Soft Matter

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Supplementary Information : Toothpicks and toy football players: frictional jamming causes locomotion of vertically-shaken assymetrical objects

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1 What is, and what is not, locomotion.

Despite that we generally 'know it when we see it', perhaps a paper on the means of achieving locomotion should have a relatively clear concept of what is, and what is not, locomotion. This question is taken up in[1], where the goal is a mathematical definition. Here, we aim for an operational definition that is reasonably, if not mathematically, precise. Namely,

Locomotion is the *self-induced translation of an object, such as an animal, vehicle, or robot, relative to an extended media (the environment) with power coming from within, or from the environment.* To be operational, all terms in this definition themselves need definition and possible restrictions.

Deformations. Generally, neither the locomoting object nor the environment is rigid. Thus, some care is needed to define translation, object motion, and the environment's motion. We do this by considering the motion of points on the object and in the environment over a time-scale over which locomotion is measured. Locomotion is generally only well defined over a long enough time-scale that deformations are relatively negligible compared to the net motion.

Translation of an object, from one time to another, is only unambiguous if:

1. The object has the same shape and orientation at the two times; or
2. The amount of translation is much larger than the amount of deformation; that is, the translation of any one point on the object differs insignificantly from the translation of any other point on the object.

For example, we can only unambiguously define the translation of a snake relative to the ground after an integer number of snake undulations, or if the translation is so large that the relative motion of parts of the snake is negligible relative to the overall translation.

The media. The media or the environment also needs to have well-defined translations. That is, either the environment is effectively rigid or, on the time-scale of measured locomotion, the displacement of the object relative to the environment needs to be unambiguous. That is, the net environmental distortion (the relative motion of points in the environment) must be negligible compared to the claimed locomotion of the object relative to the environment. Further, on average over time-scales of measured locomotion, the environment needs to have a spatially and temporally constant velocity (measured relative to a Newtonian Frame). Variations of displacement in space and time must average to something small relative to the measured locomotion, over time-scales of measured locomotion.

Multiple media. For some purposes, say sailing, one can consider two (or possibly more) media. Each of these (say, wind and water) needs a well-defined constant velocity (constant over long enough time-scales that fluctuations are negligible on average). Because, say, weather, changes in time, location depend on the existence of an intermediate time-scale that is long enough to average out fluctuations yet short enough that the weather may be considered constant.

Energy source. During relative motion dissipation is inevitable, and thus locomotion requires a mechanical energy source. This source could be, say, chemical energy in the object converted to work (*e.g.,* batteries, fat, or gasoline) or from motions of the environment (*e.g.,* vibrations, turbulence, or relative

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‡ Additional footnotes to the title and authors can be included *e.g.* Bhadra and Ghosh made the initial observations., did all of the experiments, and made the photos and videos. All authors contributed equally to the theory and writing.

motion of two fluids).

Dragging is not locomotion. Motion caused by an external force from outside the environment is *not* locomotion. A motorboat on water is locomoting; the water skier dragged behind is not. If there are two media, and the object can ride with one or the other, and that is also not locomotion. The object could also drag partially in both. That is not locomotion either. Hence seaweed being dragged with the wind is excluded from being considered as locomotion. More on this is discussed later with regard to sailing.

Key mechanical features of locomotion. Either the object, the environment, or the environment's small-scale motions need some anisotropy (some asymmetry) that aligns with and causes the motion. Even with asymmetry, locomotion also often depends on some non-linear mechanical interaction between the object and the environment such as comes from high Reynolds number flow, ratchet-like surface features, or Coulomb friction. A nice review of locomotion concepts by Radhakrishnan emphasizes that friction is often central to locomotion on land, both as something to be overcome (dissipation) and something to be used (a nonlinear response).[2].

Locomotion due to small environmental motions. We often think of locomotion as associated with self-powered objects in or on a passive stationary media: a motorboat in still water; a bird flying; a person walking on the stationary ground; or a worm tunnelling through the earth. However, there are examples of locomotion where, instead of self-power, small-scale motions of the media, which may have no directional bias, cause translation (locomotion) of an object over distance scales much larger than the media motions or media heterogeneity. For example:

- A dead fish can swim forwards powered by turbulence[3]. The asymmetry is in the thickness variation and compliance variation from head to tail in the fish, the nonlinearity is in the quadratic nature of fluid forces at high Reynolds numbers;
- A paper cone can be propelled in air, even flying up against gravity, by vibrations of a speaker[4]. The asymmetry of the object is towards the point of the cone, and the nonlinearity again from the Navier-Stokes equations;
- Gliding birds and gliding planes can fly large horizontal distances powered by vertical air currents that oscillate between upwards and downwards flow (the birds and planes are not symmetrical front to back)[5].

