Spontaneous flows and dynamics of full-integer topological defects in polar active matter
Supplementary Information

Jonas Rønning,[1] Julian Renaud,[2] Amin Doostmohammadi,[3] and Luiza Angheluta[1]

[1] *Njord Centre, Department of Physics, University of Oslo, P. O. Box 1048, 0316 Oslo, Norway*
[2] *École Normale Supérieure, PSL Research University, 45 rue d'Ulm, 75005 Paris, France*
[3] *Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, Copenhagen, Denmark*

In this supplementary material, we provide theoretical details for solving for Stokes flows with sources given by isolated polar defects. The dimensionless forms of the momentum equations are

$$(1-\zeta^2\nabla^2)\vec{u} = \vec{F}_p + \vec{F}_a - \nabla P, \tag{1}$$

$$\nabla^2 P = \nabla\cdot\vec{F}_p + \nabla\cdot\vec{F}_a, \tag{2}$$

where we denote as $\vec{F}_a$ the active dipolar force and $\vec{F}_p$ the active polar force. From the linearity of this Stokasian flow, we can use the superposition principle to write the solutions of the pressure and velocity as a sum of their polar and the dipolar force contribution, namely $\vec{u} = \vec{u}_a + \vec{u}_p$ and similar for pressure. Each contribution is an integral solution given by

$$P_{a,p}(\vec{r}) = \frac{1}{2\pi}\int d\vec{r'}\ln(|\vec{r}-\vec{r'}|)\nabla'\cdot\vec{F}_{a,p}, \tag{3}$$

$$\vec{u}_{a,p}(\vec{r}) = \frac{1}{2\pi\zeta^2}\int d\vec{r'}K_0\left(\frac{|\vec{r}-\vec{r'}|}{\zeta}\right)\left(\vec{F}_{a,p}(\vec{r'}) - \nabla' P_{a,p}(\vec{r'})\right). \tag{4}$$

Using complex analysis, we solve these integrals analytically for active forces induced by isolated polar defects as in Ref. [1, 2]. For this we use the parameterization of the polarization field for an isolated topological defect given by [3]

$$\vec{p}(r,\theta) = \chi(r)[\cos(q\theta+\phi)\vec{e}_x + \sin(q\theta+\phi)\vec{e}_y], \tag{5}$$

where (r,θ) are polar coordinates centered at the defect position (origin), $\chi(r)$ is a core function, $q=\pm1$ is topological charge and ϕ is a constant background phase. For most parts, we assume that the defects are pointwise, i.e $\chi(r)=1$. The exception is when we evaluate the flow field at the defect position in the friction dominated limit, where we use that $\chi(r)\sim r$ to avoid getting a multi-valued velocity.

I. DIPOLAR FORCES

In this section, we derive analytical expressions of the flow fields induced by the active dipolar force $\vec{F}^a = (\tilde{\alpha}_0/\Gamma)\nabla\cdot(\vec{p}\vec{p}^T - \frac{p^2}{2}\mathbf{I})$ [4–6].

A. The negative defect

We start our calculations by considering the dipolar force generated by a $q=-1$ defect. As mentioned in the main document, the effect of the uniform background polarisation on the negative defect is to rotate it, thus for simplicity we can fix $\phi=0$ and write the force components as:

$$F^-_{ax} = \frac{\tilde{\alpha}_0}{2\Gamma}[\partial_x\cos(2\theta) - \partial_y\sin(2\theta)], \tag{6}$$

$$F^-_{ay} = -\frac{\tilde{\alpha}_0}{2\Gamma}[\partial_x\sin(2\theta) + \partial_y\cos(2\theta)]. \tag{7}$$

Using that $\theta = \arctan(y/x)$, we express the force as

$$\vec{F}^-_a = -\frac{\tilde{\alpha}_0}{\Gamma r}\cos(-3\theta)\vec{e}_x - \frac{\tilde{\alpha}_0}{\Gamma r}\sin(-3\theta)\vec{e}_y. \tag{8}$$

where $\vec{e}_x = [1,0]$ and $\vec{e}_y = [0,1]$.

a. ***Flow pressure:*** By an integration by parts in Eq. (3), we find that the pressure field induced by the negative defect can be rewritten as

$$P_a^- = -\frac{1}{2\pi}\int d\vec{r'}\left(\vec{F}_x^a\frac{(x'-x)}{|\vec{r'}-\vec{r}|^2}+\vec{F}_y^a\frac{(y'-y)}{|\vec{r'}-\vec{r}|^2}\right). \tag{9}$$

To evaluate it, we first do a change of variables to the complex coordinates $z' = x'+iy'$ and its conjugate $\bar{z}' = x'-iy'$, and similar for z and $\bar{z}$. To further isolate singularities in the complex plane, we do a second variable change to complex polar forms $z' = r'\hat{w}$ with $\hat{w} = e^{i\theta'}$. The integrals then can be expressed as

$$P_a^- = \frac{i\tilde{\alpha}_0}{2\pi\Gamma}\int_0^\infty dr'r'\oint_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\left(-\frac{1}{4r'}\left(\hat{w}^3+\hat{w}^{-3}\right)\frac{(r'\hat{w}-z)+(r'\hat{w}^{-1}-\bar{z})}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})}-\frac{1}{4r'}(\hat{w}^3-\hat{w}^{-3})\frac{(r'\hat{w}-z)-(r'\hat{w}^{-1}-\bar{z})}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})}\right). \tag{10}$$

After some algebraic manipulations to simplify the integrand expression, we can reduce it to

$$P_a^- = \frac{\tilde{\alpha}_0}{4\pi i\Gamma}\int_0^\infty dr'\oint_{|\hat{w}|=1}d\hat{w}\left(\frac{1}{r'\hat{w}^4(\hat{w}-z/r')}-\frac{\hat{w}^3}{\bar{z}(\hat{w}-r'/\bar{z})}\right). \tag{11}$$

The contour integral over $\hat{w}$ can be evaluated by using the residue theorem. Note that the poles at $\hat{w} = z/r'$ and $\hat{w} = r'/\bar{z}$ are only inside the unit circle $|\hat{w}| = 1$ when $|z| = r < r'$ and $r > r'$ respectively. Thus,

$$\begin{aligned} P_a^- &= \frac{\tilde{\alpha}_0}{4\pi i\Gamma}\int_0^\infty dr'2\pi i\left(\frac{(r')^3}{z^4}|_{r<r'}-\frac{(r')^3}{z^4}-\frac{(r'^3)}{\bar{z}^4}|_{r>r'}\right) \\ &= -\frac{\tilde{\alpha}_0}{4\Gamma}\frac{x^4-6x^2y^2+y^4}{r^4}. \end{aligned} \tag{12}$$

The pressure gradient follow straightforwardly as

$$\partial_x P_a^- = \frac{\tilde{\alpha}_0}{2r\Gamma}(\cos 5\theta - \cos 3\theta), \tag{13}$$

$$\partial_y P_a^- = \frac{\tilde{\alpha}_0}{2r\Gamma}(\sin 5\theta + \sin 3\theta), \tag{14}$$

and is subtracted from the dipolar force for incompressible flows.

