Supplementary Document

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1 Cahn-Hoffman capillary vector

The main results of this section are used in Section 2.1 for Equations (3) to (7).

Cahn-Hoffman capillary vector is a map $S^2 \times S^2 \mapsto T^1$

$$\boldsymbol{\xi}(\mathbf{n}, \mathbf{k}) = \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\parallel = \gamma \mathbf{k} + \mathbf{I}_{(\sigma)} \cdot \frac{\partial \gamma}{\partial \mathbf{k}} \tag{1}$$

The surface divergence of $\boldsymbol{\xi}$ is expanded by chain rule

$$\begin{aligned}
-\nabla_{(\sigma)} \cdot \boldsymbol{\xi} &= -\operatorname{tr}\left(\mathbf{r}_\alpha g^{\alpha\beta} \nabla_{(\sigma)\beta} \boldsymbol{\xi}(\mathbf{n}, \mathbf{k})\right) &(2)\\
&= -\operatorname{tr}\left(\mathbf{r}_\alpha g^{\alpha\beta} \frac{\partial \boldsymbol{\xi}}{\partial n^i} \nabla_{(\sigma)\beta} n^i + \mathbf{r}_\alpha g^{\alpha\beta} \frac{\partial \boldsymbol{\xi}}{\partial k^j} \nabla_{(\sigma)\beta} k^j\right) &(3)\\
&= \underbrace{-\frac{\partial \boldsymbol{\xi}_\perp}{\partial \mathbf{n}} : (\nabla_{(\sigma)} \mathbf{n})^\intercal}_{0} \underbrace{-\frac{\partial \boldsymbol{\xi}_\perp}{\partial \mathbf{k}} : (\nabla_{(\sigma)} \mathbf{k})^\intercal}_{\text{Dilation Pressue}} \\
&\quad \underbrace{-\frac{\partial \boldsymbol{\xi}_\parallel}{\partial \mathbf{n}} : (\nabla_{(\sigma)} \mathbf{n})^\intercal}_{\text{Director Pressure}} \underbrace{-\frac{\partial \boldsymbol{\xi}_\parallel}{\partial \mathbf{k}} : (\nabla_{(\sigma)} \mathbf{k})^\intercal}_{\text{Rotation Pressure}} &(4)
\end{aligned}$$

where the tensor contraction nomenclature $\mathbf{A} : \mathbf{B} = A_{ij} B_{ji}$ is adopted. $\forall(\mathbf{a}, \mathbf{b}) \in (T^1 \times T^1)$ and $\forall(\mathbf{c}, \mathbf{d}) \in (T^1 \times S^2)$, the following identities are valid

$$\mathbf{ka} : (\nabla_{(\sigma)} \mathbf{b})^\intercal = (\mathbf{k} \cdot \mathbf{r}_\alpha) g^{\alpha\beta} \left(\mathbf{a} \cdot \nabla_{(\sigma)\beta} \mathbf{b}\right) = 0 \tag{5}$$

$$\mathbf{cd} : (\nabla_{(\sigma)} \mathbf{d})^\intercal = (\mathbf{c} \cdot \mathbf{r}_\alpha) g^{\alpha\beta} \left(\mathbf{d} \cdot \nabla_{(\sigma)\beta} \mathbf{d}\right) = 0 \tag{6}$$

The tangential component of the Cahn-Hoffman capillary vector is by definition

$$\boldsymbol{\xi}_{\parallel} = \mathbf{I}_{(\sigma)} \cdot \frac{\partial \gamma}{\partial \mathbf{k}} = \mathbf{I}_{(\sigma)} \cdot \sum_{j=1} 2j\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-1}\mathbf{n} \tag{7}$$