Sailing as locomotion. As noted above, we resist labelling the motion of a boat or car that is just blown with the wind as locomotion. The boat in this case is just dragging along with a different Newtonian frame (the moving air). We can avoid classing such dragging as locomotion by limiting locomotion to situations where the average translation velocity of the object is outside the convex hull of the (effective average) velocities of the environments; For example, with two environments with velocities $\vec{v}_1$ and $\vec{v}_2$ any locomotion velocity expressible as $\vec{v} = s\vec{v}_1 + (1-s)\vec{v}_2$ with $0 \le s \le 1$ is inside the convex hull of $\vec{v}_1$ and $\vec{v}_2$. So, assuming still water, sailing in the direction of the wind would not be locomotion so long as the boat's speed is not greater than the wind.

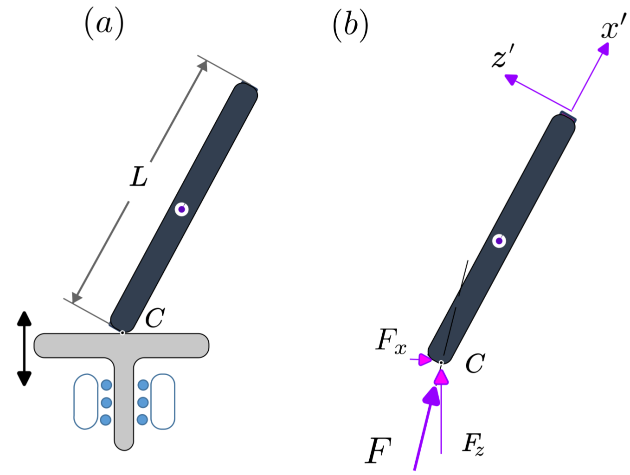

Fig. 1 (a) and (b) Dynamic mode of jamming. An uniform stick of length L on a vertically vibrating platform. The coordinates $x-z$ (horizontal-vertical) define the plane in which the stick can move. The tip angle $\theta < \tan^{-1}\mu$, puts this in the self-locking regime; point C does not slip. If there is a collision of the support with the toothpick at C, or if the platform merely accelerates vertically up, a force F is induced at an angle with the stick's axis. Here, coordinates $x'-z'$ are parallel and orthogonal to the axis of the stick.

In contrast, one generally credits as locomotion sailing situations where the boat velocity is outside the convex hull of the water and the wind velocities. This would include crosswind sailing (also known as 'reaching'); sailing upwind, directly into the wind; or sailing downwind faster than the wind. Sailing downwind slower than the wind is dragging, and not (by our definition) locomotion.

Non-linear mechanics. As for the need for non-linear mechanics, consider the shaking of a ketchup bottle to move ketchup toward the opening. To accomplish this 'locomotion' of a blob of ketchup down the length of the bottle, one uses an *asymmetrical* shake that has no net average bottle velocity but causes motion of the ketchup, locomotion, due to the non-linear fluid mechanics of ketchup.

2 Dynamic Jamming : Stick with inertia

If the stick has uniformly distributed mass m, length L, the centre of mass G at $L/2$, and moment of inertia about the centre of mass $I = mL^2/12$, the relation between contact acceleration and contact force is, in $x'-z'$ coordinates, parallel and orthogonal to the stick,

$$\begin{bmatrix} F_{Cx'} \\ F_{Cz'} \end{bmatrix} = \underbrace{\begin{bmatrix} m & 0 \\ 0 & \frac{m}{4} \end{bmatrix}}_{[m]'} \begin{bmatrix} a_{Cx'} \\ a_{Cz'} \end{bmatrix}. \tag{A.1}$$

The effective mass along the stick is m. To calculate the effective mass orthogonal to the stick, we apply a force F, orthogonal to the stick at C. The acceleration of C, a_C is due to the acceleration of the stick centre of mass, $a_G = F/m$ plus acceleration due to induced rotation $a_{C/G} = \dot{\omega}L/2$. Angular momentum balance about G gives that $\dot{\omega} = (FL/2)/I$. Adding the linear and rotational contributions gives $a_C = F/(m/4)$, thus the effective inertia (or mass) of C, orthogonal to the stick, is $m/4$. This component, although not zero, is still well smaller than m. In this case, either a vertical collision at point C (or a vertical acceleration of point C), cause a force (or impulse) with a rightwards component. The anisotropic mass matrix, with a bigger principal component along the stick, causes an acceleration of C somewhat aligned with the stick to map to a force that is more aligned with the stick. Thus a vertical acceleration causes a reaction force on the stick that has a rightwards component.