b. ***Flow velocity:*** The total flow source term can be written in the complex form as:

$$F_a^- - 2\partial_{\bar{z}}P_a^- = -\frac{\tilde{\alpha}_0}{2\Gamma}\left(\frac{\bar{z}}{z^2}+\frac{z^2}{\bar{z}^3}\right), \tag{15}$$

where we have used that the gradient in complex coordinates is $2\partial_{\bar{z}} = \partial_x + i\partial_y$. By a change of variables to complex coordinates in Eq. (4), we rewrite the integral form of the complex velocity field $u_a^- = u_{a,x}^- + iu_{a,y}^-$ as

$$u_a^- = -\frac{\tilde{\alpha}_0 i}{8\pi\Gamma\zeta^2}\int dz'd\bar{z}'K_0\left(\frac{|z'-z|}{\zeta}\right)\left(\frac{\bar{z}'}{(z')^2}+\frac{z'^2}{\bar{z}'^3}\right), \tag{16}$$

which can be transformed into

$$u_a^- = -\frac{\tilde{\alpha}_0}{4i\pi\Gamma\zeta^2}\int_0^\infty dr'r'K_0\left(\frac{r'}{\zeta}\right)\oint_{|\hat{w}|=1}d\hat{w}\left(\frac{r'+\bar{z}\hat{w}}{\hat{w}^2r'^2(\hat{w}+z/r')^2}+\frac{(\hat{w}^2(r'\hat{w}+z)^2}{\bar{z}^3(r'/\bar{z}+\hat{w})^3}\right) \tag{17}$$

The integral over $\hat{w}$ can be evaluated by the residual theorem, such that the velocity can be expressed as

$$u_a^- = -\frac{\tilde{\alpha}_0}{2\Gamma\zeta^2}\left(\zeta^2\int_0^{r/\zeta}dttK_0(t)\left(\frac{\bar{z}z}{z^3}+\frac{\bar{z}^2z^2}{\bar{z}^5}\right)-\zeta^4\int_0^{r/\zeta}dtt^3K_0(t)\left(\frac{2}{z^3}+\frac{6\bar{z}z}{\bar{z}^5}\right)+\frac{6\zeta^6}{\bar{z}^5}\int_0^{r/\zeta}dtt^5K_0(t)\right), \tag{18}$$

leading to

$$\begin{aligned} u_a^-(r,\theta) = -\frac{\tilde{\alpha}_0}{2\Gamma\zeta}\Bigg(&[1-\hat{r}K_1(\hat{r})]\left(\frac{1}{\hat{r}}e^{-3i\theta}+\frac{1}{\hat{r}}e^{5i\theta}\right)-[4-\hat{r}^3K_1(\hat{r})-2\hat{r}^2K_2(\hat{r})]\left(\frac{2}{\hat{r}^3}e^{-3i\theta}+\frac{6}{\hat{r}^3}e^{5i\theta}\right) \\ &+\frac{6}{\hat{r}^5}e^{5i\theta}\left[64-4\hat{r}^2(8+\hat{r}^2)K_0(\hat{r})-\hat{r}(8+\hat{r}^2)^2K_1(\hat{r})\right]\Bigg). \end{aligned} \tag{19}$$

Where we have scaled the radial variable to $\hat{r} = r/\zeta$. The corresponding vorticity, $\omega = \partial_y u_x - \partial_x u_y$, reduces to

$$\omega_a^-(r,\theta) = -\frac{\tilde{\alpha}_0}{\hat{r}^4\zeta^2\Gamma}\sin 4\theta\left(-48 + 4\hat{r}^2 + \hat{r}^2(24+\hat{r}^2)K_0(\hat{r}) + 8\hat{r}(6+\hat{r}^2)K_1(\hat{r})\right). \tag{20}$$

B. The positive defect

The $+1$ defect is more special in that an additional constant phase to the complex field $\psi = \chi(r)e^{i\theta+i\phi}$ changes the defect structure from vortex to spiral or aster. In the general case ($\phi \neq 0$), the dipolar force induced by $+1$ defect is

$$\vec{F}_a^+ = \frac{\tilde{\alpha}_0}{\Gamma r^2}\vec{r}\cos(2\phi) - \frac{\tilde{\alpha}_0}{\Gamma r^2}\vec{r}^{\perp}\sin(2\phi) = \tilde{F}_a^+ + \hat{F}_a^+, \tag{21}$$

where $\vec{r}^{\perp} = (y, -x)$. We notice that $\tilde{F}_a^+$ is a source/sink which gets removed by pressure for incompressible flows, $\hat{F}_a^+$ contributes to vorticity as seen below.

First, from the linearity of the main equations, we can decompose the pressure as suporposition of different contributions

$$P_a^+ = \tilde{P}_a^+\cos(2\phi) + \hat{P}_a^+\sin(2\phi). \tag{22}$$

We will start by looking at the $\tilde{P}_a^+$ term, which reduces to the integral form

$$\tilde{P}_a^+ = \frac{\tilde{\alpha}_0 i}{8\pi\Gamma}\int_0^\infty dr' \oint\limits_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\left((\hat{w}+\hat{w}^{-1})\left(\frac{1}{(r'\hat{w}-z)}+\frac{1}{(r'\hat{w}^{-1}-\bar{z})}\right)+(\hat{w}-\hat{w}^{-1})\left(\frac{1}{(r'\hat{w}-z)}-\frac{1}{(r\hat{w}^{-1}-\bar{z})}\right)\right). \tag{23}$$

which can be simplified to

$$\tilde{P}_a^+ = \frac{\tilde{\alpha}_0 i}{4\pi\Gamma}\int_0^\infty dr' \oint\limits_{|\hat{w}|=1} d\hat{w}\left(\frac{1}{r'(\hat{w}-z/r')} - \frac{1}{\bar{z}\hat{w}(\hat{w}-r'/\bar{z})}\right) = -\alpha_0\ln\frac{L}{r}. \tag{24}$$

where the core size L is introduced to remove the small-scale divergence in the limit of pointwise defects. Notice that the gradient of this pressure cancel out $\tilde{F}_a^+$ as expected from the incompressibility constraint. Similarly, we find that the other pressure contribution and show that it actually vanishes

$$\begin{aligned}\hat{P}_a^+ &= \frac{i\tilde{\alpha}_0}{2\pi\Gamma}\int_0^\infty dr'r' \oint\limits_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\left(-\frac{1}{2ir'^2}r'(\hat{w}-\hat{w}^{-1})\frac{\Re(r'\hat{w}-z)}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})} + \frac{1}{2r'^2}r'(\hat{w}+\hat{w}^{-1})\frac{\Im(r'\hat{w}-z)}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})}\right)\\ &= -\frac{\tilde{\alpha}_0}{4\pi\Gamma}\int_0^\infty dr' \oint\limits_{|\hat{w}|=1} d\hat{w}\left(\frac{1}{r'(\hat{w}-z/r')}+\frac{1}{\bar{z}\hat{w}(\hat{w}-r'/\bar{z})}\right) = 0,\end{aligned} \tag{25}$$

which implies that this rotational part of the force $\hat{F}_a^+$ does not induce any pressure. Writing this force in a complex form

$$\hat{F}_a^+ = \frac{i\tilde{\alpha}_0}{\bar{z}\Gamma}, \tag{26}$$

we express the complex velocity field as

$$\begin{aligned}u_a^+ &= \frac{i\tilde{\alpha}_0\sin(2\phi)}{4\pi\zeta^2\Gamma}\int dz'd\bar{z}'K_0\left(\frac{|z'-z|}{\zeta}\right)\frac{i}{\bar{z}'}\\ &= \frac{\tilde{\alpha}_0\sin(2\phi)}{2\pi\zeta^2\Gamma}\int_0^\infty dr'r'K_0\left(\frac{r'}{\zeta}\right)\oint\limits_{|\hat{w}|=1}\frac{d\hat{w}}{\bar{z}(r'/\bar{z}+\hat{w})}\\ &= \frac{\tilde{\alpha}_0\sin(2\phi)}{\Gamma\zeta\hat{r}}(1-\hat{r}K_1(\hat{r}))e^{i(\theta+\pi/2)}.\end{aligned} \tag{27}$$

where $\hat{r} = r/\zeta$. The corresponding vorticity field is then

$$\omega_a^+ = \frac{\tilde{\alpha}_0}{\Gamma\zeta^2} K_0(\hat{r}) \sin(2\phi), \tag{28}$$

which diverges logarithmically as $r \to 0$ since we have not included core size effects.

