Supporting Material

To preserve the nonlinear characteristics of the Herschel-Bulkley fluids while ensuring the solvability of governing equations, a coefficient g which indicates the variation of velocity with the distance along z-axis and is related to the properties of fluid is introduced into the linearization of the liquid momentum equations. Then yields that

$$\tau_{zz} = \tau_{yz} + K\left(2\frac{\partial u_{lz}}{\partial z}\right)^n = \tau_{yz} + K\left(2g + 2\frac{\partial u_{lz}}{\partial z}\right)^n . \tag{1}$$

Using binomial theorem expansion and omitting high-order small quantities, we get

$$\tau_{zz} \approx \tau_{yz} + K\left[(2g)^n + 2n(2g)^{n-1}\frac{\partial u_{lz}}{\partial z}\right]. \tag{2}$$

We assume that velocities, pressures, and stresses are equal to the initial steady state, plus the unsteady disturbances. It is assumed that the initial disturbances are only dependent on the radial position r as well.

$$\boldsymbol{u}_l = (0,0,u_l) + \boldsymbol{u}_l' = (0,0,u_l) + \boldsymbol{u}_{l0}' \exp(-i\omega t + im\theta + ikz), \tag{3}$$

where $\boldsymbol{u}_{l0}' = (u_{l0r}', u_{l0\theta}', u_{l0z}')$.

$$\boldsymbol{u}_g = (0,\frac{W_0}{r},u_g) + \boldsymbol{u}_g' = (0,\frac{W_0}{r},u_g) + \boldsymbol{u}_{g0}' \exp(-i\omega t + im\theta + ikz), \tag{4}$$

where $\boldsymbol{u}_{g0}' = (u_{g0r}', u_{g0\theta}', u_{g0z}')$.

$$P_l = p_l + p_l' = p_l + p_{l0}' \exp(-i\omega t + im\theta + ikz), \tag{5}$$

$$P_g = p_g + p_g' = p_g + p_{g0}' \exp(-i\omega t + im\theta + ikz), \tag{6}$$

Then we linearize the governing equations using the disturbances on the basis of preserving the nonlinear characteristics of the Herschel-Bulkley fluid. Substitute Eqs. (2), (3) and (5) into the liquid governing equations then gives

$$\frac{\mathrm{d}u'_{l0r}}{\mathrm{d}r}+\frac{u'_{l0r}}{r}+\frac{im}{r}u'_{l0\theta}+iku'_{l0z}=0\,, \tag{7}$$

$$\rho_l\left(-i\omega+iku_l\right)u'_{l0r}=-\frac{\mathrm{d}p'_{l0}}{\mathrm{d}r}\,, \tag{8}$$

$$\rho_l\left(-i\omega+iku_l\right)u'_{l0\theta}=-im\frac{p'_{l0}}{r}\,, \tag{9}$$

$$\rho_l\left(-i\omega+iku_l\right)u'_{l0z}=-ikp'_{l0}-2nK\left(2g\right)^{n-1}k^2u'_{l0z}\,. \tag{10}$$

In the same way, substitute Eqs. (4) and (6) into the gas governing equations then the linearized forms are expressed as:

$$\frac{\mathrm{d}u'_{g0r}}{\mathrm{d}r}+\frac{u'_{g0r}}{r}+\frac{im}{r}u'_{g0\theta}+iku'_{g0z}=0\,, \tag{11}$$

$$\rho_g\left[\left(-i\omega+im\frac{W_0}{r^2}+iku_g\right)u'_{g0r}-\frac{2W_0u'_{g0\theta}}{r^2}\right]=-\frac{\mathrm{d}p'_{g0}}{\mathrm{d}r}\,, \tag{12}$$

$$\rho_g\left[(-i\omega+im\frac{W_0}{r^2}+iku_g)u'_{g0r}\right]=-\frac{\mathrm{d}\,p'_{g0}}{\mathrm{d}\,r}\,, \tag{13}$$

$$\rho_g(-i\omega+im\frac{W_0}{r^2}+iku_g)u'_{g0z}=-ikp'_{g0}\,. \tag{14}$$

Based on the general solution forms of Bessel equation and Eqs. (8)—(10), the special solutions of initial disturbed pressure and velocities in the gel phase are yielded as:

$$p'_{l0}=a\rho_lu_l\left(-i\omega+iku_l\right)A_1I_m(lr)\,, \tag{15}$$

$$u'_{l0r}=-au_llA_1I'_m(lr)\,, \tag{16}$$

$$u'_{l0\theta}=-au_l\frac{im}{r}A_1I_m(lr)\,, \tag{17}$$

$$u'_{l0z}=\frac{au_l\rho_l\left(-i\omega+iku_l\right)ik}{\rho_l(-i\omega+iku_l)+2nK(2g)^{n-1}k^2}A_1I_m(kr)\,, \tag{18}$$

where I_m is the mth-order modified Bessel function of the first kind, l and A_1 are respectively expressed as:

$$l=k\sqrt{\frac{\rho_lg\left(-i\omega+iku_l\right)}{\rho_lg\left(-i\omega+iku_l\right)+Knk^2(2g)^n}}\,. \tag{19}$$

$$A_1 = -\frac{\varepsilon_0\left(-i\omega + iku_l\right)}{au_l l I_m'(lr)}. \tag{20}$$

