps outlined in Bi's 2013 PRB[**?**]. Substituting (13) into (14) gives

$$\frac{\partial T_{Ek}}{\partial t} + \frac{i}{\hbar}[H, T_{Ek}] + \frac{T_{Ek}}{\tau_0} = D_{Ek} - \frac{i}{\hbar}[H, \overline{S_{Ek}}] - \frac{\partial \overline{S_{Ek}}}{\partial t}\,. \tag{20}$$

This equation is identical to the equation from [**?**]. So, it is clear that the solution for the anisotropic component T_{Ek} is identical. Now let $\overline{S_{Ek}} = \frac{1}{2}\boldsymbol{\sigma} \cdot \overline{\boldsymbol{s}_{Ek}}$, $T_{Ek} = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{t}_{Ek}$ and $D_{Ek} = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{d}_{Ek}$. The full solution for the anisotropic spin density matrix is

$$\boldsymbol{t}_{Ek} = \hat{\boldsymbol{\Omega}}_k \times \left(\boldsymbol{d}_{Ek} + \frac{\overline{\boldsymbol{s}_{Ek}}}{\tau_0}\right)\frac{\Omega\tau^2}{1 + \Omega_k^2\tau_0^2} + \frac{(\boldsymbol{d}_{Ek}\tau_0)}{1 + \Omega_k^2\tau_0^2} + \frac{(\boldsymbol{d}_{Ek} \cdot \hat{\boldsymbol{\Omega}}_k)\hat{\boldsymbol{\Omega}}_k\Omega_k^2\tau_0^3}{1 + \Omega_k^2\tau_0^2} - \frac{(\mathcal{A}\overline{\boldsymbol{s}_{Ek}})\Omega_k^2\tau_0^2}{1 + \Omega_k^2\tau_0^2}\,, \tag{21}$$

where $\mathcal{A} = \mathbb{I} - \mathcal{B}$ and $\mathcal{B}_{ij} = \hat{\Omega}_{\boldsymbol{k}i}\hat{\Omega}_{\boldsymbol{k}j}$. Now, the equation for $\overline{S_{Ek}}$ is

$$\frac{\partial \overline{S_{Ek}}}{\partial t} + \frac{i}{\hbar}[\overline{H}, \overline{S_{Ek}}] + \frac{i}{\hbar}\overline{[H, T_{Ek}]} = \overline{D_{Ek}}\,, \tag{22}$$

transforming this into an equation for $\overline{\boldsymbol{s}}$ gives

$$\frac{\partial \overline{\boldsymbol{s}_{Ek}}}{\partial t} - \overline{\boldsymbol{\Omega}_k} \times \overline{\boldsymbol{s}_{Ek}} - \overline{\boldsymbol{\Omega}_k \times \boldsymbol{t}_{Ek}} = \boldsymbol{d}_{Ek}\,. \tag{23}$$

Substituting (16) into (18) gives

$$\frac{1}{\tau_0}\left[\overline{\left(\frac{\Omega_k^2\tau_0^2}{1 + \Omega_k^2\tau_0^2}\right)\mathcal{A}}\right]\overline{\boldsymbol{s}_{Ek}} - \frac{\overline{\boldsymbol{\Omega}_k} \times \overline{\boldsymbol{s}_{Ek}}}{1 + \Omega_k^2\tau_0^2} = \overline{\boldsymbol{d}_{Ek}} - \overline{\left(\frac{\Omega_k^2\tau_0^2}{1 + \Omega_k^2\tau_0^2}\right)\mathcal{A}\boldsymbol{d}_{Ek}} + \frac{\overline{(\boldsymbol{\Omega}_{\boldsymbol{k}} \times \boldsymbol{d}_{Ek})\,\tau_0}}{1 + \Omega_k^2\tau_0^2}\,. \tag{24}$$

This system of linear equations can be solved in both the weak and DP scattering limits to find the spin dependent density matrix response. In the weak scattering limit $\Omega_k\tau_0 \gg 1$

$$\boldsymbol{t}_{Ek} = \frac{\hat{\boldsymbol{\Omega}}_k}{\Omega_k} \times \left(\boldsymbol{d}_{Ek} + \frac{\overline{\boldsymbol{s}_{Ek}}}{\tau_0}\right) + (\boldsymbol{d}_{Ek} \cdot \hat{\boldsymbol{\Omega}}_k)\hat{\boldsymbol{\Omega}}_k\tau_0 - \mathcal{A}\overline{\boldsymbol{s}_{Ek}}\,, \tag{25}$$

$$\frac{1}{\tau_0}\overline{\mathcal{A}}\,\overline{\boldsymbol{s}_{Ek}} - \frac{\overline{\boldsymbol{\Omega}_k} \times \overline{\boldsymbol{s}_{Ek}}}{\Omega_k^2\tau_0^2} \approx \overline{\boldsymbol{d}_{Ek}} - \overline{\mathcal{A}\boldsymbol{d}_{Ek}}\,. \tag{26}$$