In detail, for a stick leaned at an angle θ with slope $1/q$, the relation of the vertical impulse J_v to sideways 'locomotion' impulse J_ℓ due to a collision event which changes the sticks vertical velocity is given by

$$J_\ell = \begin{cases} qJ_v & \text{Point mass model} \\ \left(\frac{3q}{4+q^2}\right) J_v & \end{cases}$$

For example, at a rod angle of $\pi/4 = 45^\circ$ (so $q = 1$), assuming a given support vertical acceleration or collision with a rod that has only vertical velocity, the point-mass model gives a horizontal locomotion impulse equal to the vertical impulse. And the uniform rod, less anisotropic, gives only 60% of that.

3 General observations about vibrations and locomotion

3.1 Vibration tends to reduce the effective friction.

Our interest here is vibration powering motion, but first consider that both horizontal and vertical vibration tends, by a variety of mechanisms, to reduce the average friction which resists motion.

3.1.1 Horizontal vibration.

Consider a symmetric sinusoidal horizontal (x direction) displacement $u_x = \delta_x \sin(\omega t)$ of a support plate, where, δ_x is the amplitude of the motion and ω is the frequency of the drive. An object supported by the plate has, at a given instant, the horizontal velocity v. The slip velocity, or relative velocity of the object and support plate is $v_{\text{rel}} = v - \omega\delta_x \cos(\omega t)$.

Non-sliding $v_{\text{rel}} = 0$ can only occur so long as the support acceleration a_{support} is small enough, namely

$$\mu g \geq |\underbrace{\delta_x \omega^2 \sin(\omega t)}_{a_{\text{support}}}|.$$

For high frequencies with $\delta_x \omega^2 \gg \mu g$, non-sliding can only happen for a small fraction of the time. Similarly, for high frequencies, the acceleration of the object is much less than the acceleration of the support so the fluctuation in object velocities is small. So, for any given velocity v of the object, the ratio of right-sliding time to left-sliding time approaches 1 as the frequency goes to ∞, and, using the friction law (Eqn. 3 of the main paper), the average force tends to zero. That is, for high-enough-frequency vibrations with large enough amplitude, and small enough object average velocity, the support friction, averaged over one or more vibration cycles, approaches (on average) frictionless behaviour. In short, rapid horizontal oscillation of support reduces the effective friction. Because object velocity biases the amount of time that relative slip is in one direction, the friction which remains is viscous in nature, with average resistance to horizontal motion increasing with the velocity of that motion and decreasing with shaking frequency or amplitude. This phenomenology is the motivation for *dither*, a vibration added to the actuator command, in control of frictional systems.

Booktop experiment. This vibration-reducing-friction effect can be demonstrated by holding a thin flat book with a coin on top. Note the tilt of the book needed to cause the coin to slip (something like $20^\circ \approx \arctan(\mu)$. Now rapidly oscillate the book with your hand (say, 5 oscillations/s). You will see that the coin will slide in the direction of any small tilt. That is the coin slides in the direction of a gravity force whose projection is well less than μmg. Friction is effectively reduced by high-frequency horizontal vibrations.

3.1.2 Vertical vibration.

Consider the support surface moving vertically according to $u_z = \delta_z \sin(\omega t)$, here δ_z is the amplitude of the motion in the vertical direction. For any supported object, as $\omega \to \infty$, for any fixed δ_z, the fraction of time in contact goes to 0. Why? Each time contact is made (even with a coefficient of restitution of zero), the object is thrown or bounces up with vertical speed $v_z \approx \omega\delta_z$ with a resulting flight time of $t_f \approx 2v_z/g$. As ω is increased, this flight time grows indefinitely in relation to the time of a single contact (on the order of π/ω). Thus, the fraction of the time in contact goes to zero. One might then think that the average friction force would go to zero. In fact, though, if there is sliding during contact, the average horizontal force, averaged over contact and flight, is μ times the average vertical force, which is mg. Thus, within the assumption of rigid objects and assuming sliding during contact, vertical oscillations do not reduce the friction force.