Similarly, the special solutions of initial disturbed pressure and velocities in the gas phase are as follows:

$$p_{g0}' = A_2 K_m(kr), \tag{21}$$

$$u_{g0r}' = -\frac{2imQ_2 A_2 K_m(kr) + krQ_1 A_2 K_m'(kr)}{r\rho_g Q_1^2}, \tag{22}$$

$$u_{g0\theta}' = \frac{-imA_2 K_m(kr)}{r\rho_g Q_1}, \tag{23}$$

$$u_{g0z}' = -\frac{ikA_2 K_m(kr)}{\rho_g Q_1}, \tag{24}$$

where K_{m} is the *m*th-order modified Bessel function of the second kind, Q_1, Q_2 and A_2 are expressed as:

$$Q_1 = -i\omega + im\frac{W_0}{r^2} + iku_g, \tag{25}$$

$$Q_2 = \frac{W_0}{r^2}, \tag{26}$$

$$A_2 = -\frac{r\varepsilon_0\rho_g Q_1^2\left(-i\omega + im\dfrac{W_0}{r} + iku_g\right)}{2imQ_2 K_m(kr) + krQ_1 K_m'(kr)}. \tag{27}$$

The manipulation to obtain the dispersion relation of a reducing form is as follows.

$$
\begin{aligned}
P_l &= p_l + p'_{l0}\exp(-i\omega t + im\theta + ikz) \\
&= p_l + a\rho_l u_l(-i\omega + iku_l)A_1 I_m(lr)\exp(-i\omega t + im\theta + ikz) \\
&= p_l + \frac{a\rho_l u_l(-i\omega + iku_l)(-\varepsilon_0(-i\omega + iku_l))I_m(lr)}{au_l l I'_m(lr)} \\
&= p_l + \frac{-\rho_l(-i\omega + iku_l)^2}{l}\frac{I_m(lr)}{I'_m(lr)}\varepsilon
\end{aligned}
\quad, \tag{28}
$$

and

$$
\begin{aligned}
P_g &= p_g + p'_{g0}\exp(-i\omega t + im\theta + ikz) \\
&= p_g + A_2 K_m(kr)\exp(-i\omega t + im\theta + ikz) \\
&= p_g - r\rho_g Q_1^2(-i\omega + im\frac{W_0}{r} + iku_g)\frac{K_m(kr)}{2imQ_2K_m(kr) + krQ_1K'_m(kr)}\varepsilon
\end{aligned}
\quad, \tag{29}
$$

thus

$$
\begin{aligned}
P_l - P_g = p_l - p_g &- \frac{\rho_l(-i\omega + iku_l)^2}{l}\frac{I_m(lr)}{I'_m(lr)}\varepsilon \\
&+ r\rho_g Q_1^2(-i\omega + im\frac{W_0}{r} + iku_g)\frac{K_m(kr)}{2imQ_2K_m(kr) + krQ_1K'_m(kr)}\varepsilon
\end{aligned}
\quad. \tag{30}
$$

Because

$$
\frac{\partial^2\varepsilon}{\partial\theta^2} = -m^2\varepsilon, \qquad \frac{\partial^2\varepsilon}{\partial z^2} = -k^2\varepsilon\,, \tag{31}
$$

$$
\rho_g\frac{W_0^2}{a^3}R = \rho_g\frac{W_0^2}{a^3}(a+\varepsilon) = \rho_g\frac{W_0^2}{a^2} + \rho_g\frac{W_0^2}{a^3}\varepsilon\,, \tag{32}
$$

$$
\begin{aligned}
\frac{du_l}{dr} &= \frac{du'_{l0r}}{dr}\exp(-i\omega t + im\theta + ikz) \\
&= -au_l l^2 A_1 I''_m(lr)\exp(-i\omega t + im\theta + ikz)\,, \\
&= l(-i\omega + iku_l)\frac{I''_m(lr)}{I'_m(lr)}\varepsilon
\end{aligned}
\tag{33}
$$

so

$$
\begin{aligned}
&-\sigma\left(\frac{1}{a^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\varepsilon + \rho_g\frac{W_0^2}{a^3}R + 2\mu_l\frac{du_l}{dr} \\
&= -\sigma\left(\frac{1}{a^2} - \frac{m^2}{a^2} - k^2\right)\varepsilon + \rho_g\frac{W_0^2}{a^3}\varepsilon + \rho_g\frac{W_0^2}{a^2} + 2\mu_l l(-i\omega + iku_l)\frac{I''_m(lr)}{I'_m(lr)}\varepsilon
\end{aligned}
\quad, \tag{34}
$$

Substitute the above into the pressure equilibrium equation in the dynamic boundary conditions, yielding that

$$
\begin{aligned}
&(-\frac{\rho_l(-i\omega+iku_l)^2}{l}\frac{I_m(lr)}{I_m'(lr)}+r\rho_g Q_1^2(-i\omega+im\frac{W_0}{r}+iku_g)\frac{K_m(kr)}{2imQ_2K_m(kr)+krQ_1K_m'(kr)} \\
&-2\mu_l l(-i\omega+iku_l)\frac{I_m''(lr)}{I_m'(lr)})\varepsilon=(-\sigma(\frac{1}{a^2}-\frac{m^2}{a^2}-k^2)+\rho_g\frac{W_0^2}{a^3})\varepsilon+\rho_g\frac{W_0^2}{a^2}+p_g-p_l
\end{aligned}
. \quad (35)
$$

The dispersion equation is obtained by dividing both sides of the equation by ε and omitting the nonlinear terms:

$$
\begin{aligned}
&-\frac{\rho_l(-i\omega+iku_l)^2}{l}\frac{I_m(lr)}{I_m'(lr)}+r\rho_g Q_1^2(-i\omega+im\frac{W_0}{r}+iku_g)\frac{K_m(kr)}{2imQ_2K_m(kr)+krQ_1K_m'(kr)} \\
&-2\mu_l l(-i\omega+iku_l)\frac{I_m''(lr)}{I_m'(lr)}=-\sigma(\frac{1}{a^2}-\frac{m^2}{a^2}-k^2)+\rho_g\frac{W_0^2}{a^3}
\end{aligned}
. \quad (36)
$$