a. ***Zero friction limit:*** We notice that the absence of friction has an interesting implication. In this limit, the flow equations reduce to

$$-\tilde{\eta}\nabla^2 \vec{u}_a^+ = -\nabla \hat{P}_a^+ + \vec{F}_a, \tag{29}$$

$$\nabla \cdot \vec{u}_a^+ = 0, \tag{30}$$

with $\tilde{\eta} = \eta/\xi^2$, and the corresponding integral solution for the velocity is modified to

$$\vec{u}_a^+ = -\frac{1}{2\pi\tilde{\eta}} \int d\vec{r'} \ln(|\vec{r} - \vec{r'}|) \left(\vec{F}_a(\vec{r'}) - \nabla P_a^+(\vec{r'})\right), \tag{31}$$

Since the pressure is the same as before, we have that

$$u_a^+ = -\frac{i\tilde{\alpha}_0}{\tilde{\eta}\bar{z}} \sin(2\phi) \int_0^r dt t \ln(t) = \frac{i\tilde{\alpha}_0 \sin(2\phi)}{4\tilde{\eta}} r(1 - 2\ln(r)) e^{i\theta} \tag{32}$$

and its corresponding vorticity is

$$\omega_a^+ = -\frac{\tilde{\alpha}_0 \sin(2\phi)}{\tilde{\eta}} \ln r. \tag{33}$$

This changes sign for $r = \sqrt{e} \approx 1.7$ giving the two counter rotating regions and it is divergent for $r \to \infty$. The reason is because the Greens function changes sign. Note that in this case there is no intrinsic hydrodynamic lengthscale. Numerical simulations shows that the change of rotation is happening on the lengthscale of the defects core, $\sim \xi$, which comes from the free energy of the polar order parameter.

C. Defect pair: zero-viscosity limit

We consider a pair of oppositely charged defects in the analytically solvable limit of zero viscosity, i.e. $\zeta \to 0$, where the incompressible flow equations reduce to

$$\Gamma \vec{u}_a = \vec{F}_a - \nabla P_a, \tag{34}$$

$$\nabla \cdot \vec{u}_a = 0. \tag{35}$$

Making use of the complex representation and evaluating the dipolar force in terms of the complex order parameter $\psi = p_x + ip_y$, we can map the flow equations into

$$u_a = -2\partial_{\bar{z}} P_a + \frac{\tilde{\alpha}_0}{\Gamma} \partial_z \psi^2, \tag{36}$$

$$4\partial_z \partial_{\bar{z}} P_a = \frac{2\tilde{\alpha}_0}{\Gamma} \Re(\partial_z^2 \psi^2). \tag{37}$$

The integral solution of the pressure reads as

$$2P_a = \frac{i\tilde{\alpha}_0}{4\pi\Gamma} \int dz' d\bar{z}' \ln[(\bar{z} - \bar{z}')(z - z')] \Re(\partial_z^2 \psi^2). \tag{38}$$

We are interested in the derivative of the pressure and to ease the notation we define the factor $(\tilde{\alpha}_0/\Gamma)\mathcal{I}_a(z, \bar{z}) = -\partial_{\bar{z}} P_a$, which is given by the integral

$$\mathcal{I}_a(z, \bar{z}) = -\frac{i}{4\pi} \int dz' d\bar{z}' \frac{1}{\bar{z} - \bar{z}'} \Re(\partial_{z'}^2 \psi^2). \tag{39}$$

Furthermore, we are only interested in its value at the defect position, we introduce

$$\mathcal{I}_a \equiv \mathcal{I}_a(0, 0) = \frac{i}{4\pi} \int dz' d\bar{z}' \frac{1}{\bar{z}'} \Re(\partial_{z'}^2 \psi^2). \tag{40}$$

The equation for the velocity at the defect position can now be written as

$$u_a = \frac{\tilde{\alpha}_0}{\Gamma} \partial_z^2 \psi^2|_{|z|=0} + \frac{\tilde{\alpha}_0}{\Gamma} \mathcal{I}_a. \tag{41}$$

a. ***At* $+1$ *defect position:*** Let us place the $+1$ defect at the center $z = 0$ and the -1 defect at the position $w_n = R_1 + iR_2$, with $\vec{R}$ being the position vector, such that we can parameterise the order parameter as

$$\psi = \chi(z, \bar{z})\sqrt{\frac{z}{\bar{z}}}\sqrt{\frac{\bar{z} - \bar{w}_n}{z - w_n}}e^{i\phi}, \tag{42}$$

with ϕ being a homogeneous background orientation. We have neglected the core function for the negative defect, while keeping it for the positive defect so that we can evaluate the active force at its position. Near the defect position $\chi(z, \bar{z}) = a\sqrt{z\bar{z}}$ and

$$\partial_z\psi^2 = a^2\partial_z\left(z^2\frac{\bar{z} - \bar{w}_n}{z - w_n}\right)e^{2i\phi} = a^2(\bar{z} - \bar{w}_n)\frac{z^2 - 2zw_n}{(z - w_n)^2}e^{2i\phi}, \tag{43}$$

which vanishes at $z = 0$. Thus, there is no contribution from the active force to the defect velocity. However, the pressure has a net effect given by

$$\mathcal{I}_a^+ = \frac{i}{4\pi}\int dz d\bar{z}\frac{1}{\bar{z}}\left(\frac{w_n(\bar{z} - \bar{w}_n)}{\bar{z}(z - w_n)^3}e^{2i\phi} + \frac{\bar{w}_n(z - w_n)}{z(\bar{z} - \bar{w}_n)^3}e^{-2i\phi}\right). \tag{44}$$

This can be solved by a similar technique of complex integration and leads to

$$\mathcal{I}_a^+ = -\int_\Lambda^{|w_n|} dr\frac{1}{2r\bar{w}_n^2}\left(\frac{6r^2}{\bar{w}_n} - 2w_n\right)e^{-2i\phi} = \frac{w_n}{\bar{w}_n^2}\ln\left(\frac{|w_n|}{\Lambda}\right)e^{-2i\phi} - \frac{3}{2}\left(\frac{|w_n|^2}{\bar{w}_n^3} - \frac{\Lambda^2}{\bar{w}_n^2}\right)e^{-2i\phi} \tag{45}$$

The term containing $\Lambda^2/\bar{w}_n^2$ can be neglected since it is assumed that the defects are well separated. It is straightforward to show that core contributions with the ansatz $\chi = a\sqrt{z\bar{z}}$ are of the same order $O(\Lambda^2)$. The positive defect velocity induced by this pressure is

$$u_a^+ = \frac{\tilde{\alpha}_0}{\Gamma}\frac{w_n}{\bar{w}_n^2}e^{-2i\phi}\left(\ln\left(\frac{|w_n|}{\Lambda}\right) - \frac{3}{2}\right) \tag{46}$$

which tends to zero when separation between defects diverges, i.e. $|w_n| \to \infty$.