The tensor contractions are

$$\frac{\partial \boldsymbol{\xi}_{\perp}}{\partial \mathbf{k}} : (\nabla_{(\sigma)}\mathbf{k})^{\intercal} = \left(\frac{\partial \gamma}{\partial \mathbf{k}}\mathbf{k} + \gamma \mathbf{I}\right) : (\nabla_{(\sigma)}\mathbf{k})^{T} = \gamma \operatorname{tr}\left(\nabla_{(\sigma)}\mathbf{k}\right) = -2H\gamma \tag{8}$$

$$\frac{\partial \boldsymbol{\xi}_{\parallel}}{\partial \mathbf{n}} : (\nabla_{(\sigma)}\mathbf{n})^{\intercal} = \sum_{j=1} 2j\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-2}\left((2j-1)\mathbf{k}\mathbf{n} : \nabla_{(\sigma)}\mathbf{n} + (\mathbf{n}\cdot\mathbf{k})\operatorname{tr}\left(\nabla_{(\sigma)}\mathbf{n}\right)\right) \tag{9}$$

$$\frac{\partial \boldsymbol{\xi}_{\parallel}}{\partial \mathbf{k}} : (\nabla_{(\sigma)}\mathbf{k})^{\intercal} = \sum_{j=1} 2j\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-2}\left((2j-1)\mathbf{n}\mathbf{n} : \nabla_{(\sigma)}\mathbf{k} + 2(\mathbf{n}\cdot\mathbf{k})^{2}H\right) \tag{10}$$

The full governing equation becomes

$$\begin{aligned} 0 = & -2H\left(\gamma_0 + \sum_{j=1}\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j}\right) \\ & + \sum_{j=1} 2j\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-2}\left((2j-1)\mathbf{k}\mathbf{n} : \nabla_{(\sigma)}\mathbf{n} + (\mathbf{n}\cdot\mathbf{k})\operatorname{tr}\left(\nabla_{(\sigma)}\mathbf{n}\right)\right) \\ & + \sum_{j=1} 2j\mu_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-2}\left((2j-1)\mathbf{n}\mathbf{n} : \nabla_{(\sigma)}\mathbf{k} + 2(\mathbf{n}\cdot\mathbf{k})^{2}H\right) \end{aligned} \tag{11}$$

Denote dimensionless parameters $\epsilon_{2j} = \mu_{2j}/\gamma_0$, then

$$\begin{aligned} 0 = & \, 2H\left(\sum_{j=1}(2j-1)\epsilon_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j} - 1\right) \\ & + \sum_{j=1} 2j(2j-1)\epsilon_{2j}(\mathbf{n}\cdot\mathbf{k})^{2j-2} \\ & \left(\mathbf{k}\mathbf{n} : \nabla_{(\sigma)}\mathbf{n} + \frac{1}{2j-1}(\mathbf{n}\cdot\mathbf{k})\operatorname{tr}\left(\nabla_{(\sigma)}\mathbf{n}\right) + \mathbf{n}\mathbf{n} : \nabla_{(\sigma)}\mathbf{k}\right) \end{aligned} \tag{12}$$

For a 6-th order model, the equation becomes

$$
\begin{aligned}
0 = 2H &\left(\epsilon_2(\mathbf{n}\cdot\mathbf{k})^2 + 3\epsilon_4(\mathbf{n}\cdot\mathbf{k})^4 + 5\epsilon_6(\mathbf{n}\cdot\mathbf{k})^6 - 1\right) \\
&+ \mathbf{kn} : \nabla_{(\sigma)}\mathbf{n}\left(2\epsilon_2 + 12\epsilon_4(\mathbf{n}\cdot\mathbf{k})^2 + 30\epsilon_6(\mathbf{n}\cdot\mathbf{k})^4\right) \\
&+ \mathrm{tr}\left(\nabla_{(\sigma)}\mathbf{n}\right)\left(2\epsilon_2(\mathbf{n}\cdot\mathbf{k})^1 + 4\epsilon_4(\mathbf{n}\cdot\mathbf{k})^3 + 6\epsilon_6(\mathbf{n}\cdot\mathbf{k})^5\right) \\
&+ \mathbf{nn} : \nabla_{(\sigma)}\mathbf{k}\left(2\epsilon_2 + 12\epsilon_4(\mathbf{n}\cdot\mathbf{k})^2 + 30\epsilon_6(\mathbf{n}\cdot\mathbf{k})^4\right) \qquad (13)
\end{aligned}
$$

2 Thermodynamic stability

The main results of this section are used in Section 2.1 for Equation (1).