However, with an applied sideways load that is less than μ times the average normal force, there is motion during the flight phase that is extinguished at the next collision. Assuming the full motion is made of flight alternating with instantaneous sticking collisions (with relatively negligible slip distance) then we have the average horizontal velocity in response to a sideways load (or 'bias' force) of F is given by, $v_{\text{ave}} = (F/2m)t_f = (Fv_z/mg)$ which imples that

$$F = mgv_{\text{ave}}/v_z \quad \Rightarrow \quad \mu_{\text{eff}} = v_{\text{ave}}/v_z$$

Here, $t_f = 2v_z/g$ is the duration of one flight phase. We have approximated that, on average, each flight is started with a vertical velocity equal to the peak vertical velocity $v_s = \delta_z\omega$ of the substrate. The effective coefficient of friction is $\mu_{\text{eff}} = F/mg$. Thus, the macroscopic average behaviour of the object in response to loads is again linear viscous $F = cv_{\text{ave}}$ (Force and mean velocity are proportional) for $F < \mu mg$ or equivalently, $\frac{v_{\text{ave}}}{v_s} < \mu$. The effective viscous constant c goes to zero as the forcing frequency $\omega \to \infty$. So, at least for slow motions, increasing the support vibration peak velocity causes a decrease in the sideways force needed to move an object sideways at a given speed. Further, no matter how small the amplitude, so long as the support has a peak vertical acceleration large enough ($\delta_z \omega^2 > g$) to cause flight, there is some sideways motion due to arbitrarily small forces F.

For example, on a modally vibrating plate that has a large enough vibration amplitude, only a small bias would be needed to guide particles to the mode's nodes to form Chladni patterns[6].

So, for both horizontal and vertical shaking of an object on planar support, the bias force needed to cause motion diminishes with shaking intensity; an object on a buzzing surface that has large enough vibration velocity and acceleration needs only small external forces, or small self-driving forces (from, say, asymmetry), to move.

3.2 Asymmetries that do and do not cause locomotion

3.2.1 Asymmetrical horizontal shaking with linear friction does not induce locomotion

Note, as mentioned above, for an object that is supported on a horizontal plate that has asymmetrical horizontal vibration ([7]) and with Coulomb friction at the support contact, horizontal motion (locomotion) is generally (generically) induced. What if we had linear viscous friction between the object and shaking support? That is, assume that the friction force F to the right on an object moving with relative velocity v_{rel} to the right is

$$F = -c v_{\text{rel}}.$$

Here, $v_{\text{rel}} = v_{\text{obj}} - v_{\text{sup}}$ with v_{obj} and v_{sup} being the instantaneous rightwards velocities of the object and shaking support, respectively. If there are no external forces, then for quasi-steady motion (periodic velocities of all parts), the average friction force F^{ave} must be zero:

$$0 = F^{\text{ave}} = \int_{t_1}^{t_2} F\, dt/(t_2 - t_1)$$

$$= -c \int_{t_1}^{t_2} v_{\text{rel}}\, dt/(t_2 - t_1) = -c(v_{\text{obj}}^{\text{ave}} - v_{\text{sup}}^{\text{ave}}).$$

We have chosen t_1 and t_2 so that an integer number of cycles of the support is included, or so that t_2 is so much greater than t_1 that there is a small error from being a non-integer number of cycles. So, independently of any asymmetry in the support motion, so long as the support has no net average velocity, $v_{\text{sup}}^{\text{ave}} = 0$, the object will have no net velocity: $v_{\text{obj}}^{\text{ave}} = 0$. For linear viscous friction, the average force is proportional to the average relative velocity and asymmetrical shaking (which has an average velocity of zero, even if unsymmetrical) does not give locomotion. A non-linear friction law changes this result.

3.2.2 Asymmetrical horizontal shaking with Coulomb friction does induce locomotion.

For Coulomb friction $F = F_f = -\mu N v_{\text{rel}}/|v_{\text{rel}}|$ when there is sliding, and $|F| \leq \mu N$ when there is no sliding. Imagine a small, wide (thus not tipping), flat-bottomed object placed on a horizontally supported plate that undergoes a sawtooth-like motion, alternating between constant velocity v_r to the right and smaller velocity $v_l > v_r$ to the left, acting for a longer period.