For the opposite limit $\Omega_k\tau_0 \ll 1$

$$\boldsymbol{t}_{Ek} = \boldsymbol{\Omega}_k \times \overline{\boldsymbol{s}_{Ek}}\tau_0 + \boldsymbol{d}_{Ek}\tau_0\,, \tag{27}$$

$$\frac{1}{\tau_0}\left[\overline{\Omega_k^2\tau_0^2\,\mathcal{A}}\right]\overline{\boldsymbol{s}_{Ek}} - \overline{\boldsymbol{\Omega}_k} \times \overline{\boldsymbol{s}_{Ek}} = \overline{\boldsymbol{d}_{Ek}} - \overline{\Omega_k^2\tau_0^2\,\mathcal{A}\boldsymbol{d}_{Ek}} + \overline{(\boldsymbol{\Omega}_{\boldsymbol{k}} \times \boldsymbol{d}_{Ek})\,\tau_0}\,. \tag{28}$$

5 Appendix E: Solving the kinetic equation

The homogeneous kinetic equation is

$$\frac{\partial\langle\rho\rangle_{Ek}}{\partial t}+\frac{i}{\hbar}\left[H_k,\langle\rho\rangle_{Ek}\right]+\hat{J}\left(\langle\rho\rangle_{Ek}\right)=\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial\langle\rho\rangle_{0k}}{\partial k}\,, \tag{29}$$

where where $\langle\rho\rangle$ is the disorder averaged homogeneous density matrix. The equilibrium density matrix $\langle\rho\rangle_{0k}$ is

$$\langle\rho\rangle_{0k}=\frac{1}{2}\left[f_{\mathrm{FD}}\left(\varepsilon_{k+}\right)+f_{\mathrm{FD}}\left(\varepsilon_{k-}\right)\right]+\frac{1}{2}\left[f_{\mathrm{FD}}\left(\varepsilon_{k+}\right)-f_{\mathrm{FD}}\left(\varepsilon_{k-}\right)\right]\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\Omega}}_{\boldsymbol{k}}\,. \tag{30}$$

The kinetic equation can be separated into a scalar and spin dependent part. The scalar equation is

$$\frac{\partial n_{Ek}}{\partial t}+\hat{J}_0\left(n_{Ek}\right)=\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial n_{0k}}{\partial k}-\hat{J}_{\Omega}\left(S_{Ek}\right)\,, \tag{31}$$

and the spin dependent equation is

$$\frac{\partial S_{Ek}}{\partial t}+\frac{i}{\hbar}\left[H_k,S_{Ek}\right]+\hat{J}_0\left(S_{Ek}\right)=\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial S_{0k}}{\partial k}-\hat{J}_{\Omega}\left(n_{Ek}\right)\,, \tag{32}$$

These coupled equations were solved perturbatively in Ω, to first order in Ω. The solution to $n_{0,k}$ will simply be

$$\begin{aligned} n_{0,Ek}=&\frac{e\tau_0(k_m)\vec{E}}{\hbar}\cdot\frac{\partial n_{0k}}{\partial k}\\ \approx&-\frac{e\tau_0(k_m)}{\hbar}\vec{E}\cdot\frac{\partial\epsilon_{0k}}{\partial k}\delta(\epsilon_0-\epsilon_F)\,, \end{aligned} \tag{33}$$

where $k_m=\sqrt{k_x^2+k_y^2+\Lambda^2k_z^2}$. Now, this can be used to solve the equation for S_{Ek} using the solution shown earlier

$$\frac{\partial S_{Ek}}{\partial t}+\frac{i}{\hbar}\left[H_k,S_{Ek}\right]+\hat{J}_0\left(S_{Ek}\right)=\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial S_{0k}}{\partial k}-\hat{J}_{\Omega}\left(n_{0,Ek}\right)\,. \tag{34}$$