The range of r_j depends on μ_2^*. The Rapini-Papoular energy is strictly positive. Denote $x = (\mathbf{n}\cdot\mathbf{k})^2$

$$\gamma^* = 1 + \mu_2^*\left(x + \sum_{j=2}^{m} r_{j-1}x^j\right) > 0, \quad \forall x \in [0,1] \qquad (14)$$

The range of r_j is required to satisfy Equation (14). In this paper, we study the stability region of (r_1, r_2). The problem reduces to the discussion of $\mu_2^* f(x) > -1$, where $f(x) = x + r_1x^2 + r_2x^3$. Equation (14) holds for both $x = 0$ and $x = 1$, resulting

$$\mu_2^*(1 + r_1 + r_2) > -1 \qquad (15)$$

2.1 If $r_2 = 0$

The function is of the following form

$$f(x) = r_1\left(x + \frac{1}{2r_1}\right)^2 - \frac{1}{4r_1} \qquad (16)$$

The extrema can only be $f(0)$, $f(1)$ or $f(x^*)$, where $x^* = -1/(2r_1)$.

Equation (15) restrains $f(0)$ and $f(1)$. The other restriction $\mu_2^* f(x^*) > -1$ occurs only if $0 < x^* < 1$. And in summary,

$$
\begin{aligned}
&r^* < 0 \quad \text{or} \quad r^* > 1 \quad \text{or} \quad \{r^* \in (0,1) \quad \text{and} \quad \mu_2^* f(x^*) > -1\} \qquad (17) \\
&\Rightarrow r_1 < 0 \quad \text{or} \quad -\frac{1}{2} < r_1 < 0 \quad \text{or} \quad r_1 < \min\left\{-0.5, \frac{\mu_2^*}{4}\right\} \qquad (18) \\
&\Rightarrow r_1 < 0 \qquad (19)
\end{aligned}
$$

In summary, the restriction is $r_1 < 0$ and $\mu_2^*(1 + r_1) > -1$.

2.2 If $r_2 \neq 0$

Let $\Delta = r_1^2 - 3r_2$. The two extrema of $f(x)$ are

$$\begin{aligned} x_l &= \frac{-r_1 - \sqrt{\Delta}}{3r_2}, \quad f(x_l) = \frac{(\sqrt{\Delta}+r_1)(r_1(\sqrt{\Delta}+r_1) - 6r_2)}{27r_2^2} \\ x_r &= \frac{-r_1 + \sqrt{\Delta}}{3r_2}, \quad f(x_r) = \frac{(\sqrt{\Delta}-r_1)(r_1(\sqrt{\Delta}-r_1) + 6r_2)}{27r_2^2} \end{aligned} \tag{20}$$

There are 3 special cases: (1) $0 < x_l < 1 < x_r$ and $\mu_2^* f(x_l) > -1$; (2) $x_l < 0 < x_r < 1$ and $\mu_2^* f(x_r) > -1$; (3) $0 < x_l < x_r < 1$ and $\mu_2^* f(x_l) > -1$ and $\mu_2^* f(x_r) > -1$. All other cases (for example, $\mathrm{Im}(x_l) \neq 0$) are restrained by Equation (15) only. In partial summary, the region allowed is composed of 7 parts:

$$\begin{aligned} \mathrm{Set}^* \Rightarrow & (1) \quad \Delta = r_1^2 - 3r_2 < 0 \\ & (2) \quad \text{OR} \quad 0 < x_l < 1 < x_r \quad \text{and} \quad \mu_2^* f(x_l) > -1 \\ & (3) \quad \text{OR} \quad x_l < 0 < x_r < 1 \quad \text{and} \quad \mu_2^* f(x_r) > -1 \\ & (4) \quad \text{OR} \quad 0 < x_l < x_r < 1 \quad \text{and} \quad \mu_2^* f(x_l) > -1 \quad \text{and} \quad \mu_2^* f(x_r) > -1 \\ & (5) \quad \text{OR} \quad x_r < 0 \\ & (6) \quad \text{OR} \quad x_l < 0 < 1 < x_r \\ & (7) \quad \text{OR} \quad 1 < x_l \end{aligned} \tag{21}$$

In summary, the stable region is the intersection of Set* and $\mu_2^*(1+r_1) > -1$.

3 Small anchoring approximation

The main results of this section are used in Section 2.2 for Equations (9) to (11).