The right moving and left moving phases have durations $t_r > t_l$, chosen so that the net displacement of the support plate in a cycle is zero ($v_r t_r = v_l t_l$). So long as the magnitude of the velocity v of the supported object is always less than v_l, at every instant in time there is a sliding relatively opposite to the direction of the instantaneous plate velocity. Because the friction force μmg acts $t_l - t_r$ longer to the left than to the right, over one cycle there is a net friction impulse $J = \int F dt = \mu mg(t_l - t_r)$ to the left. Thus, a slow-moving supported object is, on average, accelerated to the left until its speed reaches v_l for at least part of a cycle. In this case, asymmetries in the shaking lead to 'locomotion' bias to the left.

Seal fur and fish scales. Some of our intuitions about preferred sliding directions for objects with slanted outcroppings come from experience with ratchet-like animal fur and fish scales. These depend on different physics. Unlike the systems considered here, the anisotropy of slip for these surfaces does not just follow from Coulomb friction between solid objects. Rather, these have anisotropic friction, that is, large friction when rubbing the wrong way, from the hairs/scales digging into the deformable surfaces they are rubbing against.

4 Details of the mechanics model and computer simulation

Consider a CP stick of length L that is connected to a stiffness torsion spring k and damping γ placed on the platform that is vibrating , as seen in Fig. 2. The stick consists of a mass less rod that has its base at C and the mass m at P. The points $P : (x_p, z_p)$ and $C : (x_c, z_c)$ describe the top and the bottom ends of the stick.

$$\begin{bmatrix} x_c \\ z_c \end{bmatrix} = \begin{bmatrix} x_p - L\sin\theta \\ z_p - L\cos\theta \end{bmatrix} \tag{C.1}$$

$$x_c = x_p - L\sin\theta \quad \text{and} \quad z_c = z_p - L\cos\theta \tag{C.2}$$

Here $\theta : (-\pi/2, \pi/2)$ is the angle made by the stick with respect to the z-axis.

$$\begin{bmatrix} \dot{x}_c \\ \dot{z}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_p - L\dot{\theta}\cos\theta \\ \dot{z}_p + L\dot{\theta}\sin\theta \\ -\frac{k}{\gamma}(\theta - \theta_0) \end{bmatrix} \tag{C.3}$$

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{z}_c \end{bmatrix} = \begin{bmatrix} \ddot{x}_p - L\left(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta\right) \\ \ddot{z}_p - L\left(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta\right) \end{bmatrix} \tag{C.4}$$

Here θ_0 is the equilibrium angular position of the stick, in the absence of contact forces.

The platform $(\sim, z_s)$ does a periodic movement in the z axis, i.e, $z_s = \delta\cos(\omega t)$. Here δ and ω are the amplitude and the vibration frequency of the platform.

The instantaneous height difference h_c between the bottom of the stick, C, and the platform goes to zero when the bottom end of the massless stick touches the moving platform, $h_c = 0$. The stick can then either enter into a frictionally locked (pivot) state or it can slide left or right with velocity v_x.

Depending on the constraints described in the table below, one of the following four phases of motion are in effect. The motion

