ELECTRONIC SUPPLEMENTARY INFORMATION

Ridge Localization Driven by Wrinkle Packets

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Simulation.mp4: The first portion of this video file shows the evolution of wrinkles at high β (i.e. high rate) corresponding to Fig. 3A&C in the main text. The second portion of this video file shows the evolution of ridges at low β (i.e. low rate) corresponding to Fig. 3B&D in the main text

Section S1: Theoretical Analysis

Derivation of Equations

Because of the small value of the fluid layer thickness H_0 the system is highly dissipative and vertical velocities are very small compared with horizontal velocities ($V_z \ll V_x$). Under these conditions lubrication approximation gives the equation

$$0 = -\partial_x p(x,t) + \eta \partial_x^2 V_x \tag{S 1}$$

where $p = p(x,t)$ is the pressure. Note that $V_x = V_x(x,z,t)$, but the pressure is a function of x and t. Integration of this relation gives $V_x = \frac{1}{2\eta}\partial_x p(x,t) z^2 + A(x,t)z + B(x,t)$ where A and B are found by applying the two boundary conditions

$$v = V_x|_{z=0} \tag{S 2}$$

$$\tau = \eta \partial_z V_x|_{z=H} \tag{S 3}$$

where $\tau(x,t)$ is the shear stress at $z = H(x,t)$ and $v(x,t)$ is the velocity at the bottom. The general solutions for the horizontal and vertical velocity are

$$V_x(x,z,t) = \frac{1}{2\eta}\partial_x p\, z^2 - \frac{1}{\eta}\partial_x p\, Hz + \frac{1}{\eta}\tau z + v \tag{S 4}$$

$$V_z(x,z,t) = \partial_x\left(-\frac{1}{6\eta}\partial_x p\, z^3 + \frac{1}{2\eta}\partial_x p\, Hz^2 - \frac{1}{2\eta}\tau\, z^2 - vz\right) \tag{S 5}$$

Here the vertical velocity is obtained from the integration of the incompressibility condition $\partial_x V_x + \partial_z V_z = 0$ with the boundary condition $v_y|_{z=0} = 0$.

The previous equations are written in terms of eulerian coordinates; however, the description of the film deformation requires the use of lagrangian coordinates. The two dimensional deformation of the film is given by the transformation in Monge coordinates $x = s + u(s,t)$ and $z = H(s,t)$, where s gives the length along the centerline previous to the film deformation (see Fig. S1). The non-slip boundary condition connects the fluid velocities with the film deformation

$$\partial_t u(s,t) = V_x[s + u(s,t), H(s,t), t] \tag{S 6}$$

$$\partial_t H(s,t) = V_z[s + u(s,t), H(s,t), t] \tag{S 7}$$

To evaluate Eq. (S 7), we use incompressibility in the form $V_z(x,z) = -\int_0^z dz' \partial_x V_x(x,z')$ which gives $V_z[x, H(x,t)] = -\int_0^{H(x,t)} dz' \partial_x V_x(x,z') = -\partial_x \int_0^{H(x,t)} dz' V_x(x,z') - V_x[x, H(x,t)]\partial_x H(x,t)$. Equations (S 6) and (S 7) become

$$\begin{aligned}
\partial_t u(s,t) &= -\frac{1}{2\eta}\partial_x p\, H^2 + \frac{1}{\eta}\tau H + v \\
\partial_t H(s,t) &= \partial_x\left[\frac{1}{3\eta}\partial_x p\, H^3 - \frac{1}{2\eta}\tau H^2 - vH\right] + \left[-\frac{1}{2\eta}\partial_x p\, H^2 + \frac{1}{\eta}\tau H + v\right]\partial_x H
\end{aligned} \tag{S 8}$$

Note that the left hand side is written in lagrangian coordinates and the right hand side in eulerian coordinates. To compare these equations with previous formulations [S1, S2, S3], we observe that the relation $H(x,t) = H[s + u(s,t), t] = H(s,t)$ implies the kinematic relation

$$\partial_t H(s,t) = \partial_t H(x,t) + \partial_t u(s,t)\partial_x H(x,t) \tag{S 9}$$

Therefore the equation for the height in (S 8) can be written as

$$\partial_t H(x,t) = \partial_x\left[\frac{1}{3\eta}\partial_x p\, H^3 - \frac{1}{2\eta}\tau H^2 - vH\right] \tag{S 10}$$

which is consistent with the conservation law $\partial_t H + \partial_x \Theta = 0$ in eulerian coordinates, where Θ is the flow per unit of width across a vertical section of the fluid layer. Because we need to study small perturbations of the film height, the representation in Eqs. (S 8) is more convenient. Specifically, the convective term $v\partial_x H$ in Eq. (S 10), which is of the form $\sim x\partial_x H$ (see next section) and is not invariant under translations, cancels out when time derivatives are written in Lagrangian coordinates. Further simplifications of these equations give

$$\begin{aligned}\partial_t u(s,t) &= -\frac{1}{2\eta}\partial_x p\, H^2 + \frac{1}{\eta}\tau H + v \\ \partial_t H(s,t) &= \frac{1}{3\eta}\partial_x^2 p\, H^3 + \frac{1}{2\eta}\partial_x p\, \partial_x H H^2 - \frac{1}{2\eta}\partial_x \tau\, H^2 - \partial_x v H\end{aligned} \tag{S 11}$$

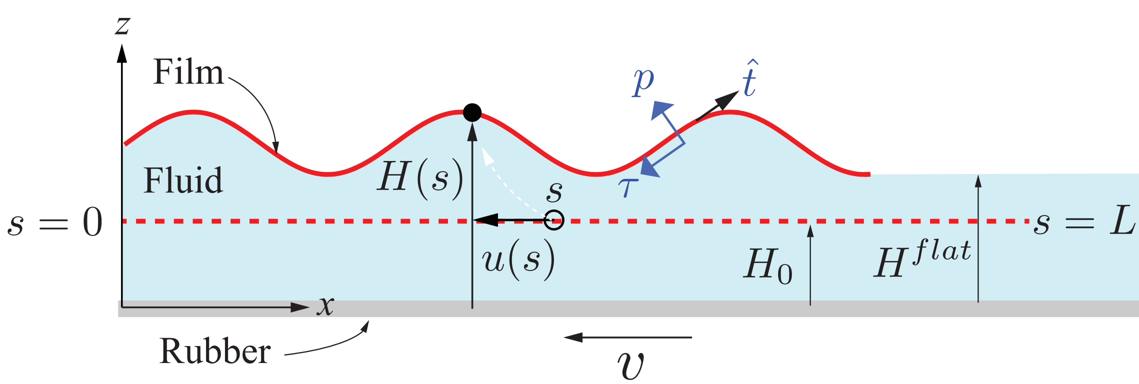

Fig. S 1. Schematic of the physical configuration for $x > 0$. The fluid layer of initial height H_0 follows the bottom that contracts with the velocity v. The part of the fluid which is free grows with the homogeneous solution H^{flat} and the part which is covered by the film wrinkles and folds. The deformation of a material point in the film is described by the transformation $(x, z) = [s + u(s,t), H(s,t)]$.