In Monge parametrization, the tensor contraction can be approximated with

$$\mathbf{n}\cdot\mathbf{k} \approx -h_x n^1 - h_y n^2 + n^3 + \mathcal{O}(\|\nabla h\|^2) \approx n^3 + \mathcal{O}(\|\nabla h\|) \tag{22}$$

$$\begin{aligned}
\mathrm{tr}\big(\boldsymbol{\nabla}_{(\sigma)}\mathbf{n}\big) &= \boldsymbol{\nabla}^x_{(\sigma)} n^1 + h_x \boldsymbol{\nabla}^x_{(\sigma)} n^3 + \boldsymbol{\nabla}^y_{(\sigma)} n^2 + h_y \boldsymbol{\nabla}^y_{(\sigma)} n^3 && (23)\\
&\approx n^1_x + n^2_y + h_x n^3_x + h_y n^3_y + \mathcal{O}(\|\nabla h\|^2) && (24)\\
&\approx n^1_x + n^2_y + \mathcal{O}(\|\nabla h\|) && (25)\\
\mathbf{kn} : \boldsymbol{\nabla}_{(\sigma)}\mathbf{n} &= k^i n^j \mathbf{e}_i\mathbf{e}_j : \boldsymbol{\nabla}^\alpha_{(\sigma)} n^t \mathbf{r}_\alpha \mathbf{e}_t && (26)\\
&= \boldsymbol{\nabla}^{y^1}_{(\sigma)} n^t (\mathbf{k}\cdot\mathbf{e}_t)(\mathbf{n}\cdot\mathbf{r}_{y^1}) + \boldsymbol{\nabla}^{y^2}_{(\sigma)} n^b (\mathbf{k}\cdot\mathbf{e}_b)(\mathbf{n}\cdot\mathbf{r}_{y^2}) && (27)\\
&\approx n^1 n^3_x + n^2 n^3_y + h_x n^3 n^3_x + h_y n^3 n^3_y + \mathcal{O}(\|\nabla h\|^2) && (28)\\
&\approx n^1 n^3_x + n^2 n^3_y + \mathcal{O}(\|\nabla h\|) && (29)\\
g^{5/2}\mathbf{nn} : \boldsymbol{\nabla}_{(\sigma)}\mathbf{k} &= g^{5/2}(\mathbf{n}\cdot\mathbf{r}_\alpha)(\mathbf{n}\cdot\boldsymbol{\nabla}^\alpha_{(\sigma)}\mathbf{k}) && (30)\\
&\approx \; -(n^1 + n^3 h_x)(n^1 h_{xx} + n^2 h_{xy}) - \frac{1}{2} n^1 n^3 g_x \\
&\quad\; -(n^2 + n^3 h_y)(n^1 h_{xy} + n^2 h_{yy}) - \frac{1}{2} n^2 n^3 g_y + \mathcal{O}(\|\nabla h\|^2) && (31)\\
&\approx \; -n^1(n^1 h_{xx} + n^2 h_{xy}) - n^2(n^1 h_{xy} + n^2 h_{yy}) + \mathcal{O}(\|\nabla h\|) && (32)
\end{aligned}$$

Let $|\mu_{2j}| \ll 1$, and $|\hat{\boldsymbol{\delta}}_z \cdot \mathbf{n}|^2 \ll 1$, the linearized PDE is

$$\begin{aligned}
0 = &\; -(h_{xx} + h_{yy}) \\
&+ \sum_j 2j\epsilon_{2j}\left((2j-1)(n^3)^{2j-2}(n^1 n^3_x + n^2 n^3_y) + (n^3)^{2j-1}(n^1_x + n^2_y)\right)
\end{aligned} \tag{33}$$

Or in a compact form

$$\nabla\cdot\nabla h = \sum_j 2j\epsilon_{2j}\left(((n^3)^{2j-1}n^1)_x + ((n^3)^{2j-1}n^2)_y\right) = \nabla\cdot\left(\sum_j 2j\epsilon_{2j}\mathcal{P}(n^3)^{2j}\right) \tag{34}$$

where $\mathcal{P} = \begin{bmatrix} n^1 & n^2 \end{bmatrix}^T$.

4 Spectral method

The main results of this section are used in Section 2.2 for Equations (9) to (11).

