Supplementary Material

1. Derivation of equations (4) and (7a,b)

Since the equations are linear, there is no loss in generality to set $c_0 = 1$. The results in the main text are then obtained by multiplying c derived below by the actual value of c_0.

Taking the Laplace transform of (3) with respect to t and using the initial condition (2), we obtain

$$p\tilde{c} = D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{c}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \tilde{c}}{\partial \theta^2}\right], \tag{A1}$$

where $\tilde{c} = \int_0^\infty e^{-pt}c(r,\theta,t)dt$ is the Laplace transform of c and p is the transform variable. The boundary conditions (1a,b) transform to

$$\tilde{c}(r,\theta = 0,\theta_0, p) = 1/p\,. \tag{A2}$$

Motivated by Jaeger, we look for a solution of the form:

$$\tilde{c} = \frac{1}{p}+\sum_{k=1}^{\infty}\sin(\nu_k\theta)\int_0^\infty f_k(u)J_{\nu_k}(ur)du\,,\; \nu_k = \frac{k\pi}{\theta_0}, \tag{A3}$$

where J_{ν_k} is the Bessel function of order ν_k and f_k is a function to be determined. Note that the boundary conditions (A2) are automatically satisfied. Substituting (A3) into (A1), we obtain

$$\frac{p}{D}\left[\frac{1}{p}+\sum_{k=1}^{\infty}\sin(\nu_k\theta)\int_0^\infty f_k(u)J_{\nu_k}(ur)du\right] = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left(\sum_{k=1}^{\infty}\sin(\nu_k\theta)\int_0^\infty f_k(u)J_{\nu_k}(ur)du\right). \tag{A4}$$

The theory of Hankel transform tells us that the RHS of (A4) is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left(\sum_{k=1}^{\infty}\sin(\nu_k\theta)\int_0^\infty f_k(u)J_{\nu_k}(ur)du\right) = -\sum_{k=1}^{\infty}\sin(\nu_k\theta)\int_0^\infty u^2 f_k(u)J_{\nu_k}(ur)du\,. \tag{A5}$$

Using (A5), (A4) is

$$-\frac{1}{D} = \sum_{k=1}^{\infty}\sin(\nu_k\theta)\left[\int_0^\infty\left(u^2+\frac{p}{D}\right)f_k(u)J_{\nu_k}(ur)du\right],\; \theta_0 > \theta > 0 \tag{A6}$$

The theory of Fourier series tells us that

$$\frac{1}{D}=\sum_{k=1}^{\infty}A_k\sin\left(\frac{k\pi}{\theta_o}\theta\right),\ A_k=\frac{2}{\theta_0 D\nu_k}\int_0^{k\pi}\sin u du=-\frac{2}{Dk\pi}\cos u\Big|_0^{k\pi}=\frac{-2}{Dk\pi}\begin{cases}0 & k=\text{even}\\ -2 & k=\text{odd}\end{cases}. \qquad \text{(A7a,b)}$$

Thus,

$$\frac{1}{D}=\frac{4}{D(2k+1)\pi}\sum_{k=0}^{\infty}\sin\left(\frac{(2k+1)\pi}{\theta_o}\theta\right). \qquad \text{(A7c)}$$

Equation (A7b) indicates that all the even terms of the series in (A3) vanish, so (A3) can be simplified to

$$\tilde{c}=\frac{1}{p}+\sum_{k=0}^{\infty}\sin(\lambda_k\theta)\int_0^{\infty}f_{2k+1}(u)J_{\lambda_k}(ur)du,\quad \lambda_k=\frac{2k+1}{\theta_0}\pi. \qquad \text{(A8)}$$

Furthermore, (A6) and (A7) imply that

$$\int_0^{\infty}\left(u^2+\frac{p}{D}\right)f_{2k+1}(u)J_{\lambda_k}(ur)du=\frac{-4}{D(2k+1)\pi}. \qquad \text{(A9)}$$

Equation (A9), together with the identity

$$\int_0^{\infty}\frac{J_{\lambda_k}(ur)}{u}du=\frac{1}{\lambda_k}=\frac{\theta_0}{(2k+1)\pi} \qquad \text{(A10)}$$

allows us to conclude that

$$f_k(u)=\frac{-4}{D\theta_0\left(u^2+p/D\right)u}. \qquad \text{(A11)}$$

Substituting (A11) into (A8) gives:

$$\tilde{c}=\frac{1}{p}-\frac{4}{\theta_0}\sum_{k=0}^{\infty}\sin\left(\frac{2k+1}{\theta_0}\pi\theta\right)\int_0^{\infty}\frac{1}{\left(Du^2+p\right)u}J_{\lambda_k}(ur)du, \qquad \text{(A12)}$$