Flat Solution

For a free surface $p = \tau = 0$ the horizontal velocity of the fluid corresponds to the velocity of the bottom $\partial_t u(s,t) = v$ and the vertical velocity is given by the incompressibility condition $\partial_t H(s,t) = -\partial_x v H$. The deformation of the bottom is defined by the transformation

$$x = s - \dot{\epsilon} t s \tag{S 12}$$

where x is the current horizontal position and s the position before deformation. The velocity of the bottom is then $v = \partial_t x = -\dot{\epsilon} s$ in lagrangian coordinates. It also allows to find the vertical velocity as $\partial_t H(s,t) = -\partial_x v H = -(\partial s/\partial x)\partial_s v H = -\dot{\epsilon} H/(1 - \dot{\epsilon} t)$. Thus, we have the two equations and homogeneous solutions

$$\begin{aligned}\partial_t u(s,t) &= -\dot{\epsilon} s & &\longrightarrow u^{flat}(s,t) = -\dot{\epsilon} t s \\ \partial_t H(s,t) &= -\tfrac{\dot{\epsilon}}{1-\dot{\epsilon} t} H & &\longrightarrow H^{flat}(s,t) = \frac{H_0}{1 - \dot{\epsilon} t}\end{aligned} \tag{S 13}$$

These solutions explain Eq. (3) in the main text. Due to liquid incompressibility, H^{flat} is determined entirely by the horizontal displacement.

Constitutive relations for p and τ.

Because the deformation of the film happens at a time scale faster than the deformation of the fluid, we can use the equilibrium equations for an elastic film to compute bending and stretching. Assuming a deformation in the plane $x - z$, the equilibrium of forces is given by the equation

$$\frac{d\vec{F}}{ds} + \vec{K} = 0 \tag{S 14}$$

where $\vec{F} = F_x\hat{x} + F_z\hat{z}$ is the force per unit of width on the cross section of the film and $\vec{K}$ is the external force per unit of area applied to the film. While this description is commonly applied to beams, it is well known that it also applies to films when they are wide enough to avoid 3D effects, such as anticlastic curvature. In our system, the external force can be approximated in the small deflection limit for $\vec{K} \approx -\tau\hat{x} + p\hat{z}$ (see Fig S1). It gives a relation between the horizontal force per unit of width and the shear, $\partial_s F_x = \tau$. The force F_x is associated with the stretching of the film through the constitutive relation $F_x = Y\partial_s u$, where $Y = Eh/(1 - \nu^2)$ represents the stiffness under plane-strain conditions. Thus, the shear force is given in terms of the displacement as

$$\tau = \partial_s F_x = Y\partial_s^2 u \tag{S 15}$$

The moment balance gives the equation

$$\frac{d\vec{M}}{ds} + \hat{t} \times \vec{F} = 0 \tag{S 16}$$

Here $\vec{M} = B\partial_s^2 H\hat{x} \times \hat{z}$, where $B = Eh^3/12(1 - \nu^2)$, and $\hat{t}$ is the tangent to the $x - z$ curve defined by the film centerline. For small deflections, the tangent is approximated by $\hat{t} \approx \hat{x} + \partial_s H\hat{z}$. The moment balance equation is then

$$B\partial_s^3 H + F_z - F_x\partial_s H = 0 \tag{S 17}$$

Applying $\partial_s(\cdot)$ to this equation and using the relation $dF_z/ds = -p$, we obtain for the pressure

$$p = B\partial_s^4 H - Y\partial_s(F_x\partial_s H) \tag{S 18}$$

where $F_x = Y\partial_s u$.

Weakly Nonlinear Model

Equations (S 15) and (S 18) are approximations for small deflections. In the same order of approximation, we can approximate $x \approx s$ in the right hand side of Eqs. (S 11). Additionally, we can neglect terms of the order $\mathcal{O}[(\partial_x H)^2]$ or higher in Eqs. (S 11) and replace the velocity of the bottom rubber by $v = \partial_t u^{flat}$ where u^{flat} is defined in Eqs. (S 13). It provides the equivalent set (where the arclength s is denoted as x in the following).

$$\begin{aligned}
\partial_t u(x,t) &= -\frac{1}{2\eta}\partial_x p\, H^2 + \frac{1}{\eta}\tau H - \dot{\epsilon}x \\
\partial_t H(x,t) &= \partial_x\left[\frac{1}{3\eta}\partial_x p\, H^3 - \frac{1}{2\eta}\tau H^2\right] + \frac{\dot{\epsilon}}{1 - \dot{\epsilon}t}H
\end{aligned} \tag{S 19}$$

Introducing the constitutive relation for the pressure and shear and neglecting again terms of the order $\mathcal{O}[(\partial_x H)^2]$, we obtain two equations for u and H.

$$\begin{aligned}
\partial_t u(x,t) &= -\frac{1}{2\eta}H^2[B\partial_x^5 H - Y\partial_x^2(\partial_x u\partial_x H)] + \frac{Y}{\eta}H\partial_x^2 u - \dot{\epsilon}x \\
\partial_t H(x,t) &= \frac{1}{3\eta}H^3[B\partial_x^6 H - Y\partial_x^3(\partial_x u\partial_x H)] - \frac{Y}{2\eta}H^2\partial_x^3 u + \frac{\dot{\epsilon}}{1 - \dot{\epsilon}t}H
\end{aligned} \tag{S 20}$$

We observe that there are two times scales in these equations. The height grows with the time scale $\dot\epsilon^{-1}$ (see solution in Eqs. (S 13)), and because our analysis is valid within the framework of linear elasticity ($\epsilon = \dot\epsilon t \ll 1$) we expect the equations to be correct for $t \ll \dot\epsilon^{-1}$. A second time scale is provided by the diffusion term in the right hand side of the first equation. It defines a diffusion coefficient $D = YH/\eta \approx YH_0/\eta$. The effect of this term was extensively studied in Chatterjee et al. [S3], where they demonstrated that a stationary solution for the strain ($\partial_t u = 0$) occurs for times on the order of $t_D = L^2/D$. Here, L represents the half-length of the film, as shown in Fig. S1. The stationary solution is obtained by balancing the last two terms of the equation for horizontal displacement after setting $\partial_x H = 0$, resulting in the equation $0 = D\partial_x^2 u - \dot\epsilon x$. It yields $\gamma = \partial_x u = -\dot\epsilon(L^2 - x^2)/(2D)$ for a system defined in the range $-L < x < L$ with the boundary condition $\gamma = 0$ at the free ends.