The dimensionless governing equation is

$$\nabla^{*2} h^* = \sum_j 2j\epsilon_{2j} \Bigg(\underbrace{(\cos^{2j-1} 2\pi x^* \sin 2\pi x^*)_{x^*} \cos^{2j} 2\pi y^*}_{I_{1j}} + \underbrace{(\cos^{2j-1} 2\pi y^* \sin 2\pi y^*)_{y^*} \cos^{2j-1} 2\pi x^*}_{I_{2j}} \Bigg) \tag{35}$$

we use the following formulae

$$\cos^{2j} x = \frac{1}{4^j}\binom{2j}{j} + \frac{1}{2^{2j-1}} \sum_{k=0}^{j-1} \binom{2j}{k} \cos 2(j-k)x \tag{36}$$

$$\cos^{2j-1} x = \frac{1}{2^{2j-2}} \sum_{k=0}^{j-1} \binom{2j-1}{k} \cos(2j-1-2k)x \tag{37}$$

Notice that

$$\sin x \cos(kx) = \frac{1}{2}(\sin(k+1)x - \sin(k-1)x) \tag{38}$$

Hence

$$(\cos^{2j-1} 2\pi x^* \sin 2\pi x^*)_{x^*} = \left(\frac{1}{2^{2j-1}} \sum_{k=0}^{j-1} \binom{2j-1}{k} (\sin 2(j-k)(2\pi)x^* - \sin 2(j-k-1)(2\pi)x^*) \right)_{x^*} \tag{39}$$

$$= \frac{2\pi}{2^{2j-1}} \sum_{k=0}^{j-1} \binom{2j-1}{k} (2(j-k) \cos 2(j-k) 2\pi x^* - 2(j-k-1) \cos 2(j-k-1) 2\pi x^*) \tag{40}$$

Denote $\cos 2\pi\omega_x x^* \cos 2\pi\omega_y y^* = \langle \omega_x, \omega_y \rangle$, then

$$\begin{aligned}
\frac{I_{1j}}{2\pi} = &\left(\frac{1}{2^{2j-1}} \sum_{k=0}^{j-1} \binom{2j-1}{k} (2(j-k)\langle 2(j-k), 0\rangle - 2(j-k-1)\langle 2(j-k-1), 0\rangle \right) \\
&\cdot \left(\frac{1}{4^j} \binom{2j}{j} + \frac{1}{2^{2j-1}} \sum_{t=0}^{j-1} \binom{2j}{t} \langle 0, 2(j-t)\rangle \right) \qquad (41) \\
= &\frac{1}{2^{4j-1}} \binom{2j}{j} \sum_{k=0}^{j-1} \binom{2j-1}{k} \underbrace{(2(j-k)\langle 2(j-k), 0\rangle - 2(j-k-1)\langle 2(j-k-1), 0\rangle)}_{\text{uniaxial}} \\
&+ \frac{1}{2^{2(2j-1)}} \sum_{k=0}^{j-1} \sum_{t=0}^{j-1} \binom{2j-1}{k} \binom{2j}{t} (2(j-k)\langle 2(j-k), 2(j-t)\rangle \\
&- 2(j-k-1)\langle 2(j-k-1), 2(j-t)\rangle) \qquad (42)
\end{aligned}$$

where the uniaxial wrinkling arises from the even order of $\cos^{2j}$ function.

$$\begin{aligned}
\frac{I_{2j}}{2\pi} = &\left(\frac{1}{2^{2j-1}} \sum_{k=0}^{j-1} \binom{2j-1}{k} (2(j-k)\langle 0, 2(j-k)\rangle - 2(j-k-1)\langle 0, 2(j-k-1)\rangle \right) \\
&\cdot \left(\frac{1}{2^{2j-2}} \sum_{t=0}^{j-1} \binom{2j-1}{t} \langle 2j-1-2t, 0\rangle \right) \qquad (43) \\
= &\frac{1}{2^{4j-3}} \sum_{k=0}^{j-1} \sum_{t=0}^{j-1} \binom{2j-1}{k} \binom{2j-1}{t} (2(j-k)\langle 2j-1-2t, 2(j-k)\rangle \\
&- 2(j-k-1)\langle 2j-1-2t, 2(j-k-1)\rangle) \qquad (44)
\end{aligned}$$

