Electronic Supplementary Material (ESI) for Soft Matter.

Appendix: derivation of the homogenised model in three dimensions

To extend the homogenised model to three dimensions, the values of the effective coefficients along the phages need to be determined.

The diffusion along the phages is taken to be uninhibited by physical barriers, so the diffusion coefficient along this direction is $\tilde{D}$.

The effective adsorption along the phages can be derived by considering antibiotic diffusion in a three-dimensional domain with the length of a phage and the cross section of a unit cell. We expand the equations governing the antibiotic dynamics in the small parameter $\hat{\eta}$, which is the difference in scale between the unit cell and the phage length $\tilde{L}_{ph}$: $\hat{\eta} = \tilde{a}/\tilde{L}_{ph}$. This results in the following non-dimensional scaling:

$$x_1 = \frac{\tilde{x}_1}{\tilde{L}_{ph}}, \quad x_2 = \frac{\tilde{x}_2}{\tilde{a}}, \quad x_3 = \frac{\tilde{x}_3}{\tilde{a}}, \quad t = \frac{\tilde{t}D}{\tilde{L}_{ph}^2}, \tag{1}$$

where the index 1 indicates the direction along the phages and 2 and 3 the directions along the unit cell. We do not consider the fast timescale $t_2 = t/\hat{\eta}^2$ because we assume that at this scale the system is at equilibrium. The other variables are non-dimensionalised as before and.the resulting non-dimensional equations are

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{\hat{\eta}^2}\frac{\partial^2 u}{\partial x_2^2} + \frac{1}{\hat{\eta}^2}\frac{\partial^2 u}{\partial x_3^2} + \frac{q}{\hat{\eta}^2}\frac{\partial}{\partial x_2}\left(u\frac{\partial \phi}{\partial x_2}\right) + \frac{q}{\hat{\eta}^2}\frac{\partial}{\partial x_3}\left(u\frac{\partial \phi}{\partial x_3}\right), \qquad \boldsymbol{x} \in \mathcal{D}_{3D}, \tag{2a}$$

$$-n_1\frac{\partial u}{\partial x_1} = 0, \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{2b}$$

$$-n_2\left(\frac{\partial u}{\partial x_2} + qu\frac{\partial \phi}{\partial x_2}\right) - n_3\left(\frac{\partial u}{\partial x_3} + qu\frac{\partial \phi}{\partial x_3}\right) = \gamma_{ph}(\mu u - v), \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{2c}$$

$$\frac{\partial v}{\partial t} = \frac{\gamma_{ph}}{\hat{\eta}^2}(\mu u - v), \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{2d}$$

where $\gamma_{ph} = \frac{\kappa \tilde{L}_{ph}^2}{D}\hat{\eta}^2$ and the subscripts $3D$ indicate the extension of the domains along the x_1 direction between 0 and L_{ph}. There are no electric field-induced drift terms in the x_1-direction because the field around the phages is radial. All boundary conditions that are not specified are non-flux conditions.

Expanding in $\hat{\eta}^2$ gives at $O(1)$:

$$\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{\hat{\eta}^2}\frac{\partial^2 u_0}{\partial x_2^2} + \frac{1}{\hat{\eta}^2}\frac{\partial^2 u_0}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + \frac{q}{\hat{\eta}^2}\frac{\partial}{\partial x_2}\left(u_0\frac{\partial \phi}{\partial x_2}\right) + \tag{3a}$$

$$\frac{q}{\hat{\eta}^2}\frac{\partial}{\partial x_3}\left(u_0\frac{\partial \phi}{\partial x_3}\right) + q\frac{\partial}{\partial x_2}\left(u_2\frac{\partial \phi}{\partial x_2}\right) + q\frac{\partial}{\partial x_3}\left(u_2\frac{\partial \phi}{\partial x_3}\right), \qquad \boldsymbol{x} \in \mathcal{D}_{3D}, \tag{3b}$$

$$-n_1\frac{\partial u_0}{\partial x_1} = 0, \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{3c}$$