b. ***At the* -1 *defect position:*** Now, we place the negative defect t the origin and the positive defect at the position w_p, such that the order parameter takes the form

$$\psi = \chi(r)\sqrt{\frac{\bar{z}}{z}}\sqrt{\frac{z - w_p}{\bar{z} - \bar{w}_p}}e^{i\phi}, \tag{47}$$

from which we see that the active force at the defect position vanishes since

$$\partial_z\psi_{in}^2 = \frac{a^2\bar{z}^2}{(\bar{z} - \bar{w}_p)}e^{2i\phi}. \tag{48}$$

However, the pressure contribution is given by

$$\mathcal{I}_a^- = -\frac{1}{2\pi i}\int dr\oint d\hat{z}\left(\frac{\hat{z}^4\bar{w}_p}{r^3(\hat{z} - w_p/r)}e^{-2i\phi} - \frac{w_p}{r^2\bar{w}_p\hat{z}^3(\hat{z} - r/\bar{w}_p)}e^{2i\phi}\right). \tag{49}$$

which is solved to

$$\mathcal{I}_a^- = -\frac{w_p^4\bar{w}_p}{6|w_p|^6}e^{-2i\phi} - \frac{w_p\bar{w}_p^2}{4|w_p|^4}e^{2i\phi}. \tag{50}$$

Therefore, the velocity of the negative defect induced by this pressure is

$$u_a^- = -\frac{\tilde{\alpha}_0}{\Gamma}\left(\frac{w_p}{6\bar{w}_p^2}e^{-2i\phi} + \frac{1}{4w_p}e^{2i\phi}\right), \tag{51}$$

Which also decrease inversely proportional with the distance between the defects, similarly to that of the positive defect.

II. POLAR FORCE

We are now studying the effect of the polar force, $\vec{F}_p = \tilde{\alpha}_p \vec{p}$ [6] on the ± 1 defects. The order parameter is then given by

$$\vec{p} = \cos(q\theta + \phi)\vec{e}_x + \sin(q\theta + \phi)\vec{e}_y. \tag{52}$$

A. Positive defect

Using the parameterization of the order parameter for an isolated +1 defect, we evaluate the polar force as

$$\vec{F}_p^+ = \frac{\tilde{\alpha}_p}{r}[\vec{r}\cos\phi - \vec{r}^\perp \sin\phi], \tag{53}$$

the first term is a source eliminated by the incompressibility constraint through the pressure contribution $P_p^+ = r\cos\theta$, while the second term is a rotational contribution. The flow velocity induced by this polar force is given by the integral

$$\vec{u}_p^+ = -\frac{\tilde{\alpha}_p}{2\pi\zeta^2}\int d\vec{r'} K_0\left(\frac{|\vec{r}-\vec{r'}|}{\zeta}\right)\frac{\vec{r'}^\perp}{r'}\sin\phi, \tag{54}$$

which in the complex form reads as

$$u_p^+ = -\frac{\tilde{\alpha}_p}{4\pi\zeta}\sin\phi\int dw d\bar{w} K_0\left(\frac{|w|}{\zeta}\right)\sqrt{\frac{w+z}{\bar{w}+\bar{z}}}, \tag{55}$$

and the complex coordinates r' and $\hat{w} = e^{i\theta'}$,

$$u_p^+ = \frac{\tilde{\alpha}_p}{2\pi\zeta}\sin\phi\int dr' r' K_0(r'/\zeta)\oint_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\sqrt{\frac{r'\hat{w}+z}{r'\hat{w}^{-1}+\bar{z}}}. \tag{56}$$

a. ***Contour integral:*** The contour integral over $\hat{w}$

$$I = \oint_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\sqrt{\frac{r'\hat{w}+z}{r'\hat{w}^{-1}+\bar{z}}}, \tag{57}$$

has branch cuts which can be isolated by the keyhole contour. Using that z/r' and $r'/\bar{z}$ have the same argument $e^{i\theta}$ and changing variables to $u = e^{-i\theta}\hat{w}$, i.e rotating the system so that the branch cuts are on the real axis, we rewrite the above integral as

$$I = \sqrt{\frac{r'}{\bar{z}}}e^{i\theta/2}\oint_{|u|=1} du\frac{\sqrt{u+r/r'}}{\sqrt{u}\sqrt{u+r'/r}}. \tag{58}$$

There are three branch points that we have to consider: $u = 0$, $u = -r/r'$ and $u = -r'/r$ as shown in Fig. 1. The two branch points inside unit disk are isolated by the two keyhole contour. Therefore, the contour integral equals the keyhole integral

$$I = \lim_{\epsilon\to 0}\left(\int_{-a-i\epsilon}^{-i\epsilon} + \int_{i\epsilon}^{-a+i\epsilon} + \int_{C_1} + \int_{C_2}\right)\sqrt{\frac{r'}{\bar{z}}}e^{i\theta/2}du\frac{\sqrt{u+r/r'}}{\sqrt{u}\sqrt{u+r'/r}}. \tag{59}$$

The integrals over C_1 and C_2 vanishes regardless whether a is r'/r or r/r', thus

$$I = -2\sqrt{\frac{r'}{\bar{z}}}e^{i\theta/2}\int_{-a}^{0} du\frac{\sqrt{u+r/r'}}{\sqrt{u}\sqrt{u+r'/r}}. \tag{60}$$

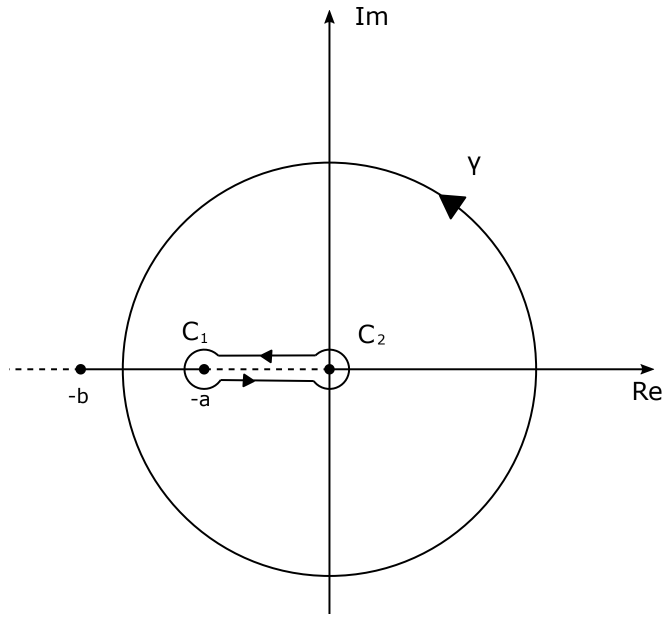

FIG. 1: Contour and branch points.