Neglect the constant term, and

$$\nabla^{*2} h^* = \sum_{m=1} a_j \langle \omega_{mx}, \omega_{my} \rangle \Rightarrow h^* = -\frac{1}{4\pi^2} \sum_{m=1} \underbrace{\frac{a_j}{\omega_{mx}^2 + \omega_{my}^2}}_{\mathbf{Q}_{xy}} \langle \omega_{mx}, \omega_{my} \rangle \qquad (45)$$

with notation $h^* = \mathbf{w}_x^T \mathbf{Q} \mathbf{w}_y$, where

$$\mathbf{Q} = \begin{bmatrix} \dots & \leftarrow & \omega_y & \rightarrow & \dots \\ \uparrow & & & & \\ \omega_x & & & & \\ \downarrow & & & & \\ \dots & & & & \end{bmatrix}_{(2m+1)^2}$$

$$\mathbf{w}_x = \begin{bmatrix} \cos 0 \\ \cos x^* \\ \dots \\ \cos(2m-1)x^* \\ \cos 2mx^* \end{bmatrix}_{2m+1}, \quad \mathbf{w}_y = \begin{bmatrix} \cos 0 \\ \cos y^* \\ \dots \\ \cos(2m-1)y^* \\ \cos 2my^* \end{bmatrix}_{2m+1} \tag{46}$$

For a 6[th]-order model, we need to compute $I_{1j}/2\pi$ and $I_{2j}/2\pi$ for $j = 1, 2, 3$, which correspond to $\epsilon_2, \epsilon_4, \epsilon_6$. The values are computed by a computer program. The $\mathbf{Q}$-matrices for $j = 1, 2, 3$ are

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/5 \\ 1/8 & 0 & 1/16 \end{bmatrix}, \quad \mathbf{Q}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/40 & 0 & 3/136 \\ 3/64 & 0 & 1/32 & 0 & 1/320 \\ 0 & 0 & 1/104 & 0 & 1/200 \\ 3/256 & 0 & 1/80 & 0 & 1/512 \end{bmatrix}$$

$$\mathbf{Q}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/128 & 0 & 5/272 & 0 & 15/4736 \\ 25/1024 & 0 & 75/4096 & 0 & 3/1024 & 0 & 1/4096 \\ 0 & 0 & 25/3328 & 0 & 1/160 & 0 & 1/768 \\ 5/512 & 0 & 3/256 & 0 & 3/1024 & 0 & 1/3328 \\ 0 & 0 & 5/7424 & 0 & 1/1312 & 0 & 3/15616 \\ 5/3072 & 0 & 9/4096 & 0 & 9/13312 & 0 & 1/12288 \end{bmatrix} \tag{47}$$

In the general governing equation:

$$\frac{1}{2\pi} \nabla^{*2} h^* = \sum_j 2j\epsilon_{2j} \left(\frac{I_1}{2\pi} + \frac{I_2}{2\pi} \right) \tag{48}$$

we have

$$h^* = -\frac{1}{2\pi} \sum_j^m 2j\epsilon_{2j} \mathbf{w}_x^T \mathbf{Q}_{2j} \mathbf{w}_y + c_m \tag{49}$$

where c_m is a constant, we treat it as zero here. If $m=3$, and $\epsilon_2 \neq 0$, let $\epsilon_4/\epsilon_2 = r_1$ and $\epsilon_6/\epsilon_2 = r_2$.

$$-\pi h^* = \mathbf{w}_x^T \left(\epsilon_2 \mathbf{Q}_2 + 2\epsilon_4 \mathbf{Q}_4 + 3\epsilon_6 \mathbf{Q}_6\right) \mathbf{w}_y \tag{50}$$

$$= \epsilon_2 \mathbf{w}_x^T \left(\mathbf{Q}_2 + 2r_1 \mathbf{Q}_4 + 3r_2 \mathbf{Q}_6\right) \mathbf{w}_y = \epsilon_2 \mathbf{w}_x^T \mathbf{C}(r_1, r_2) \mathbf{w}_y \tag{51}$$