To understand how the flat state solution defined in Eqs. (S 13) is a valid base solution before buckling, we observe that derivatives are of the order $\partial_x \sim 1/L$ and $\partial_t \sim 1/t_D$ and rescale the equations by using the dimensionless variables $\bar x = x/L$, $\bar u = u/L$, $\bar t = t/t_D$ and $\bar H = H/H_0$. It yields

$$\begin{aligned}
\partial_{\bar t}\bar u(\bar x,\bar t) &= -\frac{1}{2}\bar H^2\left[\frac{\Lambda^4}{N^2}\partial_{\bar x}^5\bar H - \Lambda^2\partial_{\bar x}^2(\partial_{\bar x}\bar u\partial_{\bar x}\bar H)\right] + \bar H\partial_{\bar x}^2\bar u - \epsilon_D\bar x \\
\partial_{\bar t}\bar H(\bar x,\bar t) &= \frac{1}{3}\bar H^3\left[\frac{\Lambda^4}{N^2}\partial_{\bar x}^6\bar H - \Lambda^2\partial_{\bar x}^3(\partial_{\bar x}\bar u\partial_{\bar x}\bar H)\right] - \frac{1}{2}\bar H^2\partial_{\bar x}^3\bar u + \frac{\epsilon_D}{1-\epsilon_D\bar t}\bar H
\end{aligned} \tag{S 21}$$

where $\Lambda = H_0/L$ and $\epsilon_D = \dot\epsilon t_D$. Taking the limit $\Lambda \to 0$, we obtain the set of equations

$$\partial_{\bar t}\bar u(\bar x,\bar t) = \bar H\partial_{\bar x}^2\bar u - \epsilon_D\bar x \tag{S 22}$$

$$\partial_{\bar t}\bar H(\bar x,\bar t) = -\frac{1}{2}\bar H^2\partial_{\bar x}^3\bar u + \frac{\epsilon_D}{1-\epsilon_D\bar t}\bar H \tag{S 23}$$

that must be solved with the boundary conditions $\bar u|_{\bar x=0} = 0$ (center fixed) and $\partial_{\bar x}\bar u|_{\bar x=\pm1} = 0$ (free ends). The solution is straightforward when taking the approximation $\bar H \approx 1$ in the interaction term in both equations. It yields

$$\begin{aligned}
\bar u(\bar x,\bar t) &= -\frac{\epsilon_D}{2}(\bar x - \bar x^3/3) + \sum_{m=1}^{\infty} A_m\sin(k_m\bar x)e^{-k_m^2\bar t} \\
\bar H(\bar x,\bar t) &= \frac{1}{1-\epsilon_D\bar t} - \frac{1}{2}\frac{1}{1-\epsilon_D\bar t}\left(\epsilon_D\bar t(1-\epsilon_D\bar t/2) - \sum_{m=1}^{\infty}k_m^3A_m\cos(k_m\bar x)\int_0^{\bar t}dt'(1-\epsilon_D t')e^{-k_m^2t'}\right)
\end{aligned} \tag{S 24}$$

where $k_m = (2m-1)\pi/2$ $(m = 1, 2, ..)$ and $A_m = \epsilon_D\int_0^1 dx'(x' - x'^3/3)\sin(k_m x')$. Using repetitively the identity $\sum_{m=1}^{\infty}A_m\sin(k_m\bar x) = \frac{\epsilon_D}{2}(\bar x - \bar x^3/3)$, these solutions can be approximated for $\bar t \ll 1$ to

$$\begin{aligned}
\bar u(\bar x,\bar t) &= -\epsilon_D\bar t\,\bar x + \mathcal{O}(\bar t^2) \\
\bar H(\bar x,\bar t) &= \frac{1}{1-\epsilon_D\bar t} + \frac{\mathcal{O}(\bar t^2)}{1-\epsilon_D\bar t}
\end{aligned} \tag{S 25}$$

which corresponds to the flat solution given in Eqs. (S 13).

In dimensional terms, we conclude that for short times ($t \ll t_D$) the film is under constant compression and flat. Shear and pressure can be neglected under these assumptions. Therefore, in the limit of large systems ($t_D \to \infty$), buckling occurs for a base state described by Eqs. (S 13) (Eq. (3) in the main text). A simple explanation of the flat solution is provided by observing that displacements and elastic forces are small for short times. Hence, the first terms on the right-hand side of Eqs. (S 22) and (S 23) can be neglected, and the dynamics is dictated by the source terms.

Linear Analysis

To study small perturbation of the equations, it is more convenient to use the strain $\gamma = \partial_x u$ as a variable. We take the derivative of the first equation and include the explicit value of u^{flat} in both equations. It yields

$$\begin{aligned}
\partial_t\gamma(x,t) &= \partial_x\left[-\frac{1}{2\eta}\partial_x p\,H^2 + \frac{1}{\eta}\tau H\right] - \dot\epsilon \\
\partial_t H(x,t) &= \partial_x\left[\frac{1}{3\eta}\partial_x p\,H^3 - \frac{1}{2\eta}\tau H^2\right] + \frac{\dot\epsilon}{1-\dot\epsilon t}H
\end{aligned} \tag{S 26}$$

Introducing the constitutive relation for the pressure and shear and neglecting again terms of the order $\mathcal{O}[(\partial_x H)^2]$, we obtain two equations for γ and H. They are

$$\begin{aligned}
\partial_t \gamma(x,t) &= -\frac{1}{2\eta} H^2 [B\partial_x^6 H - Y\partial_x^3(\gamma\partial_x H)] + \frac{Y}{\eta} H \partial_x^2 \gamma - \dot{\epsilon} \\
\partial_t H(x,t) &= \frac{1}{3\eta} H^3 [B\partial_x^6 H - Y\partial_x^3(\gamma\partial_x H)] - \frac{Y}{2\eta} H^2 \partial_x^2 \gamma + \frac{\dot{\epsilon}}{1-\dot{\epsilon}t} H
\end{aligned} \tag{S 27}$$

Equations (S 27) are valid for time scales smaller than $\dot{\epsilon}^{-1}$, so that $\dot{\epsilon}^{-1}$ must be our largest time scale. It implies that the diffusion time must be shorter than $\dot{\epsilon}^{-1}$, or $\epsilon_D = \dot{\epsilon}t_D \ll 1$. Thus, the strain ϵ_D at the time scale t_D must be small. All our finite element simulations satisfy this condition. Because we are interested in the instability of the solution (u^{flat}, H^{flat}) that approximates infinite film conditions, the present analysis is valid for times shorter than the diffusion time ($t \ll t_D$) or, equivalently, $\epsilon \ll \epsilon_D$.

For times smaller than the diffusion time $t \ll t_D$, we expect the effect of the boundaries to be negligible and the flat solution in Eqs. (S 13) to be dominant. To study the stability of this solution, we perform a perturbation analysis of Eqs. (S 27) in the form

$$\begin{aligned}
\gamma &= \gamma^{flat} + \omega(t)\cos(kx) \\
H &= H^{flat} + H_0 \xi(t) \cos(kx)
\end{aligned} \tag{S 28}$$

where $\gamma^{flat} = \partial_x u^{flat} = -\dot{\epsilon}t$. Equations (S 28) correspond to the perturbation defined in Eqs. (9) in the main text. The equations for ω and ξ are

$$\frac{d\omega}{dt} = \frac{1}{2\eta}\chi^2 H_0^2 k^2 (Bk^4 - Yk^2\dot{\epsilon}t)\xi - \frac{Y}{\eta}\chi H_0 k^2 \omega \tag{S 29}$$

$$\frac{d\xi}{dt} = -\frac{1}{3\eta}\chi^3 H_0^2 k^2 (Bk^4 - Yk^2\dot{\epsilon}t)\xi + \frac{Y}{2\eta}\chi^2 H_0 k^2 \omega + \chi\dot{\epsilon}\xi \tag{S 30}$$