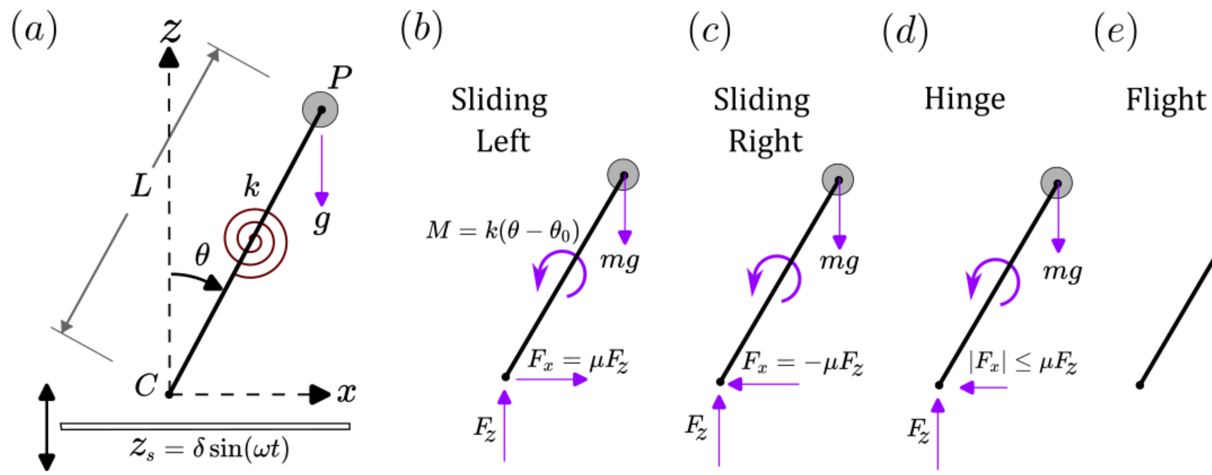

Fig. 2 (a) Schematic description of a stick CP of length L that is connected to a stiffness torsion spring k placed on the platform that is vibrating. The sub-panels (b $\cdots$ e) show the free body diagrams for the sliding left (b), sliding right (c), hinge (d) and flight (e) phases.

continues in a particular phase until one or more conditions are violated.

Table 1 Tabular form of the constraints. The equalities above are enforced by the equations of motion in that state. The inequalities are checked for violation. If an inequality is violated, the state transitions to another state. The new state is (the) one in which the inequalities are satisfied for the present values of the state variables.

Phase	h_c	v_x	F_z	F_x
Flight	>0		$=0$	$=0$
Pivot	$=0$	$=0$	≥ 0	$\lvert F_x\rvert \leq \mu F_z$
Sliding Left	$=0$	<0	≥ 0	μF_z
Sliding Right	$=0$	<0	≥ 0	$-\mu F_z$

4.1 Flight

In this phase the stick is not making a contact with the platform, i.e., $h_c > 0$ and hence both the contact forces F_x and F_z are set to zero. The equation of motion of the point P is

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{z}_p \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{C.5}$$

4.2 Pivot

In the pivot mode the horizontal velocity v_x of the point C is zero and $z_c(t) = z_s(t)$. Thus in the pivot mode the system has only one degree of freedom. From the conservation of angular momentum $(mL^2\ddot{\theta} = H_{/C})$about the hinged point C we have

$$mL^2\ddot{\theta} = (g + \ddot{z}_s)mL\sin\theta - k(\theta - \theta_0) - \gamma\dot{\theta} \tag{C.6}$$

In the pivoting mode $\ddot{x}_c = 0$ and $\ddot{z}_c = \ddot{z}_s$. Thus from Eqn. C.4 we obtain,

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{z}_p \end{bmatrix} = \begin{bmatrix} L(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta) \\ \ddot{z}_s + L(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta) \end{bmatrix} \tag{C.7}$$

4.3 Sliding

In the horizontal forces F_x in the hinge mode exceeds the frictional force, the bottom of the stick looses its frictional grip and begins to slide.

Sliding Left

In the sliding-left mode the horizontal velocity v_x of the point C is negative and $z_c(t) = z_s(t)$. Thus in the pivot mode the system has two degrees of freedom, i.e., it can slide as well as change its angle. The resultant motion is that the point C is constrained to be on the platform but moves to the left while the point P moves down vertically.

Since there is no net moment acting on the system, the sum of the moments $\Sigma H_{/P}$ about the point P must be zero

$$0 = k(\theta - \theta_0) + \gamma\dot{\theta} + F_x(z_p - z_c) - F_z(x_p - x_c) \tag{C.8}$$

Here $F_x = \mu F_z$, thus

$$F_z = \frac{k(\theta - \theta_0) + \gamma\dot{\theta}}{L(-\mu\cos\theta + \sin\theta)} \tag{C.9}$$

Therefore the equation of motion of the point $P : (x_p, z_p)$

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{z}_p \end{bmatrix} = \begin{bmatrix} \frac{\mu F_z}{m} \\ \frac{F_z}{m} - g \end{bmatrix} \tag{C.10}$$

The variation of the tilt angle θ is obtained from Eqns. C.10, C.4, thus

Sliding Right

In the sliding-right mode the horizontal velocity v_x of the point C is positive and $z_c(t) = z_s(t)$. Since there is no net moment acting on the system, the sum of the moments $\Sigma H_{/P}$ about the point P must be zero

$$0 = k(\theta - \theta_0) + \gamma\dot{\theta} + F_x(z_p - z_c) - F_z(x_p - x_c)$$