We can now preform the integral. For $r < r'$, we have that $a = r/r'$ is inside the contour. Therefore

$$I_{r<r'} = 4ie^{i\theta}\left(\left[\frac{r}{r'} - \frac{r'}{r}\right] K\left(\frac{r^2}{r'^2}\right) + \frac{r'}{r} E\left(\frac{r^2}{r'^2}\right)\right), \tag{61}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, and their infinite power series representations are

$$K(x) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 x^n, \tag{62}$$

$$E(x) = \frac{\pi}{2}\left(1 - \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{x^n}{2n-1}\right). \tag{63}$$

Similarly, for $r > r'$, then $a = r'/r$ and $b = r/r'$, so that the integral becomes

$$I_{r>r'} = 4ie^{i\theta} E\left(\frac{r'^2}{r^2}\right). \tag{64}$$

The expression for the complex velocity is then:

$$u_p^+ = \frac{2i\tilde{\alpha}_p}{\pi\zeta^2} e^{i\theta}\left(\int_0^r dr' r' K_0(r'/\zeta) E\left(\frac{r'^2}{r^2}\right) + \int_r^\infty dr' r' K_0(r'/\zeta)\left(\left[\frac{r}{r'} - \frac{r'}{r}\right] K\left(\frac{r^2}{r'^2}\right) + \frac{r'}{r} E\left(\frac{r^2}{r'^2}\right)\right)\right) \tag{65}$$

Inserting the series expansions, the velocity field is a series expansions in moments of the Bessel function

$$u_p^+ = i\tilde{\alpha}_p e^{i\theta}\sin\phi\Bigg(1 - \hat{r}K_1(\hat{r}) - \Bigg(\int_0^{\hat{r}} dt t K_0(t)\left(\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{1}{2n-1}\left(\frac{t}{\hat{r}}\right)^{2n}\right) + \int_{\hat{r}}^{\infty} dt t K_0(t)\left(\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{1}{2n+2}\left(\frac{\hat{r}}{t}\right)^{2n+1}\right)\Bigg)\Bigg). \tag{66}$$

using that $t = r'/\zeta$, and $\hat{r} = r/\zeta$.

*b. **Far-field asymptotic:*** In the limit of $\hat{r} \to \infty$, since the Bessel function decays exponential, we can ignore the factor $\hat{r}K_1(\hat{r})$ and second integral in Eq. (66). We replace the upper bound with ∞ and perform the integration up to second order in $1/\hat{r}$

$$u_p^+ = i\tilde{\alpha}_p e^{i\theta} \sin\phi \left(1 - \frac{1}{\hat{r}^2}\right) \tag{67}$$

There are two things to note. Firstly, the length of u becomes constant in the far-field as expected. Secondly, this is the flow flied of a vortex centered at the defect position.

*c. **Series expansion:*** The integrals over the Bessel functions from Eq. (66). were evaluated in Ref. [1] and given by

$$\int_0^r dt K_0(t) t^{2n+1} = \frac{1}{4}\hat{r}^{2n+2} F_1(\hat{r}, n) \tag{68}$$

$$\int_r^\infty dt K_0(t) t^{-2n} = \frac{1}{4}\hat{r}^{-2n+1} F_2(\hat{r}, n) \tag{69}$$

with the functions given as

$$F_1(n, \hat{r}) = \sum_{k=0}^{\infty} \left[\frac{1}{(n+k+1)} - 2\ln\left(\frac{\hat{r}}{2}\right) + 2\psi^{(0)}(k+1)\right] \frac{1}{(n+k+1)(k!)^2} \left(\frac{\hat{r}}{2}\right)^{2k} \tag{70}$$

and

$$F_2(n, \hat{r}) = 4\sum_{k=0}^{\infty} \left[\psi^{(0)}(k+1) - \ln\left(\frac{\hat{r}}{2}\right) - \frac{1}{2n-1-2k}\right] \frac{1}{(2n-1-2k)(k!)^2} \left(\frac{\hat{r}}{2}\right)^{2k} + \frac{2\pi}{((2n-1)!!)^2} (\hat{r})^{2n-1}. \tag{71}$$

Inserting these expressions into Eq. (66) and after some algebram, we express the velocity field as

$$u_p^+ = i\tilde{\alpha}_p e^{i\theta} \sin\phi \left(1 - \hat{r}K_1(\hat{r}) + \frac{\pi}{2} I_1(\hat{r}) + \sum_{k,n=0}^{\infty} \kappa_1(n,k) \frac{\hat{r}^{2k+2}}{((2k)!!)^2} + \sum_{k,n=0}^{\infty} \kappa_2(n,k) \left[\ln\left(\frac{\hat{r}}{2}\right) - \psi^{(0)}(k+1)\right] \frac{\hat{r}^{2k+2}}{((2k)!!)^2}\right) \tag{72}$$

with the coefficients

$$\kappa_1(n,k) = -\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{1}{2n+2} \left(\frac{1}{(2n-1-2k)^2} + \frac{2n+1}{4(2n+2)(n+2+k)^2}\right) \tag{73}$$

$$\kappa_2(n,k) = \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{1}{2n+2} \left(\frac{2n+1}{2(2n+2)(n+2+k)} - \frac{1}{2n-1-2k}\right). \tag{74}$$

B. The negative defect

The polar force for the negative defect is

$$\vec{F}_p^- = \tilde{\alpha}_p[\vec{e}_x(\cos\theta\cos\phi + \sin\theta\sin\phi) + \vec{e}_y(\cos\theta\sin\phi - \sin\theta\cos\phi)]. \tag{75}$$

or equivalently

$$\vec{F}_p^- = \tilde{\alpha}_p \mathcal{R}[\vec{e}_x \cos\theta - \vec{e}_y \sin\theta] \tag{76}$$

where the rotation matrix R is given as

$$\mathcal{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{77}$$

Therefore, we can rotate the basis so that the phase is $\phi = 0$.

a. ***Pressure:*** The integral solution of the pressure can be written as

$$P_p^- = \frac{i}{2\pi}\int dr' r' \oint_{|\hat{w}|=1} \frac{d\hat{w}}{\hat{w}}\left(\mathbf{F}_x^p(r',\hat{w})\frac{\Re(r'\hat{w}-z)}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})} + \mathbf{F}_y^p(r',\hat{w})\frac{\Im(r'\hat{w}-z)}{(r'\hat{w}-z)(r'\hat{w}^{-1}-\bar{z})}\right), \tag{78}$$

with the polar force is given as

$$\vec{F}_p^-(r',\hat{w}) = \tilde{\alpha}_p[\vec{e}_x\Re(\hat{w}) - \vec{e}_y\Im(\hat{w})]. \tag{79}$$

Inserting this into the pressure and doing some manipulations we find

$$\begin{aligned} P_p^- &= \frac{\tilde{\alpha}_p i}{4\pi}\int_0^\infty dr' r' \oint_{|\hat{w}|=1} d\hat{w}\left(\frac{1}{r'\hat{w}^2(\hat{w}-z/r')} - \frac{\hat{w}}{\bar{z}(\hat{w}-r'/\bar{z})}\right) \\ &= \frac{\tilde{\alpha}_p}{3}\frac{(x^2-y^2)}{r}. \end{aligned}$$

Hence, the velocity field induced by the polar force becomes

$$u_p^- = -\frac{i\tilde{\alpha}_p}{12\pi\zeta^2}\int dr' r' K_0\left(\frac{r'}{\zeta}\right)\oint_{|\hat{w}|=1}\frac{d\hat{w}}{\hat{w}}\left(3\sqrt{\frac{r'\hat{w}^{-1}+\bar{z}}{r'\hat{w}+z}} + \left(\frac{r'\hat{w}+z}{r'\hat{w}^{-1}+\bar{z}}\right)^{3/2}\right). \tag{80}$$

b. ***The contour:*** The contour integral has the same type of branch points as for discussed before, thus we use the contour drawn in Fig. 1. We start by looking at the component

$$3\oint_\gamma \frac{d\hat{w}}{\hat{w}}\sqrt{\frac{r'\hat{w}^{-1}+\bar{z}}{r'\hat{w}+z}}. \tag{81}$$

This one is more straightforward to solve. By changing variables to $t = 1/\hat{w}$ with $dt = -d\hat{w}/\hat{w}^2$, and inserting an extra negative sign since we have to reverse the contour

$$3\oint_\gamma \frac{dt}{t}\sqrt{\frac{r't+\bar{z}}{r't^{-1}+z}}, \tag{82}$$

which is the contour integral we solved for the positive defect in section II A 0 a with z and $\bar{z}$ switched. We can then use the solution we found replacing $\theta \to -\theta$.