where $\chi(t) = 1/(1-\dot{\epsilon}t)$. Because the equations are valid for $t \ll \dot{\epsilon}^{-1}$, we assume $\chi \approx 1$ in the following. These equations have the characteristic time scale $t_\alpha = \eta H_0/Y$ and length scale H_0. We make the time and wavenumber dimensionless by defining $t_* = t/t_\alpha$ and $k_* = kH_0$, and we introduce the dimensionless numbers $N^2 = H_0^2 Y/B$ and $\alpha = \dot{\epsilon}t_\alpha$ to transform the equations into

$$\begin{aligned}
\frac{d\omega}{dt_*} &= \frac{1}{2}{k_*}^2 (N^{-2}{k_*}^4 - {k_*}^2\alpha t_*)\xi - {k_*}^2\omega \\
\frac{d\xi}{dt_*} &= -\frac{1}{3}{k_*}^2 (N^{-2}{k_*}^4 - {k_*}^2\alpha t_*)\xi + \frac{1}{2}{k_*}^2\omega + \alpha\xi
\end{aligned} \tag{S 31}$$

The parameter $N = \sqrt{H_0^2 Y/B} = \sqrt{12}(H_0/h)$ is the Von Kármán number for this system. Most of our simulations are conducted at $H_0/h = 9.84$, for which $N \approx 34$. The dimensionless parameter $\alpha = \dot{\epsilon}t_\alpha$ corresponds to the strain at the characteristic time t_α. The relation between the experimental parameters, β and H_0/h, and the theoretical parameters, N^2 and α, are

$$\alpha = \beta(H_0/h) \tag{S 32}$$

$$N = \sqrt{12}(H_0/h) \tag{S 33}$$

Equations (S 31) can be formally written as

$$\dot{\omega} = a_1(t_*)\xi + b_1\omega \tag{S 34}$$

$$\dot{\xi} = a_2(t_*)\xi + b_2\omega \tag{S 35}$$

where $\dot{\omega} = d\omega/dt_*$ to reduce notation. Taking the time derivative of the second equation gives the expression

$$\ddot{\xi} = \dot{a}_2\xi + a_2\dot{\xi} + b_2\dot{\omega} \tag{S 36}$$

The same equations give $\dot{\omega} = a_1\xi + b_1(\dot{\xi} - a_2\xi)/b_2$, so that we obtain the equation for ξ

$$\ddot{\xi} + 2\mu\dot{\xi} - q^2\xi = 0 \tag{S 37}$$

where

$$\mu = -(a_2 + b_1)/2 \tag{S 38}$$

$$q^2 = \dot{a}_2 + b_2 a_1 - b_1 a_2 \tag{S 39}$$

WKB approximation

The transformation $\xi = e^{-\int_0^{t_*} dt'\,\mu}\psi$ gives the equation $\ddot{\psi} - Q^2\psi = 0$ where $Q = \sqrt{q^2 + \dot{\mu} + \mu^2}$. For the condition $dQ/dt_* \ll Q^2$, we can use the WKB approximation to solve this equation. In terms of the original amplitude the solution is

$$\xi = c_1\frac{e^{S_+(t_*,k_*)}}{Q^{1/2}} + c_2\frac{e^{S_-(t_*,k_*)}}{Q^{1/2}} \tag{S 40}$$

where $S_\pm(t_*,k_*) = \int_0^{t_*} dt'[-\mu \pm Q]$.

To make further progress, we constrain the analysis to the condition of large Von Kármán number ($N^2 \gg 1$) and small strain rate $\alpha \ll 1$ which are also the assumptions in all our numerical work. Under these conditions, we observe that $Q^2 = q^2 + \dot{\mu} + \mu^2 \approx \mu^2$ so that Q^2 is positive. To prove that, we study the ratio

$$r = \frac{q^2 + \dot{\mu}}{\mu^2} = \frac{3N^2{k_*}^2[-{k_*}^6 + \alpha N^2(12 + {k_*}^2(2 + {k_*}^2 t_*)]}{[3N^2{k_*}^2 + {k_*}^6 - \alpha N^2(3 + {k_*}^4 t_*)]^2} \tag{S 41}$$

Assuming that the instability occurs at a time $t_* \sim 1$ with a wavenumber of order $k_* \sim 1$, and that the product of α and N^2 is of order $\alpha N^2 \sim 1$, we estimate $r \sim 1/N^2$. Furthermore, under the same assumptions, we can demonstrate that $\mu \gg 1$. We conclude that $S_-(t_*,k_*) < 0$, and, therefore, the instability is explained by the growth exponent $S_+(t_*,k_*)$, which could be positive or negative. The condition $r \ll 1$ leads to the following approximation for the growth function:

$$S_+(t_*,k_*) = \int_0^{t_*} dt'[-\mu + \sqrt{q^2 + \dot{\mu} + \mu^2}] \approx \int_0^{t_*} dt'\,\frac{q^2 + \dot{\mu}}{2\mu} \tag{S 42}$$

The explicit result is

$$S_+(t_*,\zeta) = -\frac{\zeta t_*}{4} + \left(\frac{1}{2} + \frac{6}{\zeta} + \frac{3}{4\alpha} + \frac{3\alpha}{\zeta^2}\right)\ln\left[\frac{3\zeta + \zeta^3/N^2}{3\zeta + \zeta^3/N^2 - \zeta^2\alpha t_*}\right] \tag{S 43}$$

where we use $\zeta = {k_*}^2$ to simplify the expression.

Asymptotic Analysis

To understand Eq. (S 43), we start the analysis for short timescales by taking the limit $t_* \to 0$. It yields

$$S_+/t_* \;\approx\; \frac{-\zeta^4 + 2N^2\alpha\zeta(12 + \zeta) + 12N^2\alpha^2}{4\zeta(3N^2 + \zeta^2)} \tag{S 44}$$

In our numerical simulations we are interested in the limit of large Von Kármán number and small values of α; hence, we take the limits $N^2 \to \infty$ and $\alpha \to 0$. Assuming that the maximum is reached for a ζ_m such that $\zeta_m \sim 1$ and neglecting terms of order $\mathcal{O}(\alpha^2)$, we obtain the approximate growth function

$$\begin{aligned} S_+/t_* &\approx \frac{-\zeta^4 + 2N^2\alpha\zeta(12 + \zeta)}{12zN^2} = \frac{-\zeta^3}{12N^2} + 2\alpha + \frac{\alpha\zeta}{6} \\ &\approx 2\alpha + \frac{-\zeta^3}{12N^2} + \frac{\alpha\zeta}{6} \end{aligned} \tag{S 45}$$