Here $F_x = -\mu F_z$, thus

$$F_z = \frac{k(\theta - \theta_0) + \gamma\dot{\theta}}{L(\mu\cos\theta + \sin\theta)} \tag{C.11}$$

$$\tag{C.12}$$

Therefore the equation of motion of the point $P : (x_p, z_p)$

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{z}_p \end{bmatrix} = \begin{bmatrix} -\frac{\mu F_z}{m} \\ \frac{F_z}{m} - g \end{bmatrix} \tag{C.13}$$

The variation of the tilt angle θ is obtained from Eqns. C.13, C.4, thus

$$\begin{bmatrix} \ddot{z}_c \\ \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} \ddot{z}_s \\ \frac{1}{\sin\theta}\left(\frac{\ddot{z}_c - \ddot{z}_p}{L} - \dot{\theta}^2\cos\theta\right) \\ -\frac{\mu F_z}{m} - L\left(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta\right) \end{bmatrix} \tag{C.14}$$

4.4 Collision Event

A collision event happens when the stick exits the flight phase. After collision $\theta = \theta_0$ while $\dot{\theta}$ needs to be evaluated by equating the angular moment H about the point C before and after the collision, i.e.,

$$\vec{H}^-_{/c} = \vec{H}^+_{/c}$$

$$\left(\vec{r}_{P/C}\right)^- \times (\vec{v}^-_p - \dot{z}_s\hat{k})m = ((x_p - x_c)(\dot{z}^+_p - \dot{z}_c) - (z_p - z_c)\dot{x}^+_p)m\hat{k}$$

$$= -mL^2\dot{\theta}^+$$

$$\dot{\theta}^+ = -\frac{\left(\vec{r}_{P/C}\right)^- \times (\vec{v}^-_p - \dot{z}_s\hat{k})}{L^2} \tag{C.15}$$

Here $\left(\vec{r}_{P/C}\right)^- = L(\sin\theta_0\hat{i} + \cos\theta_0\hat{k})$ and $\vec{v}^-_p = \dot{x}^-_p\hat{i} + \dot{z}^-_p\hat{k}$

5 State Machine

Notes and references

1 J. Steigenberger, Multibody System Dynamics, 2011, **26**, 81–90.

2 V. Radhakrishnan, Proceedings of the National Academy of Sciences, 1998, **95**, 5448–5455.

3 D. N. Beal, F. S. Hover, M. S. Triantafyllou, J. C. Liao and G. V. Lauder, Journal of Fluid Mechanics, 2006, **549**, 385–402.

4 S. Childress, N. Vandenberghe and J. Zhang, Physics of fluids, 2006, **18**, 117103.

5 C. Pennycuick, Ibis, 1972, **114**, 178–218.

6 Q. Zhou, V. Sariola, K. Latifi and V. Liimatainen, Nature communications, 2016, **7**, 12764.

7 D. Reznik and J. Canny, Proceedings. 1998 IEEE International Conference on Robotics and Automation (Cat. No. 98CH36146), 1998, pp. 1471–1477.

Algorithm 1 HINGE:

```
case HINGE:
    switch Select Next Phase do
        if μF_z + F_x > 0 then
            if |θ| = 0 then
                assert(Next Phase ← 'FLIGHT')
                z_c ← z_s
            else
                assert(Next Phase ← 'SLIDE RIGHT')
            end if
        end if
        if μF_z − F_x > 0 then
            assert(Next Phase ← 'SLIDE LEFT')
        end if
        if F_z > 0 then
            assert(Next Phase ← 'FLIGHT')
            z_c ← z_s
        end if
        if z_c < 0 or z_p < 0 then
            assert(Next Phase ← 'ABORT')
        end if
```

