AFM-based spherical indentation of a brush-coated cell: Modeling the bottom effect

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Supplementary Information

A. Asymptotic constants for a bonded layer

The asymptotic constants $a_0$ and $a_1$ can be evaluated as follows (Alexandrov and Pozharskii, 2001):

$$a_m = \frac{(-1)^m}{[(2m)!]^2} \int_0^\infty [1 - L(u)] u^{2m} du. \quad (A.1)$$

In the case of an isotropic elastic layer bonded to a rigid base, the kernel function is given by

$$L(u) = \frac{2\kappa \sinh(2u) - 4u}{2\kappa \cosh(2u) + 1 + \kappa^2 + 4u^2}, \quad (A.2)$$

where $\kappa = 3 - 4\nu$ is Kolosov’s constant, and $\nu$ is Poisson’s ratio.

The normalized asymptotic constants are defined as

$$\alpha_0 = \frac{8a_0}{3\pi}, \quad \alpha_1 = \frac{-8a_1}{3\pi}.$$

In view of (A.1), we have

$$\alpha_0 = \frac{8}{3\pi} \int_0^\infty [1 - L(u)] \, du, \quad \alpha_1 = \frac{2}{3\pi} \int_0^\infty [1 - L(u)] u^2 \, du, \quad (A.3)$$

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where \( L(u) \) is given by (A.2).

Practically, the improper integrals (A.1) and (A.3) can be evaluated numerically by replacing the infinite upper limit of the integral by a finite upper limit, that is

\[
a_m \approx \frac{(-1)^m}{[(2m)!]^2} \int_0^M [1 - L(u)] u^{2m} du.
\] (A.4)

The upper bound for the error of such approximation is illustrated in Fig. 1 for the case \( \nu = 0.5 \). The error is exponentially decaying with the increase of the upper limit \( M \) in the definite integral (A.4).

**B. Accuracy of the approximate solutions**

Let us introduce the notation

\[
\varepsilon = \frac{a}{H}, \quad \varpi = \frac{\sqrt{R\delta}}{H}, \quad \tilde{F} = \frac{RF}{E^* H^3}, \quad \tilde{\delta} = \frac{R\delta}{H^2}.
\] (B.1)

The fourth-order asymptotic solution obtained by [Vorovich et al., 1974] has the following form:

\[
F = \frac{4}{3} \frac{E^*}{R} a^3 \left( 1 - \varepsilon^3 \frac{8a_1}{3\pi} \right),
\] (B.2)

\[
F = \frac{4}{3} E^* a_0 \left\{ 1 + \varepsilon \frac{4a_0}{3\pi} + \varepsilon^2 \left( \frac{4a_0}{3\pi} \right)^2 + \varepsilon^3 \left( \frac{4a_0}{3\pi} \right)^3 + \varepsilon^4 \left( \frac{4a_0}{3\pi} \right)^4 + \varepsilon^3 \frac{8a_1}{15\pi} \left( 1 + \varepsilon \frac{8a_0}{3\pi} \right) \right\}.
\] (B.3)
Using the asymptotic solution (B.2), (B.3) and the dimensionless variables (B.1), we can represent the approximate force-displacement relation in the parametric form as

\[ \tilde{F} = \frac{4}{3} \varepsilon^3 \left(1 - \varepsilon^3 \frac{8a_1}{3\pi}\right), \]  
\[ \tilde{\delta} = \varepsilon^2 \left(1 - \varepsilon^3 \frac{8a_1}{3\pi}\right) \]  

(B.4)

The fourth-order asymptotic solution derived by Argatov and Sabina (2013), which is asymptotically equivalent to (B.2), (B.3), has the form

\[ \tilde{F} = \frac{4}{3} \varepsilon^3 \left(1 - \varepsilon^3 \frac{8a_1}{3\pi}\right), \]  
\[ \tilde{\delta} = \varepsilon^2 \left(1 - \varepsilon^3 \frac{16a_1}{5\pi} + \varepsilon^4 \frac{32a_0a_1}{9\pi^2}\right). \]  

(B.6)

The second-order asymptotic solution is recovered from Eqs. (B.6) and (B.7) by dropping the terms containing \( a_1 \), that is as follows (Argatov, 2010):

\[ \tilde{F} = \frac{4}{3} \varepsilon^3, \quad \tilde{\delta} = \varepsilon^2 \left(1 - \varepsilon^4 \frac{4a_0}{3\pi}\right). \]  

(B.8)

Observe that Eqs. (B.2)–(B.7) utilize only the first two asymptotic constants \( a_0 \) and \( a_1 \). The sixth-order asymptotic solution derived by Argatov (2001), which incorporates also the third asymptotic constant \( a_2 \), has the following form:

\[ \tilde{F} = \frac{4}{3} \varepsilon^3 \left(1 - \varepsilon^3 \frac{8a_1}{3\pi} - \varepsilon^5 \frac{128a_2}{15\pi}\right), \]  
\[ \tilde{\delta} = \varepsilon^2 \left(1 - \varepsilon^3 \frac{16a_1}{5\pi} + \varepsilon^4 \frac{32a_0a_1}{9\pi^2}\right) \]  