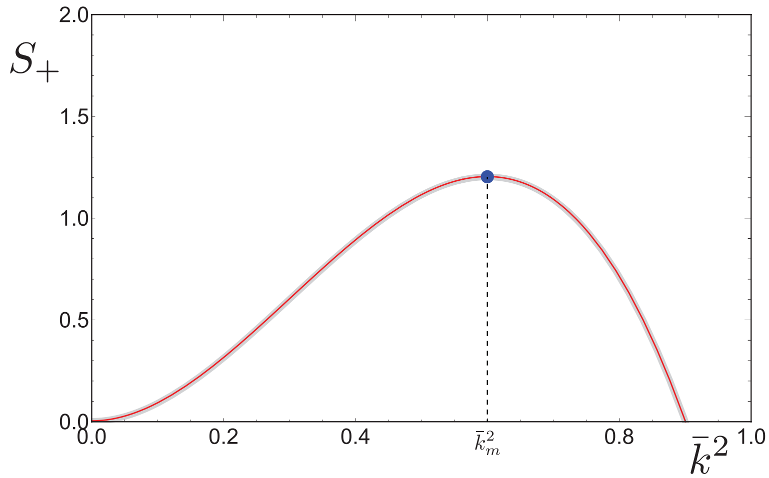

Fig. S 2. Growth S_+ as a function of k_*^2 for $\epsilon = 0.002$. The other parameters are $N^2 = 900$ and $\alpha = 1.67 \times 10^{-8}$ (equivalent to $H_0/h = 10$, $\nu = 0.5$, and $\beta = 1.25 \times 10^{-9}$). The thick gray line corresponds to the exact value of S_+ given in Eq. (S 42) and the red line is the approximation obtained in Eq. (S 49). The blue dot marks the wavenumber at the maximum which is predicted by Eq. (S 50).

The growth function has a maximum which is

$$\frac{d(S_+/t_*)}{d\zeta} = 0 \longrightarrow \frac{-\zeta_m^2}{4N^2} + \frac{\alpha}{6} = 0 \longrightarrow \zeta_m = \frac{1}{6}\sqrt{24\alpha N^2} \tag{S 46}$$

A linear expansion in time of the growth exponent is not sufficient to capture the time behavior of the maximum $\zeta_m(t_*)$. Because of the logarithmic dependence with time of the growth exponent, we expect that a second order expansion in t_* should be sufficient to capture the time behavior. It yields

$$S_+/t_* \approx \frac{-\zeta^4 + 2N^2\alpha\zeta(12+\zeta) + 12N^2\alpha^2}{4\zeta(3N^2+\zeta^2)} + \frac{\alpha t_* N^4(3\zeta^2 + 2\zeta(12+\zeta)\alpha + 12\alpha^2)}{8(3N^2+\zeta^2)^2}$$

To avoid dangerous terms of the form αN^2, we do a systematic expansion with respect to the α variable using the scaled parameters $\tilde{\zeta} = \zeta/\sqrt{\alpha N^2}$ and $\tilde{N}^2 = \alpha N^2$ so that they can be assumed to be of order $\mathcal{O}(1)$. It gives

$$S_+/(\alpha t_*) \approx 2 - \frac{1}{12}\tilde{\zeta}^3\sqrt{\tilde{N}^2} + \frac{1}{6}\tilde{\zeta}\sqrt{\tilde{N}^2} + \frac{1}{24}\tilde{N}^2\tilde{\zeta}^2 t_* + \mathcal{O}(\alpha, t_*^2) \tag{S 47}$$

In terms of the original variables and the strain $\epsilon = \alpha t_*$, this is

$$S_+/\epsilon \approx 2 - \frac{1}{12}\frac{\zeta^3}{\alpha N^2} + \frac{1}{6}\zeta + \frac{1}{24}\frac{\epsilon\zeta^2}{\alpha} \tag{S 48}$$

We can verify that this relation fits the growth exponent for both short and long times. Figure S2 compares this approximation for the growth function with the exact expression given in Eq. (S 42), showing an excellent agreement. Maximization of this relation gives

$$\frac{d(S_+/\epsilon)}{d\zeta} = 0 \longrightarrow \frac{-\zeta_m^2}{4\alpha N^2} + \frac{1}{6} + \frac{1}{12}\frac{\epsilon\zeta_m}{\alpha} = 0 \longrightarrow \zeta_m = \frac{1}{6}\sqrt{\alpha N^2}\left[(N^2/\alpha)^{1/2}\epsilon + \sqrt{24 + N^2\epsilon^2/\alpha}\right]$$

In summary, the growth exponent and wavenumber at the maximum are

$$S_+(\epsilon, k)/\epsilon \approx 2 - \frac{1}{12}\frac{(kH_0)^6}{\alpha N^2} + \frac{1}{6}(kH_0)^2 + \frac{1}{24}\frac{\epsilon(kH_0)^4}{\alpha} \tag{S 49}$$

$$(k_m H_0)^2 = \frac{1}{6}\left[N^2\epsilon + \sqrt{24N^2\alpha + N^4\epsilon^2}\right] \tag{S 50}$$

in terms of the original variables. For a large ratio $\varphi = N\epsilon/\sqrt{\alpha}$ between the two terms in the square root of Eq. (S 50), we obtain Eq. (12) in the main text.

“Instantaneous” compression

Here we adapt the analysis of Huang et al. [S1, S2] for an instantaneous compression applied by thermal expansion or swelling. Under plain-strain conditions, the force displacement relation is reduced to

$$F_x = Y(\partial_x u - \epsilon_0) \tag{S 51}$$

The film is initially compressed by a force $F_x = -Y\epsilon_0$ and relaxation takes place until the strain reaches the value $\partial_x u = \epsilon_0$. However, the film can buckle before relaxation is complete. Assuming the same configuration defined in Fig. S1 with $v = 0$, we obtain the equations

$$\begin{aligned}\partial_t u(x,t) &= -\frac{1}{2\eta}\partial_x p\, H^2 + \frac{1}{\eta}\tau H\\ \partial_t H(x,t) &= \partial_x\left[\frac{1}{3\eta}\partial_x p\, H^3 - \frac{1}{2\eta}\tau H^2\right]\end{aligned} \tag{S 52}$$

Equations (S 15) and (S 18) give the shear and pressure in terms of the force in the section F_x; however, the constitutive relation (S 51) must be used for this case. Neglecting nonlinear terms involving derivatives in the height ($(\partial_x H)^2, \partial_x H\partial_x^2 H$, etc.), the equations become

$$\begin{aligned}\partial_t u(x,t) &= -\frac{1}{2\eta}H^2[B\partial_x^5 H - Y\partial_x^2((\partial_x u - \epsilon_0)\partial_x H)] + \frac{Y}{\eta}H\partial_x^2 u\\ \partial_t H(x,t) &= \frac{1}{3\eta}H^3[B\partial_x^6 H - Y\partial_x^3((\partial_x u - \epsilon_0)\partial_x H)] - \frac{Y}{2\eta}H^2\partial_x^3 u\end{aligned} \tag{S 53}$$

The same analysis applied to the previous problem before buckling reveals that, for small values of $\Lambda = H_0/L$, these equations in dimensionless variables become

$$\partial_{\bar t}\bar u(\bar x,\bar t) = \bar H\partial_{\bar x}^2\bar u + \mathcal{O}(\Lambda^2, \Lambda^4/N^2) \tag{S 54}$$

$$\partial_{\bar t}\bar H(\bar x,\bar t) = -\frac{1}{2}\bar H^2\partial_{\bar x}^3\bar u + \mathcal{O}(\Lambda^2, \Lambda^4/N^2) \tag{S 55}$$