For the other integral

$$I = \oint \frac{d\hat{w}}{\hat{w}}\left(\frac{r'\hat{w}+z}{r'\hat{w}^{-1}+\bar{z}}\right)^{3/2} = \left(\frac{r'}{\bar{z}}\right)^{3/2}\oint d\hat{w}\sqrt{\hat{w}}\frac{(\hat{w}+e^{i\theta}r/r')^{3/2}}{(\hat{w}+e^{i\theta}r'/r)^{3/2}}. \tag{83}$$

As for the positive defect we rotate the integral domain so that the branch points falls on the real axis. That is we change variables to $u = e^{-i\theta}\hat{w}$ and get

$$I = \left(\frac{r'}{r}\right)^{3/2} e^{3i\theta}\oint du\sqrt{u}\frac{(u+r/r')^{3/2}}{(u+r'/r)^{3/2}}. \tag{84}$$

It have three branch points, but we can only use the keyhole integration that we did for the positive defect when $r < r'$. This is because we need the integrand to approach zero when we approach the branch cut. In the other limit, we need to take a binomial expansion. We first do the $r < r'$ case. Then we have

$$I_{r<r'} = -2\left(\frac{r'}{r}\right)^{3/2} e^{3i\theta}\int_{-r/r'}^0 du\sqrt{u}\frac{(u+r/r')^{3/2}}{(u+r'/r)^{3/2}} \tag{85}$$

$$= -2\frac{4i}{3}\frac{r'}{r}e^{3i\theta}\left(\left[3\left(\frac{r}{r'}\right)^2 - 11 + 8\left(\frac{r'}{r}\right)^2\right]K\left(\left(\frac{r}{r'}\right)^2\right) + \left[7 - 8\left(\frac{r'}{r}\right)^2\right]E\left(\left(\frac{r}{r'}\right)^2\right)\right). \tag{86}$$

In the other case, $r > r'$, it is the r'/r pole that is inside the contour. We need to change variables to $h = \sqrt{u}$ so that the integral reads:

$$I_{r>r'} = b^{3/2}a^{3/2}e^{3i\theta}\oint_\gamma dh\frac{(1+h^2/a)^{3/2}}{h(1+b/h^2)^{3/2}}. \tag{87}$$

We can expand the exponents as

$$e^{3i\theta}\oint_{\gamma} dh\frac{1}{h}\left(\sum_{k=0}^{\infty}\binom{3/2}{k}\left(\frac{h^2}{a}\right)^k\right)\left(\sum_{n=0}^{\infty}\binom{-3/2}{n}\left(\frac{b}{h^2}\right)^n\right). \tag{88}$$

We use the residue theorem to solve the integral term by term. Conveniently, only the terms where $n = k$ contribute to the integral. We therefore have

$$2\pi i e^{3i\theta}\sum_{n=0}^{\infty}\binom{3/2}{n}\binom{-3/2}{n}\left(\frac{r'}{r}\right)^{2n} = 6\pi i e^{3i\theta}\sum_{n=0}^{\infty}\frac{2n+1}{(3-2n)(1-2n)}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{r'}{r}\right)^{2n}. \tag{89}$$

Where in the last relation we have used the definition of the generalized binomial coefficient.

c. ***Velocity field*** The corresponding velocity field is

$$u_p^- = -\frac{i\tilde{\alpha}_p}{12\pi\zeta^2}\int_0^{\infty} dr'r'K_0\left(\frac{r'}{\zeta}\right)\oint_{\gamma}\frac{d\hat{w}}{\hat{w}}\left(3\sqrt{\frac{r'\hat{w}^{-1}+\bar{z}}{r'\hat{w}+z}}+\left(\frac{r'\hat{w}+z}{r'\hat{w}^{-1}+\bar{z}}\right)^{3/2}\right), \tag{90}$$

which becomes

$$\begin{aligned} u_p^- = -\frac{i\tilde{\alpha}_p}{12\pi\zeta^2}\int_0^{\infty} dr'r'K_0\left(\frac{r'}{\zeta}\right)\Bigg(3\Bigg[4ie^{-i\theta}\left(\left[\frac{r}{r'}-\frac{r'}{r}\right]K\left(\frac{r^2}{r'^2}\right)+\frac{r'}{r}E\left(\frac{r^2}{r'^2}\right)\right)|_{r<r'} + 4ie^{-i\theta}E\left(\frac{r'^2}{r^2}\right)|_{r>r'}\Bigg] \\ + \left[\frac{4i}{3}\frac{r'}{r}e^{3i\theta}\left(\left[3\left(\frac{r}{r'}\right)^2-11+8\left(\frac{r'}{r}\right)^2\right]K\left(\left(\frac{r}{r'}\right)^2\right)+\left[7-8\left(\frac{r'}{r}\right)^2\right]E\left(\left(\frac{r}{r'}\right)^2\right)\right)\right]_{r<r'} \\ + \left[6\pi i e^{3i\theta}\sum_{n=0}^{\infty}\frac{2n+1}{(3-2n)(1-2n)}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{r'}{r}\right)^{2n}\right]_{r>r'}\Bigg). \end{aligned} \tag{91}$$

The $e^{-i\theta}$ is strait forward because it is the same as the $e^{i\theta}$ term for the positive defect, but with factors $2i\tilde{\alpha}_p/(\pi\zeta^2)$ in stead of $\tilde{\alpha}_p/(\pi\zeta^2)$. Inserting the expressions for E and K given in eq. (62) and (63) one can find the relation

$$\begin{aligned} &\left[3\left(\frac{r}{r'}\right)^2-11+8\left(\frac{r'}{r}\right)^2\right]K\left(\left(\frac{r}{r'}\right)^2\right)+\left[7-8\left(\frac{r'}{r}\right)^2\right]E\left(\left(\frac{r}{r'}\right)^2\right) \\ &\qquad = -\frac{18}{8}\pi\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{n}{(n+2)(n+1)}\left(\frac{r}{r'}\right)^{2n+2}, \end{aligned} \tag{92}$$

so that the velocity become

$$\begin{aligned} u_p^- = \frac{\tilde{\alpha}_p}{2}\Bigg(e^{-i\theta}\Bigg[\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{1}{2n+2}\int_{\hat{r}}^{\infty} dttK_0(t)\left(\frac{\hat{r}}{t}\right)^{2n+1} + \\ \left(\int_0^{\hat{r}} dttK_0(t) - \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{1}{2n-1}\int_0^{\hat{r}} dttK_0(t))\left(\frac{t}{\hat{r}}\right)^{2n}\right)\Bigg] \\ + e^{3i\theta}\Bigg[\sum_{n=0}^{\infty}\frac{2n+1}{(3-2n)(1-2n)}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\int_0^{\hat{r}} dttK_0(t)\left(\frac{t}{\hat{r}}\right)^{2n} \\ - \sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{2n}{(2n+4)(2n+2)}\int_{\hat{r}}^{\infty} dttK_0(t)\left(\frac{\hat{r}}{t}\right)^{2n+1}\Bigg]\Bigg) \end{aligned} \tag{93}$$