The two driving terms in this equation are

$$\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial S_{0k}}{\partial k}\approx-\boldsymbol{\sigma}\cdot\left[\frac{e\boldsymbol{E}}{2}\cdot\frac{\partial\boldsymbol{\Omega}_k}{\partial\vec{k}}\delta(\epsilon_0-\epsilon_F)+\boldsymbol{\Omega}_k\left(\frac{e\boldsymbol{E}}{2}\cdot\frac{\partial\epsilon_0}{\partial\vec{k}}\right)\frac{\partial}{\partial\epsilon_0}\delta(\epsilon_0-\epsilon_F)\right]\,, \tag{35}$$

and

$$\begin{aligned} \hat{J}_{\Omega}\left(n_{Ek}\right)=&\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\left[\boldsymbol{\Omega}_k\left(\frac{n_{Ek}}{k_m\tau_0(k_m)}\right)\frac{\partial k_m}{\partial\epsilon_0}-\boldsymbol{\Omega}_k\frac{\partial k_m}{\partial\epsilon_0}\frac{\partial}{\partial k_m}\left(\frac{\overline{n_{Ek}}}{\tau_0(k_m)}\right)\right.\\ &\left.-n_{Ek}\frac{\partial k_m}{\partial\epsilon_0}\frac{\partial}{\partial k_m}\left(\frac{\overline{\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)+\frac{\partial k_m}{\partial\epsilon_0}\frac{\partial}{\partial k_m}\left(\frac{\overline{n_{Ek}\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)\right]\\ =&\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\left[\boldsymbol{\Omega}_k\left(\frac{n_{Ek}}{k_m\tau_0(k_m)}\right)\frac{\partial k_m}{\partial\epsilon_0}-n_{Ek}\frac{\partial k_m}{\partial\epsilon_0}\frac{\partial}{\partial k_m}\left(\frac{\overline{\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)\right]. \end{aligned} \tag{36}$$

Now the kinetic equation was as shown in the previous section

$$\frac{1}{\tau_0}\left[\overline{\left(\frac{\Omega_k^2\tau_0^2}{1+\Omega_k^2\tau_0^2}\right)\mathcal{A}}\right]\overline{\boldsymbol{s}_{Ek}}-\frac{\overline{\boldsymbol{\Omega}_k}\times\overline{\boldsymbol{s}_{Ek}}}{1+\Omega_k^2\tau_0^2}=\overline{\boldsymbol{d}_{Ek}}-\overline{\left(\frac{\Omega_k^2\tau_0^2}{1+\Omega_k^2\tau_0^2}\right)\mathcal{A}\boldsymbol{d}_{Ek}}+\frac{\overline{\left(\boldsymbol{\Omega}_{\boldsymbol{k}}\times\boldsymbol{d}_{Ek}\right)\tau_0}}{1+\Omega_k^2\tau_0^2}\,. \tag{37}$$

The driving term is purely anisotropic, so all the components of on the right hand side of this equation vanish and hence

$$\overline{\boldsymbol{s}_{Ek}}=0\,. \tag{38}$$

The anisotropic component is found using the solution discussed in Appendix B. For our system the $\boldsymbol{t}_{Ek}$ is

$$\begin{aligned} \boldsymbol{t}_{Ek}=&\frac{1}{2}\frac{1}{1+\Omega_k^2\tau_0^2}\left(\hbar\tau_0^2n_{Ek}\frac{\partial k_m}{\partial\epsilon_0}\left(\boldsymbol{\Omega}_k\times\left(\frac{\partial}{\partial k_m}\frac{\overline{\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)\right)-\tau_0^2\left[\boldsymbol{\Omega}_k\times\left(e\boldsymbol{E}\cdot\frac{\partial\boldsymbol{\Omega}_k}{\partial\vec{k}}\right)\right]\delta(\epsilon_0-\epsilon_F)-\frac{\hbar n_{Ek}\boldsymbol{\Omega}_k}{k_m}\frac{\partial k_m}{\partial\epsilon_0}\right.\\ &+\hbar n_{Ek}\tau_0\left(\frac{\partial}{\partial k_m}\frac{\overline{\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)\frac{\partial k_m}{\partial\epsilon_0}-\tau_0\left(e\boldsymbol{E}\cdot\frac{\partial\boldsymbol{\Omega}_k}{\partial\vec{k}}\right)\delta(\epsilon_0-\epsilon_F)-\tau_0\boldsymbol{\Omega}_k\left(e\boldsymbol{E}\cdot\frac{\partial\epsilon_0}{\partial\vec{k}}\right)\frac{\partial}{\partial\epsilon_0}\delta(\epsilon_o-\epsilon_F)-\\ &\Omega_k\tau_0^3\hat{\boldsymbol{\Omega}}_k\,\boldsymbol{\Omega}_k\cdot\left(\hbar\boldsymbol{\Omega}_k\left(\frac{n_{Ek}}{k_m\tau_0(k_m)}\right)\frac{\partial k_m}{\partial\epsilon_0}-\hbar n_{Ek}\frac{\partial k_m}{\partial\epsilon_0}\frac{\partial}{\partial k_m}\left(\frac{\overline{\boldsymbol{\Omega}_k}}{\tau_0(k_m)}\right)+\frac{e\boldsymbol{E}}{2}\cdot\frac{\partial\boldsymbol{\Omega}_k}{\partial\vec{k}}\delta(\epsilon_0-\epsilon_F)+\right.\\ &\left.\left.\boldsymbol{\Omega}_k\left(\frac{e\boldsymbol{E}}{2}\cdot\frac{\partial\epsilon_0}{\partial\vec{k}}\right)\frac{\partial}{\partial\epsilon_0}\delta(\epsilon_0-\epsilon_F)\right)\right). \end{aligned} \tag{39}$$