The boundary conditions to solve these equations are $\bar u|_{\bar x=0} = 0$ (center fixed) and $\partial_{\bar x}\bar u|_{\bar x=\pm 1} = \epsilon_0$ (free ends). The solution for the horizontal displacement when taking the approximation $\bar H \approx 1$ in the interaction term in both equations is

$$\bar u(\bar x,\bar t) = -\epsilon_0\bar x + \sum_{m=1}^{\infty} A_m \sin(k_m\bar x)e^{-k_m^2\bar t} \tag{S 56}$$

where $k_m = (2m-1)\pi/2$ ($m = 1, 2, ..$) and $A_m = 2\epsilon_0\int_0^1 dx'x'\sin(k_m x')$. Using repetitively the identity $\sum_{m=1}^{\infty} A_m\sin(k_m\bar x) = \epsilon_0\bar x$, these solutions can be approximated for $\bar t \ll 1$ to

$$\begin{aligned}\bar u(\bar x,\bar t) &= 0 + \mathcal{O}(\bar t^2)\\ \bar H(\bar x,\bar t) &= 1 + \mathcal{O}(\bar t^2)\end{aligned} \tag{S 57}$$

Thus, the film is initially undeformed but compressed with the constant cross section force $F_x = -Y\epsilon_0$. Boundary layers propagate to the center of the film in such a way that $F_x = 0$ and $\partial_x u = \epsilon_0$ for $t \gg t_D$.

Repeating the analysis for the case of the moving wall in terms of the strain $\gamma = \partial_x u$, we transform Eqs. (S 53) into

$$\begin{aligned}\partial_t\gamma(x,t) &= -\frac{1}{2\eta}H^2[B\partial_x^6 H - Y\partial_x^3((\gamma - \epsilon_0)\partial_x H)] + \frac{Y}{\eta}H\partial_x^2\gamma\\ \partial_t H(x,t) &= \frac{1}{3\eta}H^3[B\partial_x^6 H - Y\partial_x^3((\gamma - \epsilon_0)\partial_x H)] - \frac{Y}{2\eta}H^2\partial_x^2\gamma\end{aligned} \tag{S 58}$$

To study the stability of this solution for short times when the base state is $\gamma = 0$ and $H = H_0$, we do a perturbation analysis of Eqs. (S 58) of the form

$$\begin{aligned}\gamma &= \omega(t)\cos(kx)\\ H &= H_0 + H_0\xi(t)\cos(kx)\end{aligned} \tag{S 59}$$

The equations for ω and ξ are

$$\frac{d\omega}{dt} = \frac{1}{2\eta}\chi^2 H_0^2 k^2 (Bk^4 - Yk^2\epsilon_0)\xi - \frac{Y}{\eta}\chi H_0 k^2 \omega \tag{S 60}$$

$$\frac{d\xi}{dt} = -\frac{1}{3\eta}\chi^3 H_0^2 k^2 (Bk^4 - Yk^2\epsilon_0)\xi + \frac{Y}{2\eta}\chi^2 H_0 k^2 \omega \tag{S 61}$$

where $\chi = 1/(1-\epsilon)$. Because the equations are valid for small strain, we assume $\chi \approx 1$ in the following. Defining the dimensionless time scale $t_* = Yt/(\eta H_0)$, wavenumber $k_* = kH_0$, and number $N^2 = H_0^2 Y/B$, we obtain

$$\begin{aligned}\frac{d\omega}{dt_*} &= \frac{1}{2}{k_*}^2(N^{-2}{k_*}^4 - {k_*}^2\epsilon_0)\xi - {k_*}^2\omega \\ \frac{d\xi}{dt_*} &= -\frac{1}{3}{k_*}^2(N^{-2}{k_*}^4 - {k_*}^2\epsilon_0)\xi + \frac{1}{2}{k_*}^2\omega\end{aligned} \tag{S 62}$$

An equivalent set of equations was obtained by Huang and Suo [S1, S2] but under biaxial compression. These equations can be transformed into a second order equation with constant coefficients

$$\ddot{\xi} + 2\mu\dot{\xi} - q^2\xi = 0 \tag{S 63}$$

that has the solution

$$\xi = c_1 e^{S_+(t_*,k_*)} + c_2 e^{S_-(t_*,k_*)} \tag{S 64}$$

where $S_\pm(t_*,k_*) = (-\mu \pm \sqrt{q^2+\mu^2})t_*$. The instability is explained by the growth exponent $S_+(t_*,k_*)$ that could be positive or negative. Moreover, we can approximate $S_+(t_*,k_*)$ as

$$S_+(t_*,k_*) = (-\mu + \sqrt{q^2+\mu^2})t_* \approx \frac{q^2}{2\mu}t_* \tag{S 65}$$

The explicit result is

$$S_+(t_*,\zeta) = \frac{(\epsilon_0 - \zeta/N^2)\zeta^2}{4(3-\zeta\epsilon_0) + \zeta^2/N^2}t_* \tag{S 66}$$

where $\zeta = k_*^2$. Taking the scaled parameter $\bar{N}^2 = \epsilon_0 N^2$ and scaled variable $\bar{\zeta} = \zeta/(\epsilon_0 N^2)$, we obtain for small strain

$$S_+(t_*,\bar{\zeta}) = \bar{\zeta}^2\frac{1-\bar{\zeta}}{12\bar{N}^2}\epsilon_0 t_* + \mathcal{O}(\epsilon_0^2) \tag{S 67}$$

In terms of the original variables

$$S_+(t_*,\zeta) = \frac{1}{12}(\epsilon_0 - \zeta/N^2)\zeta^2 t_* \longrightarrow S_+(t_*,\zeta) = \frac{1}{12}(\epsilon_0 - (kH_0)^2/N^2)(kH_0)^4 t_* \tag{S 68}$$

which is Eq. (19) in the main text.

Compression of a filament

Chopin et al. [S4] validated a model predicting the deformation of a filament immersed in a viscous fluid undergoing compression. The source of dissipation for this case is explained by the Stokes flow around the filament and can be computed estimated as $\vec{F}_d = -\mu\vec{v}$, where $\mu = 4\pi\eta/\ln(\phi)$ and ϕ is the aspect ratio of the filament. Although the effective value of ϕ is unknown after buckling, the logarithmic dependence makes this parameter unimportant. The computation of the drag force requires a completely 3D treatment of the fluid-structure interactions; however, we can assume the deformation is 2D in the plane $x-z$. Equations (S 14) and (S 16) are for this case

$$\frac{d\vec{F}}{ds} + \vec{F}_d = 0 \tag{S 69}$$

$$\frac{d\vec{M}}{ds} + \hat{t}\times\vec{F} = 0 \tag{S 70}$$

where for a filament $\vec{M} = EI\partial_s^2 H\hat{x} \times \hat{z}$ and $\hat{t}$ is the tangent to the $x-z$ curve defined by the film centerline. Here I is the moment of inertia which is $I = Wh^3/12$ for a rectangular filament of thickness h and width W. For small deflections, the tangent is approximated by $\hat{t} \approx \hat{x} + \partial_s H\hat{z}$. The moment balance equation is then