Here we have changed variables to $t = r'/\zeta$ and introduced $\hat{r} = r/\zeta$. The integrals over the Bessel functions are the same as we considered in section II A 0 c. Inserting their expression and doing some work we can write the velocity as:

$$u_p^- = \frac{\tilde{\alpha}_p}{2}\left(f_1^p(\hat{r})e^{-i\theta}+f_3^p(\hat{r})e^{3i\theta}\right). \tag{94}$$

Where the function f_1^p is given as

$$f_1^p(\hat{r}) = 1 - \hat{r}K_1(\hat{r}) + \frac{\pi}{2}I_1(\hat{r}) + \sum_{k,n=0}^{\infty}\kappa_1(n,k)\frac{\hat{r}^{2k+2}}{((2k)!!)^2} + \sum_{k,n=0}^{\infty}\kappa_2(n,k)\left[\ln\left(\frac{\hat{r}}{2}\right) - \psi^{(0)}(k+1)\right]\frac{\hat{r}^{2k+2}}{((2k)!!)^2}, \tag{95}$$

with the coefficients defined in eq. (73) and (74). The other function f_3^p is given as

$$f_3^p(r) = -\frac{\pi}{2}I_3(\hat{r}) + \sum_{k,n=0}^{\infty}\kappa_3(n,k)\frac{r^{2k+2}}{((2k)!!)^2} + \sum_{k,n=0}^{\infty}\kappa_4(n,k)\frac{r^{2k+2}}{((2k)!!)^2}\left[\ln\left(\frac{\hat{r}}{2}\right) - \psi^{(0)}(k+1)\right], \tag{96}$$

and the new coefficients are defined as

$$\kappa_3(n,k) = \left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{2n+1}{4(3-2n)(1-2n)(n+k+1)^2} + \frac{2n}{(2n+4)(2n+2)(2n-1-2k)^2}\right), \tag{97}$$

$$\kappa_4(n,k) = \left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{2n}{(2n+4)(2n+2)(2n-1-2k)} - \frac{2n+1}{2(3-2n)(1-2n)(n+k+1)}\right). \tag{98}$$

The asymptotic velocity from Eq. (93) is obtained by the same procedure as used for positive defect in Eq. (67) and given by

$$u_p^-(\hat{r},\theta) = \frac{\tilde{\alpha}_p}{2}\left[\left(e^{-i\theta} + \frac{1}{3}e^{3i\theta}\right) - \frac{1}{\hat{r}^2}\left[e^{-i\theta} + 3e^{3i\theta}\right]\right]. \tag{99}$$

As for the positive defect we see that the velocity tends to a constant non-zero value fare away from the defect core.

C. Defect pair: zero-viscosity limit

Similar to the flow equation in the frictionless limit for the dipolar force, we can find the equation for the polar flow,

$$u_p = \tilde{\alpha}_p\psi + \tilde{\alpha}_p\mathcal{I}_p, \tag{100}$$

where $\psi = p_x + ip_y$ is the complex order parameter and the integral factor at the origin is given as

$$\mathcal{I}_p = \frac{i}{4\pi}\int dz'd\bar{z}'\frac{1}{\bar{z}'}\Re(\partial_{z'}\psi). \tag{101}$$

We are now going to use this to find the flow at the centre of the defects for a dipole configuration.

a. ***At the positive defect position:*** We consider first a positive defect placed at the orgin with a negative defect placed at w_n for which complex order parameter is

$$\psi = \chi(r)\sqrt{\frac{z}{\bar{z}}}\sqrt{\frac{\bar{z}-\bar{w}_n}{z-w_n}}e^{i\phi}, \tag{102}$$

Where χ is the core function given as $\chi(r) = 1$ for $r \gg a$ and $\chi(r \ll a) = a\sqrt{z\bar{z}}$. It is straightforward to show that the polar force at the centre of the positive defect $\tilde{\alpha}_p\psi(z=0) = 0$, so the only contribution to the flow at the defects center comes from the pressure gradient given by the integral factor eq. (101). To find it we start by finding the derivative of the order parameter

$$\partial_z\psi = \frac{1}{2}\frac{w_n}{z(w_n-z)}\sqrt{\frac{z}{\bar{z}}}\sqrt{\frac{\bar{z}-\bar{w}_n}{z-w_n}}e^{i\phi} \tag{103}$$

So that the integral we need to solve is

$$\mathcal{I}_p^+ = \frac{i}{16\pi}\int dzd\bar{z}\frac{1}{\bar{z}}\left(\frac{w_n}{z(w_n-z)}\sqrt{\frac{z}{\bar{z}}}\sqrt{\frac{\bar{z}-\bar{w}_n}{z-w_n}}e^{i\phi} + \frac{\bar{w}_n}{\bar{z}(\bar{w}_n-\bar{z})}\sqrt{\frac{\bar{z}}{z}}\sqrt{\frac{z-w_n}{\bar{z}-\bar{w}_n}}e^{-i\phi}\right). \tag{104}$$

We now scale the integral variable so that we use $z' = z/w_n$ and $\bar{z}' = \bar{z}/\bar{w}_n$ and rewrite the integral to

$$\mathcal{I}_p^+ = \frac{i}{16\pi}\int dz'd\bar{z}'\frac{1}{\bar{z}'}\frac{1}{z'(1-z')}\sqrt{\frac{z'}{\bar{z}'}}\sqrt{\frac{\bar{z}'-1}{z-1}}e^{i\phi} + \frac{iw_n}{16\pi\bar{w}_n}\int dz'd\bar{z}'\frac{1}{\bar{z}'}\frac{1}{\bar{z}'(1-\bar{z}')}\sqrt{\frac{\bar{z}'}{z'}}\sqrt{\frac{z'-1}{\bar{z}'-1}}e^{-i\phi}. \tag{105}$$

To get rid of the square roots we change variables to $u^2 = z'/(z'-1)$ and $\bar{u}^2 = \bar{z}'/(\bar{z}'-1)$,

$$\mathcal{I}_p^+ = -\frac{i}{8\pi}\int dud\bar{u}\frac{1}{\bar{u}^2(\bar{u}^2-1)}e^{i\phi} - \frac{iw_n}{8\pi\bar{w}_n}\int dud\bar{u}\frac{(\bar{u}^2-1)}{\bar{u}^2(u^2-1)^2}e^{-i\phi}. \tag{106}$$

We now change variables to complex polar form,

$$\mathcal{I}_p^+ = -\frac{1}{4i\pi}\int drr\oint\frac{d\hat{u}}{\hat{u}}\frac{1}{r^2\hat{u}^{-2}(r^2\hat{u}^{-2}-1)}e^{i\phi} - \frac{w_n}{4i\pi\bar{w}_n}\int drr\oint\frac{d\hat{u}}{\hat{u}}\frac{(r^2\hat{u}^{-2}-1)}{r^2\hat{u}^{-2}(r^2\hat{u}^2-1)^2}e^{-i\phi}. \tag{107}$$

and apply the residue theorem to evaluate it. Thus,

$$u_p^+ = \frac{\tilde{\alpha}_p}{4}e^{i\phi}\left(1 - e^{2i(\varphi-\phi)}\right). \tag{108}$$