This is simplified by taking it in either the weak scattering or DP limit.

6 Appendix F: Inhomogeneous density matrix response

The kinetic equation to first order in the gradient is

$$\frac{\partial f_{Ek}}{\partial t} + \frac{i}{\hbar}\left[H_k, f_{Ek}\right] + \hat{J}\left(f_{Ek}\right) = -\frac{1}{2\hbar}\left\{\frac{\partial H}{\partial \vec{q}}, \vec{\nabla} f_h\right\} + \frac{1}{2\hbar}\left\{\vec{\nabla} H, \frac{\partial f_h}{\partial \vec{q}}\right\}. \tag{40}$$

Similarly to the homogeneous equation, this equation can be separated into a scalar and spin dependent part. The scalar equation will be

$$\frac{\partial n_\nabla}{\partial t} + \hat{J}_0\left(n_\nabla\right) = -\hat{J}_\Omega\left(S_\nabla\right) - \frac{1}{\hbar}\left[\frac{\partial \epsilon_0}{\partial \vec{k}} \cdot \vec{\nabla} n_{Ek} + \frac{\hbar}{2}\frac{\partial \boldsymbol{\Omega}}{\partial \vec{k}} \cdot \vec{\nabla} \boldsymbol{t}_{Ek}\right] + \frac{1}{2}\left[\frac{\partial \boldsymbol{t}_{Ek}}{\partial \vec{k}} \cdot \vec{\nabla} \boldsymbol{\Omega}\right], \tag{41}$$

and the spin dependent equation will be

$$\frac{\partial S_\nabla}{\partial t} + \frac{i}{\hbar}\left[H_k, S_\nabla\right] + \hat{J}_0\left(S_\nabla\right) = -\hat{J}_\Omega\left(n_\nabla\right) + \frac{1}{2}\left[\left(\vec{\nabla}\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}\right) \cdot \frac{\partial}{\partial \vec{k}} n_{Ek}\right] - \frac{1}{\hbar}\left[\frac{\hbar}{2}\vec{\nabla} n_{Ek} \cdot \left(\frac{\partial \boldsymbol{\Omega}}{\partial \vec{k}} \cdot \boldsymbol{\sigma}\right) + \frac{\partial \epsilon_0}{\partial \vec{k}} \cdot \left(\vec{\nabla} \boldsymbol{t}_{Ek} \cdot \boldsymbol{\sigma}\right)\right]. \tag{42}$$

This equation was again solved perturbatively in Ω_k, to first order in Ω_k. The coupled equations to first order in Ω_k are

$$\frac{\partial n_\nabla}{\partial t} + \hat{J}_0\left(n_\nabla\right) \approx 0\,, \tag{43}$$

and

$$\frac{\partial S_\nabla}{\partial t} + \frac{i}{\hbar}\left[H_k, S_\nabla\right] + \hat{J}_0\left(S_\nabla\right) = \frac{1}{2}\left(\vec{\nabla}\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}\right) \cdot \frac{\partial}{\partial \vec{k}} n_{Ek} - \frac{1}{\hbar}\frac{\partial \epsilon_0}{\partial \vec{k}} \cdot \left(\vec{\nabla} \boldsymbol{t}_{Ek} \cdot \boldsymbol{\sigma}\right). \tag{44}$$

This was solved using the method outlined in Appendix D.

Note: In order to solve the kinetic equations the following assumptions were made $\frac{A_0^2}{M_0^2}, \frac{B_0^2}{M_0^2}, \frac{A_0 B_0}{M_0^2} \ll 1$. This is necessarily true where our SW transform is valid.

References

[1] R. Winkler, *Spin–Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (2003).