$$EI\partial_s^3 H + F_z - F_x\partial_s H = 0 \tag{S 71}$$

The non-slip boundary condition for a velocity written in lagrangian coordinates gives

$$\partial_t u(s,t) = v_x(s,t) \tag{S 72}$$

$$\partial_t H(s,t) = v_z(s,t) \tag{S 73}$$

and the force balance yields the dynamics

$$\frac{dF_x}{ds} = \mu\partial_t u \tag{S 74}$$

$$\frac{dF_z}{ds} = \mu\partial_t H \tag{S 75}$$

We now consider the situation where the filament experiences thermal compression of strain $\epsilon = \alpha_l \Delta T$ where α_l is the linear expansion coefficient and ΔT is the increase in temperature. The force-strain relation gives

$$F_x = EA(\partial_s u - \epsilon) \tag{S 76}$$

where A is the section area of the filament. Applying $\partial_s(\cdot)$ to Eq. (S 71) and using the previous relation, we obtain the equations

$$\mu\partial_t u = EA\partial_s^2 u \tag{S 77}$$

$$\mu\partial_t H = -EI\partial_s^4 H + EA\partial_s[(\partial_s u - \epsilon)\partial_s H] \tag{S 78}$$

It yields the linear set of equations

$$\mu\partial_t u = EA\partial_s^2 u \tag{S 79}$$

$$\mu\partial_t H = -EI\partial_s^4 H - EA\epsilon\partial_s^2 H \tag{S 80}$$

Note that the horizontal and vertical displacements are decoupled which represents a significant simplification from the problem studied in the previous sections. For a linear change in temperature with time, we can write the strain as $\epsilon = \dot{\epsilon}t$. Again, we do a perturbation analysis of the form

$$H = \xi(t)\cos(kx) \tag{S 81}$$

which is valid for long filaments since it does not fulfill the exact boundary conditions. It gives the equation and solution

$$\mu\partial_t\xi = (-EIk^4 + EA\dot{\epsilon}tk^2)\xi \longrightarrow \xi = \xi_0 e^{S(t,k)} \tag{S 82}$$

where

$$S(t,k) = \frac{1}{\mu}\left(-EIk^4 t + EA\dot{\epsilon}k^2\frac{t^2}{2}\right) \tag{S 83}$$

which is Eq. (23) in the main text.

Section S2: Simulation methods

Simulations were conducted under plane strain conditions using the ABAQUS explicit solver. The simulation geometry is shown in Fig. 2 in the main text, with a finite length along the x-direction, $2L \gg H_0$. The film and the rubber substrate, of thickness h and h_r respectively, were both modeled by neo-Hookean 1D beam elements with shear modulus G and G_r, respectively. The modulus of the film referred to in the main text is $E = 2G(1+\nu)$ where $\nu = 0.49$ to approximate incompressibility. Similarly, the Young modulus of the rubber layer is $E_r = 2G_r(1+\nu_r)$ where $\nu_r = \nu$. The liquid layer was modeled as a 2D viscoelastic material with a modulus that decreases exponentially in time (see below and [S5]). The length of the liquid layer is much longer than the length of the film and coincides with the length of the rubber layer defined by $2L_r$ ($2L_r > 2L$).

The free surface of the film was set to be stress-free. The ends of the film were set to have zero force and moment. The ends of the rubber layer were translated inwards at fixed velocity v_{end}, and out-of-plane deformation of this layer was forbidden via a roller boundary condition. The modulus of the rubber was set to be much larger than of the film ensuring that the in-plane strain, ϵ, at all locations in the rubber was equal to the nominal value, i.e. at all locations x,

$$\epsilon = \frac{end\ displacement}{rubber\ length} = \dot{\epsilon}t \tag{S 47}$$

where $\dot{\epsilon} = v_{end}/L_r$ is the imposed strain rate. The corresponding compression of the fluid layer then induced buckles over the mid-section of the film.

To model the viscous fluid layer, we use a viscoelastic material with an exponentially-decaying modulus. The time-dependent modulus is defined as:

$$G(t) = G_0 \exp\left(-\frac{t}{\tau_v}\right) \tag{S 48}$$

Accordingly, the corresponding fluid part has a viscosity η

$$\eta = \int_0^\infty G(t)dt = G_0\tau_v \tag{S 49}$$

It is expected that for the condition $\dot{\epsilon}\tau_v \ll 1$ the system is dominated by viscosity and displacements become important. To compute large displacements correctly, we use a hyperelastic material for the liquid layer. Specifically, a Neo-Hookean model defined by the parameters $C_1 = G_0/2$ and $D_1 = 3(1-2\nu_0)/[(1+\nu_0)G_0]$ (see Ref. [S5]). Since the elastic part is irrelevant for the large time behavior of the fluid, the initial modulus G_0 was set as equal to the film modulus, G. Although the fluid is incompressible, numerical instability is observed for a Poisson ratio $\nu_0 = 1/2$; hence, the Poisson ratio of the viscoelastic material was set to a safe value of ν_0 =0.475. To summarize, Table S1 provides all the parameters used in the simulations.

To validate the modeling of the fluid part, we tested the material by studying different flows under well-known boundary conditions in fluid mechanics (Couette flow, squeeze film flow, etc.) [S6]. In all our numerical experiments, we re-obtained the classical solutions for Stokes flow as long the characteristic time of the problem is larger than the relaxation time of the viscoelastic material and the Reynolds number is $Re \ll 1$.

Table S1: Simulation Parameters

Property	Dimensional value used in simulation	Non-dimensional value mentioned in the main text
Rubber layer modulus	$E_r = 1.8 \times 10^7$ Pa	
Rubber layer thickness (1D elements)	$h_r = 0.0254$ mm	
Length of rubber	$L_r = 50$ mm	$L_r/H_0 = 200$
Film modulus	$E = 1.8 \times 10^6$ Pa	
Film Poisson ratio	$\nu = 0.49$	
Film thickness (1D elements)	$h = 0.0254$ mm	
Film half length	$L = 47.5$ mm	$L\,/H_0 = 190$
Liquid layer thickness (2D elements)	$H_0 = 0.25$ mm	$H_0/h = 9.84$
Initial shear modulus of the liquid	$G_0 = 0.6 \times 10^6$ Pa	
Relaxation time of the liquid	$\tau_v = 5 \times 10^{-4}$ s	
Liquid layer Poisson ratio	$\nu_0 = 0.475$	
Viscosity of the liquid layer	$\eta = G_0\tau_v = 300$ Pa. s	
Compression rate (fixed H_0)	$\dot{\epsilon} = 0.005\ \mathrm{s}^{-1}$	$\beta = (1-\nu^2)\eta\dot{\epsilon}/E = 6.3 \times 10^{-7}$ $\dot{\epsilon}\tau_v = 2.5 \times 10^{-6}$

Section S3:

Correlation functions for calculating wavelength and inter-ridge distance

The amplitude and velocity autocorrelation functions are defined as

$$C_a(b) = \frac{\int \Delta H(s)\Delta H(s+b)ds}{\int [\Delta H]^2 ds} \tag{S 50}$$

$$C_v(b) = \frac{\int \Delta\dot{u}(s)\Delta\dot{u}(s+b)ds}{\int [\Delta\dot{u}(s)]^2 ds} \tag{S 51}$$

where s is the arclength along the film. These are shown below for the same data as Fig. 3 in the main text. As with all quantitative analysis of the simulations in this paper, we use the central portion of film of contour length $1200h$, i.e. the portion of the film that, at $t = 0$, lies between $-\frac{rh}{2} < x < \frac{rh}{2}$ with $r = 1200$.