This is independent on the distance between the defects, which is not surprising since the velocity field of individual defects are also independent on the distance in the far-field. φ is here the angle of the position vector for the negative defect. The motion of one defect is therefore dependent on the uniform orientation field and its relative position with respect to the other defect. In real coordinates, this becomes

$$\vec{u}_p^+ = \frac{\tilde{\alpha}_p}{2R^2}\vec{R}^\perp(\vec{R}^\perp\cdot\hat{p}_0), \tag{109}$$

where $\vec{R} = \vec{r}_- - \vec{r}_+$ with $\vec{r}_-$ and $\vec{r}_+$ is the position vector for the negative and positive defect respectively, and $\vec{p}_0 = \cos(\phi)\vec{e}_x + \sin(\phi)\vec{e}_y$ which we will assume is a constant vector. Notice that the velocity of the positive defect is always perpendicular to the relative position vector $\vec{R}$.

b. ***At negative position:*** We now place the negative defect at the origin and the positive defect at w_p so that the order parameter can be represented as

$$\psi = \chi(r)\sqrt{\frac{\bar{z}}{z}}\sqrt{\frac{z-w_p}{\bar{z}-\bar{w}_p}}e^{i\phi}, \tag{110}$$

with

$$\psi(0) = 0. \tag{111}$$

In the far-field the derivative is given as

$$\partial_z\psi = -\frac{1}{2}\frac{w}{z(w-z)}\sqrt{\frac{\bar{z}}{z}}\sqrt{\frac{z-w}{\bar{z}-\bar{w}}}e^{i\phi}, \tag{112}$$

so that the integral is

$$\begin{aligned}
\mathcal{I}_p^- &= -\frac{i}{8\pi}\int dzd\bar{z}\frac{1}{\bar{z}}\Re\left(\frac{w_p}{z(w_p-z)}\sqrt{\frac{\bar{z}}{z}}\sqrt{\frac{z-w_p}{\bar{z}-\bar{w}_p}}e^{i\phi}\right)\\
&= \frac{i}{8\pi}\int dud\bar{u}\frac{1}{u^2(\bar{u}^2-1)}e^{i\phi} + \frac{iw_p}{8\pi\bar{w}_p}\int dud\bar{u}\frac{u^2(\bar{u}^2-1)}{\bar{u}^4(u^2-1)^2}e^{-i\phi}\\
&= -\frac{1}{4\pi i}\int drr\oint\frac{d\hat{u}}{\hat{u}}\frac{1}{r^2(\hat{u}-r)(\hat{u}+r)}e^{i\phi} + \frac{w_p}{4\pi i\bar{w}_p}\int drr\oint d\hat{u}\frac{\hat{u}^3(r^2-\hat{u}^2)}{r^6(\hat{u}-1/r)^2(\hat{u}+1/r)^2}e^{-i\phi}.
\end{aligned}$$

using the same kind of variable transformations: $z' = z/w_p$ and $\bar{z}' = \bar{z}/\bar{w}_p$, $u^2 = z'/(z'-1)$ and $\bar{u}^2 = \bar{z}'/(\bar{z}'-1)$ and complex polar form. Solving these using the residual theorem we end up with a velocity of

$$u_p^- = \frac{\tilde{\alpha}_p}{4}e^{i\phi}(1 + \frac{1}{3}e^{2i(\varphi_p-\phi)}), \tag{113}$$

Where $\varphi_p = \pi + \varphi$ is the angle of the position vector of the positive defect $-\vec{R}$. In real coordinates this is

$$\vec{u}_p^- = \frac{\tilde{\alpha}_p}{6R^2}\vec{R}^\perp(\vec{R}^\perp \cdot \vec{p}_0) + \frac{\tilde{\alpha}_p}{3R^2}\vec{R}(\vec{R}\cdot\vec{p}_0). \tag{114}$$

This has a component that is perpendicular to the relative position vector $\vec{R}$ and one that is parallel to it. Note that the perpendicular velocity is in the same direction as the one for the positive defect, but it is always smaller in magnitude. This means that the vector is rotating. This rotation has two zeros. To see this easier, we consider the evolution of the relative position due to these velocities. The evolution of the relative position of the defects is

$$\dot{\vec{R}} = \frac{\tilde{\alpha}_p}{3R^2}(\vec{R}\cdot\vec{p}_0)\vec{R} - \frac{\tilde{\alpha}_p}{3R^2}(\vec{R}^\perp\cdot\vec{p}_0)\vec{R}^\perp. \tag{115}$$

We notice that one component is parallel and the other is perpendicular to $\vec{R}$, so that their effect is to stretch and rotate the relative position. The rotation has two zeros, when $\vec{R}$ is parallel or antiparallel to $\vec{p}_0$. To check whether these are attractive or repulsive we write $\vec{R} = R(\cos\Omega, \sin\Omega)$, with Ω being the angle of $\vec{R}$ relative to $\vec{p}_0$. Assuming that they are almost pointing in the same direction, $\Omega \ll 1$, we have the equations

$$\dot{R} = \frac{\tilde{\alpha}_p}{3}, \tag{116}$$

$$\dot{\Omega} = \frac{\tilde{\alpha}_p}{3R}\Omega, \tag{117}$$

The equation for Ω shows that $\Omega = 0$ is an unstable zero, meaning that $\vec{R}$ tends to rotate away from this direction. The other zero is when $\vec{R}$ is close to antiparallel with $\vec{p}_0$. Then we can write the angle as $\Omega = \pi - \Omega_a$, where Ω_a a small angle. The equations are in this configuration approximately

$$\dot{R} = -\frac{\tilde{\alpha}_p}{3}, \tag{118}$$

$$\dot{\Omega}_a = -\frac{\tilde{\alpha}_p}{3R}\Omega_a \tag{119}$$

Meaning that we have a stable zero at $\Omega = \pi$. Notice that the unstable zero corresponds to repulsion force and the stable one corresponds to attraction. This means that, as long as $\vec{R}$ and $\vec{p}$ are not pointing in the same direction, the defects will eventually annihilate.

---

[1] Jonas Rønning, Cristina M Marchetti, Mark J Bowick, and Luiza Angheluta. Flow around topological defects in active nematic films. Proceedings of the Royal Society A, 478(2257):20210879, 2022.
[2] Luiza Angheluta, Zhitao Chen, M Cristina Marchetti, and Mark J Bowick. The role of fluid flow in the dynamics of active nematic defects. New Journal of Physics, 23(3):033009, 2021.
[3] Len M Pismen et al. Vortices in nonlinear fields: From liquid crystals to superfluids, from non-equilibrium patterns to cosmic strings, volume 100. Oxford University Press, 1999.
[4] M Cristina Marchetti, Jean-François Joanny, Sriram Ramaswamy, Tanniemola B Liverpool, Jacques Prost, Madan Rao, and R Aditi Simha. Hydrodynamics of soft active matter. Rev. Mod. Phys., 85(3):1143, 2013.
[5] R Aditi Simha and Sriram Ramaswamy. Hydrodynamic fluctuations and instabilities in ordered suspensions of self-propelled particles. Physical review letters, 89(5):058101, 2002.
[6] Aboutaleb Amiri, Romain Mueller, and Amin Doostmohammadi. Unifying polar and nematic active matter: emergence and co-existence of half-integer and full-integer topological defects. Journal of Physics A: Mathematical and Theoretical, 55(9):094002, feb 2022.