Algorithm 2 FLIGHT:

case FLIGHT:
  switch *Select Next Phase* **do**
    if $z_c - z_s = 0$ **then**
      if $\tan(\theta) \leq \mu$ **then**
        if $((\dot{x}_p\hat{i} + (\dot{z}_p - \dot{z}_s)\hat{k}) \cdot \vec{r}_{CP}) < 0$ **then**
          Compute F_x and F_z in the HINGE Phase
          if $F_z > 0$ **then**
            if $\mu F_z < |F_x|$ **then**
              `assert`(Next Phase ← 'HINGE')
            end if
            if $F_x > 0$ **then**
              `assert`(Next Phase ← 'SLIDE LEFT')
            else
              `assert`(Next Phase ← 'SLIDE RIGHT')
            end if
          else // Tensional ground reaction implies flight //
            `assert`(Next Phase ← 'FLIGHT')
            $z_c \leftarrow z_s$
          end if
        else // Rod not shortening //
          `assert`(Next Phase ← 'HINGE')
          $\dot{x}_c \leftarrow 0$
          $z_c \leftarrow z_s$
          $\dot{\theta} = (\vec{r}_{CP} \times (\vec{v}_p - \vec{v}_s))/L^2$
          $\dot{x}_p \leftarrow \dot{x}_c + L(\dot{\theta}\cos(\theta))$
          $\dot{z}_p \leftarrow \dot{z}_s - L(\dot{\theta}\sin(\theta))$
          if $\dot{x}_c > 0$ **then**
            `assert`(Next Phase ← 'SLIDE RIGHT')
            $z_c \leftarrow z_s$
            $\dot{\theta} \leftarrow -(\dot{z}_p - \dot{z}_s)/(L\sin(\theta))$
          else
            `assert`(Next Phase ← 'SLIDE LEFT')
            $z_c \leftarrow z_s$
            $\dot{\theta} \leftarrow -(\dot{z}_p - \dot{z}_s)/(L\sin(\theta))$
          end if
        end if
      else
        if $\dot{x}_c > 0$ **then**
          `assert`(Next Phase ← 'SLIDE RIGHT')
          $\dot{x}_c \leftarrow \dot{x}_p + (\dot{z}_p - \dot{z}_s)\cot(\theta)$
          $z_c \leftarrow z_s$
          $\dot{\theta} \leftarrow -(\dot{z}_p - \dot{z}_s)/(L\sin(\theta))$
        else
          `assert`(Next Phase ← 'SLIDE LEFT')
          $\dot{x}_c \leftarrow \dot{x}_p + (\dot{z}_p - \dot{z}_s)\cot(\theta)$
          $z_c \leftarrow z_s$
          $\dot{\theta} \leftarrow -(\dot{z}_p - \dot{z}_s)/(L\sin(\theta))$
        end if
      end if
    end if
    if $z_c < 0 || z_p < 0$ **then**
      `assert`(Next Phase ← 'ABORT')
    end if

Algorithm 3 SLIDE RIGHT:

case SLIDE RIGHT:
  switch *Select Next Phase* **do**
    Flag = 0
    Compute F_x and F_z in the HINGE Phase
    if $\mu F_z \geq |F_x|$ **then**
      assert(Next Phase ← 'HINGE')
      $\dot{x}_c \leftarrow 0$
      $z_c \leftarrow z_s$
      Flag ← 1
    end if
    if Flag = 0 **then**
      Compute F_x and F_z in the SLIDE LEFT Phase
      if $F_z > 0$ **then**
        assert(Next Phase ← 'SLIDE LEFT')
        $z_c \leftarrow z_s$
      else
        assert(Next Phase ← 'ABORT')
      end if
    end if
    if $F_z = 0$ or $F_x = 0$ **then**
      assert(Next Phase ← 'FLIGHT')
      $z_c \leftarrow z_s$
    end if
    if $z_c < 0$ or $z_p < 0$ **then**
      assert(Next Phase ← 'ABORT')
    end if

Algorithm 4 SLIDE LEFT:

case SLIDE LEFT:
  switch *Select Next Phase* **do**
    Flag = 0
    Compute F_x and F_z in the HINGE Phase
    if $\mu F_z \geq |F_x|$ **then**
      assert(Next Phase ← 'HINGE')
      $\dot{x}_c \leftarrow 0$
      $z_c \leftarrow z_s$
      Flag ← 1
    end if
    if Flag = 0 **then**
      Compute F_x and F_z in the SLIDE RIGHT Phase
      if $F_z > 0$ **then**
        assert(Next Phase ← 'SLIDE RIGHT')
        $z_c \leftarrow z_s$
      else
        assert(Next Phase ← 'ABORT')
      end if
    end if
    if $F_z = 0$ or $F_x = 0$ **then**
      assert(Next Phase ← 'FLIGHT')
      $z_c \leftarrow z_s$
    end if
    if $z_c < 0$ or $z_p < 0$ **then**
      assert(Next Phase ← 'ABORT')
    end if