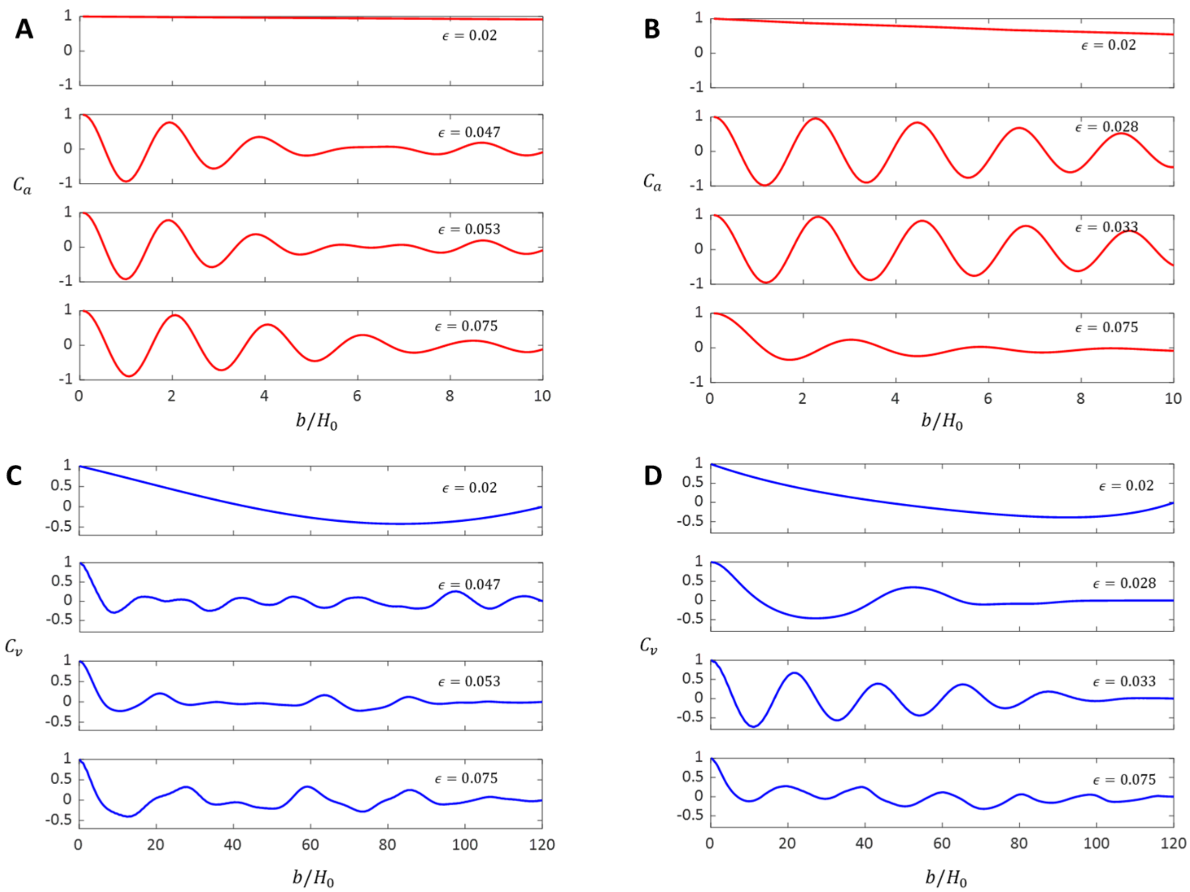

Fig. S 3: A. Amplitude and C. velocity autocorrelation functions at the same strains as the wrinkling simulation in Fig. 3C in the main text. B and D are the autocorrelation functions for the ridge localization simulation in Fig. 3D in the main text. Wavelength is identified as the location of first peak in C_a. The inter-ridge distance is identified as the location of the first peak in C_v. As explained in the main paper, for simulations such as C above, the peak in C_v is not well defined and hence the inter-ridge distance is not reported. Such simulations are deemed to be in the uniformly-wrinkled state.

Critical strain for wrinkling versus β and H_0/h

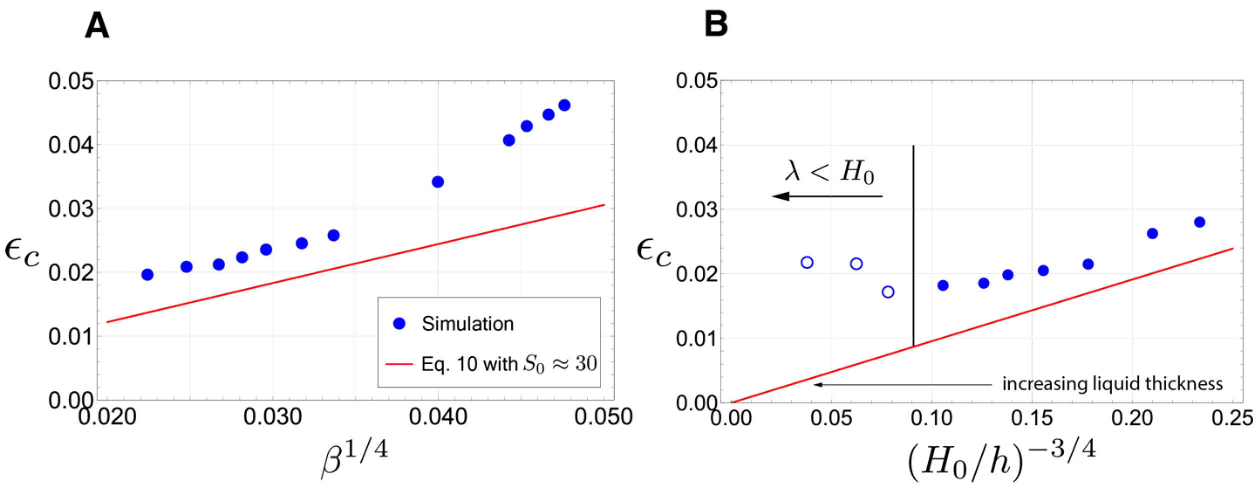

Fig. S 4: A. Critical strain versus dimensionless rate for $H_0/h = 9.84$. B. Critical strain versus aspect ratio H_0/h for $\beta = 6.3 \times 10^{-7}$. Red line corresponds to the parameter $S_0 \approx 30$ used to fit the data in Fig. 4 (main text). The theory is based on the lubrication approximation which is only valid for $\lambda \gg H_0$, and hence increasing deviations are expected as H_0 increases. Indeed, the simulations for which $\lambda < H_0$ (shown in open symbols) deviate most severely from the expected trends.

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