$$-n_2\left(\frac{\partial u_0}{\partial x_2} + qu_0\frac{\partial \phi}{\partial x_2}\right) - n_3\left(\frac{\partial u_0}{\partial x_3} + qu_0\frac{\partial \phi}{\partial x_3}\right) = \gamma_{ph}(\mu u_0 - v_0), \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{3d}$$

$$\frac{\partial v_0}{\partial t} = \frac{\gamma_{ph}}{\hat{\eta}^2}(\mu u_0 - v_0) + \gamma_{ph}(\mu u_2 - v_2), \qquad \boldsymbol{x} \in \Gamma_{3D}. \tag{3e}$$

At $O(\hat{\eta}^{-2})$ the equations are

$$\frac{\partial^2 u_0}{\partial x_2^2} + \frac{\partial^2 u_0}{\partial x_3^2} + q\frac{\partial}{\partial x_2}\left(u_0\frac{\partial \phi}{\partial x_2}\right) + q\frac{\partial}{\partial x_3}\left(u_0\frac{\partial \phi}{\partial x_3}\right) = 0, \qquad \boldsymbol{x} \in \mathcal{D}_{3D}, \tag{4a}$$

$$-n_2\left(\frac{\partial u_0}{\partial x_2} + qu_0\frac{\partial \phi}{\partial x_2}\right) - n_3\left(\frac{\partial u_0}{\partial x_3} + qu_0\frac{\partial \phi}{\partial x_3}\right) = \gamma_{ph}(\mu u_0 - v_0), \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{4b}$$

$$\gamma_{ph}(\mu u_0 - v_0) = 0, \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{4c}$$

which as before implies $\mu u_0 = v_0$ and $u_0 = \langle u_0 \rangle m_{-q}$. At order $O(1)$

$$\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + q\frac{\partial}{\partial x_2}\left(u_2\frac{\partial \phi}{\partial x_2}\right) + q\frac{\partial}{\partial x_3}\left(u_2\frac{\partial \phi}{\partial x_3}\right), \qquad \boldsymbol{x} \in \mathcal{D}_{3D}, \tag{5a}$$

$$-n_1\frac{\partial u_0}{\partial x_1} = 0, \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{5b}$$

$$-n_2\left(\frac{\partial u_2}{\partial x_2} + qu_2\frac{\partial \phi}{\partial x_2}\right) - n_3\left(\frac{\partial u_2}{\partial x_3} + qu_2\frac{\partial \phi}{\partial x_3}\right) = \gamma_{ph}(\mu u_2 - v_2), \qquad \boldsymbol{x} \in \Gamma_{3D}, \tag{5c}$$

$$\frac{\partial v_0}{\partial t} = \gamma_{ph}(\mu u_2 - v_2), \qquad \boldsymbol{x} \in \Gamma_{3D}. \tag{5d}$$

Integrating over the unit cell and using Gauss' law yields

$$\int_{\mathcal{C}} \frac{\partial u_0}{\partial t} d^3\boldsymbol{x} = \int_{\mathcal{C}} \frac{\partial^2 u_0}{\partial x_1^2} d^3\boldsymbol{x} + \int_{\Gamma} \frac{\partial v_0}{\partial t} d^2\boldsymbol{x}. \tag{6}$$

This results in the macroscopic diffusion equation along the phages:

$$(|\mathcal{C}| + \frac{\langle e^{-q\phi}\rangle_\Gamma}{\langle e^{-q\phi}\rangle}\mu|\Gamma|)\frac{\partial \langle u_0 \rangle}{\partial t} = |\mathcal{C}|\frac{\partial^2 \langle u_0 \rangle}{\partial x_1^2}. \tag{7}$$

Apart from the diffusion coefficient, this is identical to the macroscopic effective equation across the phages, which means that the effective adsorption coefficient is equal in all directions, as opposed to the effective diffusion coefficient which is an anisotropic tensor in three dimensions.