Then

$$\mathbf{C}(r_1,r_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{128+96r_1+75r_2}{640} & 0 & \frac{12r_1+15r_2}{272} & 0 & \frac{45}{4736}r_2 \\ \frac{128+96r_1+75r_2}{1024} & 0 & \frac{256(1+r_1)+225r_2}{4096} & 0 & \frac{32r_1+45r_2}{5120} & 0 & \frac{3}{4096}r_2 \\ 0 & 0 & \frac{64r_1+75r_2}{3328} & 0 & \frac{8r_1+15r_2}{800} & 0 & \frac{1}{256}r_2 \\ \frac{12r_1+15r_2}{512} & 0 & \frac{32r_1+45r_2}{1280} & 0 & \frac{4r_1+9r_2}{1024} & 0 & \frac{3}{3328}r_2 \\ 0 & 0 & \frac{15}{7424}r_2 & 0 & \frac{3}{1312}r_2 & 0 & \frac{9}{15616}r_2 \\ \frac{5}{1024}r_2 & 0 & \frac{27}{4096}r_2 & 0 & \frac{27}{13312}r_2 & 0 & \frac{1}{4096}r_2 \end{bmatrix} \tag{52}$$

There are in total 21 possible non-vanishing modes.

(1) Uniaxial wrinkling. Nonvanishing mode: [6,0]. The others vanish at

$$\text{mode [2,0]}: \quad r_2 = -\frac{96}{75}r_1 - \frac{128}{75} \tag{53}$$

$$\text{mode [4,0]}: \quad r_2 = -\frac{4}{5}r_1 \tag{54}$$

(2) Equi-biaxial wrinkling. Nonvanishing mode: [6,6]. The others vanish at

$$\text{mode [2,2]}: \quad r_2 = -\frac{256}{225}r_1 - \frac{256}{225} \tag{55}$$

$$\text{mode [4,4]}: \quad r_2 = -\frac{4}{9}r_1 \tag{56}$$

(3) Non-symmetric biaxial wrinkling. Nonvanishing mode: [1,6], [2,6], [3,6], [4,6], [5,2], [5,4], [5,6], [6,2], [6,4]. The others vanish at

$$\text{mode [1,2]}: \quad r_2 = -\frac{96}{75}r_1 - \frac{128}{75} \tag{57}$$

$$\text{mode [1,4]}: \quad r_2 = -\frac{4}{5}r_1 \tag{58}$$

$$\text{mode [2,4]}: \quad r_2 = -\frac{32}{45}r_1 \tag{59}$$

$$\text{mode [3,2]}: \quad r_2 = -\frac{64}{75}r_1 \tag{60}$$

$$\text{mode [3,4]}: \quad r_2 = -\frac{8}{15}r_1 \tag{61}$$

$$\text{mode [4,2]}: \quad r_2 = -\frac{32}{45}r_1 \tag{62}$$

In summary, there are three special lines along which certain wrinkling modes disappear at the same time:

$$\text{mode [2,0], [1,2]}: \quad r_2 = -\frac{96}{75}r_1 - \frac{128}{75} \tag{63}$$

$$\text{mode [4,0], [1,4]}: \quad r_2 = -\frac{4}{5}r_1 \tag{64}$$

$$\text{mode [2,4], [4,2]}: \quad r_2 = -\frac{32}{45}r_1 \tag{65}$$

5 Curvatures

The main results of this section are used in Section 2.3 for Equations (12) to (13).

The Gaussian curvature is

$$K^* \approx \det \nabla^* \nabla^* h^* = \left(\frac{\epsilon_2}{\pi}\right)^2 \det \nabla\nabla C_{ij} \cos 2\pi(i-1)x^* \cos 2\pi(j-1)y^* \tag{66}$$

The metric tensor is

$$\begin{aligned}\frac{g-1}{\pi^2} &= (i-1)(a-1)C_{ij}C_{ab}(\langle i-a,0\rangle - \langle i+a-2,0\rangle)(\langle 0,j-b\rangle + \langle 0,j+b-2\rangle) \\ &\quad + C_{ij}C_{ab}(\langle i-a,0\rangle + \langle i+a-2,0\rangle)(\langle 0,j-b\rangle - \langle j+b-2\rangle)\end{aligned} \tag{67}$$

Denote the metric tensor as $g = 1 + \pi^2\lambda^*(x^*, y^*)$. The mean value is

$$
\begin{aligned}
c_m = & \left(\iint_\Omega \mathrm{d}A^*\right)^{-1} \iint_\Omega \sqrt{g} h^* \mathrm{d}x^* \mathrm{d}y^* && (68) \\
= & -\epsilon_2 \int_0^1 \mathrm{d}x^* \int_0^1 \mathrm{d}y^* \left(\sum_{k,l}^{2m+1} C_{kl} \langle k-1, l-1\rangle\right) \sqrt{\frac{1}{\pi^2} + \lambda^*(x^*, y^*)} && (69)
\end{aligned}
$$