Taking the inverse Laplace transform of (A12) inside the integral, we have

$$c=1-\frac{4}{\theta_0}\sum_{k=0}^{\infty}\sin(\lambda_k\theta)\int_0^{\infty}\frac{e^{-Dtu^2}}{u}J_{\lambda_k}(ur)du=1-\frac{4}{\theta_0}\sum_{k=0}^{\infty}\sin(\lambda_k\theta)\int_0^{\infty}\frac{e^{-Dtw^2/r^2}}{w}J_{\lambda_k}(w)dw. \qquad \text{(A13)}$$

Next, the property of Bessel function [1] tells us that

$$\int_0^{\infty}\frac{e^{-Dtw^2/r^2}}{w}J_{\lambda_k}(w)dw=\frac{\Gamma(\lambda_k/2)}{2\Gamma(\lambda_k+1)}\left[\frac{r}{2\sqrt{Dt}}\right]^{\lambda_k}M\left[\frac{\lambda_k}{2},\lambda_k+1,-\frac{r^2}{4Dt}\right], \qquad \text{(A14)}$$

where $M\left[\frac{\lambda_k}{2},\lambda_k+1,-\frac{r^2}{4Dt}\right]$ is the Kummer function. Combining (A14) and (A13) results in (4), i.e.,

$$c=1-\frac{2}{\theta_0}\sum_{k=0}^{\infty}\sin\left(\lambda_k\theta\right)\frac{\Gamma\left(\lambda_k/2\right)}{\Gamma\left(\lambda_k+1\right)}\left[\frac{r}{2\sqrt{Dt}}\right]^{\lambda_k} M\left[\frac{\lambda_k}{2},\lambda_k+1,-\frac{r^2}{4Dt}\right]. \tag{A15}$$

To study how c behaves for small or large r, we note the following asymptotic behavior of the Kummer function: as $Z\equiv\frac{r^2}{4Dt}$ goes to zero (near the tip of wedge or long times)

$$M\left[\frac{\lambda_k}{2},\lambda_k+1,-\frac{r^2}{4Dt}\right]=1-\frac{\lambda_k}{2\left(\lambda_k+1\right)}\frac{r^2}{4Dt}+O\left(\frac{r^2}{4Dt}\right)^2. \tag{A16}$$

Substituting (A16) into (A15), we obtain the leading order behavior

$$c\approx 1-\frac{4}{\theta_0}\sum_{k=0}^{\infty}\sin\left(\lambda_k\theta\right)\frac{\Gamma\left(\lambda_k/2\right)}{2\Gamma\left(\lambda_k+1\right)}\left[\frac{r}{2\sqrt{Dt}}\right]^{\lambda_k}\approx 1-\frac{4}{\theta_0}\sin\left(\frac{\pi\theta}{\theta_0}\right)\frac{\Gamma\left(\frac{\pi}{2\theta_0}\right)}{2\Gamma\left(\frac{\pi}{\theta_0}+1\right)}\left[\frac{r}{2\sqrt{Dt}}\right]^{\frac{\pi}{\theta_0}}. \tag{A17}$$

Consider the special cases $\theta_0=\pi/2$ and $\theta_0=3\pi/2$. The near tip solutions are, respectively,

$$c_{\theta_0=\pi/2}\approx 1-\frac{2}{\pi}\sin\left(2\theta\right)\left[\frac{r}{2\sqrt{Dt}}\right]^2,\quad c_{\theta_0=3\pi/2}\approx 1-\frac{4}{3\pi}\sin\left(\frac{2\theta}{3}\right)\frac{\Gamma\left(1/3\right)}{\Gamma\left(5/3\right)}\left[\frac{r}{2\sqrt{Dt}}\right]^{\frac{2}{3}}. \tag{A18a,b}$$

2. Derivation of equation (11)

The differential equation is:

$$D\nabla^2 c=\partial c/\partial t \tag{A19}$$

The boundary conditions are:

$$c(x=\pm L,y<0,t>0)=c_0,\quad c(|x|<L,y=0,t>0)=c_0, \tag{A20a,b}$$

with initial condition

$$c(x,|y|<L,t=0)=0. \tag{A21}$$

We first subtract the solution of an *infinite* strip occupying $y \in (-\infty, \infty)$ whose boundary at $x = \pm L$ is subjected to the same boundary condition (A20a) and initial condition (A21), we denote this solution by $c_\infty(x,t)$. It can be easily verified that this solution is

$$c_\infty(x,t)/c_0 = 1 - \sum_{n=0}^{\infty} a_n e^{\frac{-(2n+1)^2 \pi^2}{4L^2} Dt} \cos\left(\frac{2n+1}{2L}\pi x\right). \tag{A22a}$$

where a_k are the Fourier coefficients of the $\cos\left(\frac{2n+1}{2L}\pi y\right)$ series, they are determined by the initial condition

$$c_\infty(x,t=0) = 0 \Rightarrow 1 = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n+1}{2L}\pi x\right) \Rightarrow a_n = \frac{(-1)^n 4}{(2n+1)\pi}. \tag{A22b}$$