(B.9)

The fourth-order asymptotic approximation for the force-displacement relation in the explicit form, which was obtained by Argatov (2011), has the form

\[ \tilde{F} = \frac{4}{3} \omega^3 \left\{1 + \omega \frac{2a_0}{\pi} + \omega^2 \frac{14a_0^2}{3\pi^2} + \omega^3 \left(286a_0^4 + 320a_0^3 \frac{32a_0}{27\pi^3} + \frac{64a_0a_1}{5\pi^2}\right)\right\} \]  

(B.11)

and simplifies to the second-order approximation as follows (Argatov, 2011):

\[ \tilde{F} = \frac{4}{3} \omega^3 \left(1 + \omega \frac{2a_0}{\pi} + \omega^2 \frac{14a_0^2}{3\pi^2}\right). \]  

(B.12)
We recall that for a bonded incompressible layer, we have $a_0 = 1.77022$, $a_1 = -0.95777$, and $a_2 = 0.43736$. In this special case, the following approximate solution was obtained by Dimitriadis et al. (2002):\[ F = \frac{4}{3} \delta^{3/2} \left( 1 + 1.133 \delta^{3/2} + 1.283 \delta + 0.769 \delta^{3/2} + 0.0975 \delta^2 \right). \] (B.13)

A much more accurate solution was derived by Garcia and Garcia (2018) in the form\[ \tilde{F} = \frac{4}{3} \delta^{3/2} \left( 1 + 1.133 \tilde{\delta}^{3/2} + 1.497 \tilde{\delta} + 1.469 \delta^{3/2} + 0.755 \delta^2 \right), \] (B.14) which differs from (B.13) only by the expansion coefficients.

The accuracy of the analytical solutions outlined above have been tested using the following accurate analytical approximation obtained by Hermanowicz (2021):\[ \tilde{F} = \begin{cases} \frac{4}{3} \delta^{3/2} \left( 1 + 1.105 \delta^{3/2} + 1.607 \delta + 1.602 \delta^{3/2} \right), & \tilde{\delta} \leq 0.4 \\ 0.616 - 3.114 \delta^{1/2} + 6.693 \delta - 7.17 \delta^{3/2} + 8.228 \delta^{3/2} + \frac{\pi}{2} \delta^3, & 0.4 < \tilde{\delta}. \end{cases} \] (B.15)

![Graph](a) Percentage relative error % vs. Relative contact radius $a/H$ (b) Percentage relative error % vs. Relative contact radius $a/H$

Figure 2: Accuracy of the approximate solutions as a function of the relative contact radius.

The results of the comparison are presented in Figs. 2 and 3 where the following legend applies: Curve 1 corresponds to the fourth-order asymptotic approximation in the explicit form (B.11) [Argatov 2011]; Curve 2 corresponds to the analytical approximation (B.14) [Garcia and Garcia 2018]; Curve 3 corresponds to the fourth-order asymptotic approximation in the parametric form (B.6); Curve 4 corresponds to the fourth-order asymptotic approximation in the parametric form (B.2), (B.3) [Vorovich et al., 1974]; Curve 5 corresponds to the analytical approximation (B.13) [Dimitriadis et al., 2002]; Curve 6 corresponds to the sixth-order asymptotic approximation in the parametric form (B.9), (B.10) [Argatov 2001]; Curve 7
Figure 3: Accuracy of the approximate solutions as a function of the relative indentation depth.

corresponds to the second-order asymptotic approximation in the explicit form (B.12) [Argatov 2011]; Curve 8 corresponds to the second-order asymptotic approximation in the parametric form (B.8) [Argatov 2010].

C. Indentation scaling factor

According to the fourth-order asymptotic solution (B.11) obtained by Argatov (2011), the indentation scaling factor can be evaluated as

\[
f(\varpi) = 1 + \varpi \frac{2a_0}{\pi} + \varpi^2 \frac{14a_0^2}{3\pi^2} + \varpi^3 \left( \frac{320a_0^3}{27\pi^3} + \frac{32a_1}{15\pi} \right) + \varpi^4 \left( \frac{286a_0^4}{9\pi^4} + \frac{64a_0a_1}{5\pi^2} \right). \tag{B.16}\]

In view of (A.1) and (A.2), the variation of the scaling factor \( f \) as a function of \( \varpi \) depends on the layer Poisson’s ratio \( \nu \). This is illustrated in Fig. 4.
Figure 4: Indentation scaling factor for a paraboloidal indentation of a bonded elastic layer.

References