Define

$$\phi \equiv \frac{c - c_\infty(x,t)}{c_0}, \tag{A23}$$

then ϕ satisfies the following BCs and IC

$$\phi(|x| \le L, y=0, t>0) = \sum_{n=0}^{\infty} a_n e^{-\frac{(2n+1)^2 \pi^2}{4L^2} Dt} \cos\left(\frac{2n+1}{2L}\pi x\right), \tag{A24a}$$

$$\phi(x = \pm L, y<0, t>0) = 0, \tag{A24b}$$

$$\phi(|x| < L, y<0, t=0) = 0. \tag{A24c}$$

Let us look for a solution of the form

$$\phi = \sum_{n=0}^{\infty} f_n(y,t) \cos(\lambda_n x), \ \lambda_n = \frac{2n+1}{2L}\pi. \tag{A25}$$

Substituting (A25) into (A19), we obtain

$$D\left[\frac{\partial^2 f_n}{\partial y^2} - \lambda_n^2 f_n\right] = \frac{\partial f_n}{\partial t}. \tag{A26}$$

Taking the Laplace transform of (A26) and using $f_n(x,t=0) = 0$ which satisfies the IC (A24c), (A26) is

$$\left[\frac{\partial^2 \tilde{f}_n}{\partial y^2} - \left(\lambda_n^2 + \frac{p}{D}\right)\tilde{f}_n\right] = 0 \Rightarrow \tilde{f}_n = A_n(p) \exp\left(-|y|\sqrt{\lambda_n^2 + \frac{p}{D}}\right), \tag{A27}$$

where $\tilde{f}_n$ is the Laplace transform of f_n and we have used the fact that $f_n(y,t)$ is bounded as $y \to -\infty$, hence one of the linearly independent solution drops out. At y=0, using (A24a)

$$f_n(y=0,t>0) = a_n e^{-\lambda_n^2 Dt} \Rightarrow \tilde{f}_n(y=0,p) = A_n(p) = \frac{a_n}{p+\lambda_n^2 D}. \tag{A28}$$

Thus,

$$\tilde{f}_n = \frac{a_n}{p+\lambda_n^2 D} \exp\left(-\frac{|y|}{\sqrt{D}}\sqrt{D\lambda_n^2 + p}\right). \tag{A29}$$

Inverting $\tilde{f}_n$ gives

$$f_n(y,t) = a_n e^{-D\lambda_n^2 t} erfc\left(\frac{|y|}{2\sqrt{Dt}}\right). \tag{A30}$$

Combining (A30) and (A25), we have

$$\phi = erfc\left(\frac{|y|}{2\sqrt{Dt}}\right)\sum_{n=0}^{\infty} a_n e^{-D\lambda_n^2 t}\cos(\lambda_n x),\ \lambda_n = \frac{2n+1}{2L}\pi. \tag{A31}$$

The exact solution is, according to (A25),

$$c \equiv c_0\phi + c_\infty(x,t) = c_0\left[erfc\left(\frac{|y|}{2\sqrt{Dt}}\right)\sum_{n=0}^{\infty} a_n e^{-D\lambda_n^2 t}\cos(\lambda_n x) + 1 - \sum_{n=0}^{\infty} a_n e^{-D\lambda_n^2 t}\cos(\lambda_n x)\right]. \tag{A32}$$

Therefore,

$$c/c_0 = 1 - \left[1 - erfc\left(\frac{|y|}{2\sqrt{Dt}}\right)\right]\sum_{n=0}^{\infty} a_n e^{-D\lambda_n^2 t}\cos(\lambda_n x) = 1 - erf\left(\frac{|y|}{2\sqrt{Dt}}\right)\sum_{n=0}^{\infty} a_n e^{-D\lambda_n^2 t}\cos(\lambda_n x), \tag{A33}$$

where $a_n = \dfrac{(-1)^n 4}{(2n+1)\pi}$.

Reference

[1] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions: with formulas, graphs, and mathematical tables, Courier Corporation, 1964. Page 486,11.4.28.