The Gaussian curvature is therefore

$$
\begin{aligned}
\frac{K^*}{4\pi^2\epsilon_2^2} \approx & \, C_{ij} C_{ab} (i-1)^2 (b-1)^2 \\
& \cdot \left(\langle i+a-2, j+b-2\rangle + \langle i+a-2, j-b\rangle + \langle i-a, j+b-2\rangle + \langle i-a, j-b\rangle\right) \\
& - (k-1)(l-1)(m-1)(n-1) C_{kl} C_{mn} \\
& \cdot \left(\langle k+m-2, l+n-2\rangle - \langle k+m-2, l-n\rangle - \langle k-m, l+n-2\rangle + \langle k-m, l-n\rangle\right) \\
& && (70)
\end{aligned}
$$

The mean curvature is simply

$$
\begin{aligned}
H^* \approx & \frac{1}{2} \operatorname{tr} \nabla^* \nabla^* h^* = -\frac{\epsilon_2}{2\pi} C_{ij} \cdot (-4\pi^2)((i-1)^2 + (j-1)^2) \langle i-1, j-1\rangle \\
& && (71) \\
= & \, 2\epsilon_2 \pi C_{ij} ((i-1)^2 + (j-1)^2) \langle i-1, j-1\rangle && (72)
\end{aligned}
$$

The deciatoric curvature D^*, the Casorati curvature C^*, and the shape parameter S

$$
D^* = \sqrt{H^{*2} - K^*}, \quad C^* = \sqrt{H^{*2} + D^{*2}}, \quad S = \frac{2}{\pi} \arctan\left(\frac{H^*}{D^*}\right) \tag{73}
$$

The mean curvature square is

$$
\begin{aligned}
H^{*2} = & \, 4\epsilon_2^2 \pi^2 C_{ij} C_{ab} ((i-1)^2 + (j-1)^2)((a-1)^2 + (b-1)^2) \\
& \cos 2\pi(i-1)x^* \cos 2\pi(a-1)x^* \cos 2\pi(j-1)y^* \cos 2\pi(b-1)y^* && (74) \\
= & \, \epsilon_2^2 \pi^2 C_{ij} C_{ab} ((i-1)^2 + (j-1)^2)((a-1)^2 + (b-1)^2) \\
& \left(\langle i+a-2, 0\rangle + \langle i-a, 0\rangle\right) \left(\langle 0, j+b-2\rangle + \langle 0, j-b\rangle\right) && (75)
\end{aligned}
$$

The surface integral is

$$\frac{1}{\epsilon_2^2\pi^2}\iint_\Omega H^{*2}\,\mathrm{d}\,A^* = \frac{1}{\epsilon_2^2\pi^2}\iint_\Omega \sqrt{g}H^2\,\mathrm{d}\,x^*\,\mathrm{d}\,y^* \tag{76}$$

$$\approx \sum_{i,j,a,b} C_{ij}C_{ab}((i-1)^2+(j-1)^2)((a-1)^2+(b-1)^2)$$
$$\int_0^1 (\langle i+a-2,0\rangle + \langle i-a,0\rangle)\,\mathrm{d}\,x^* \int_0^1 (\langle 0,j+b-2\rangle + \langle 0,j-b\rangle)\,\mathrm{d}\,y^* \tag{77}$$

$$= \sum_{i,j,a,b} C_{ij}C_{ab}((i-1)^2+(j-1)^2)((a-1)^2+(b-1)^2)$$
$$(\delta^0_{i+a-2}+\delta^0_{i-a})(\delta^0_{j+b-2}+\delta^0_{j-b}) \tag{78}$$

Similarly,

$$\frac{1}{4\pi^2\epsilon_2^2}\iint_\Omega K^*\,\mathrm{d}\,A^* = \iint_\Omega \sqrt{g}C_{ij}C_{ab}(i-1)^2(b-1)^2$$
$$\cdot\,(\langle i+a-2,0\rangle + \langle i-a,0\rangle)(\langle 0,j+b-2\rangle + \langle 0,j-b\rangle)\,\mathrm{d}\,x^*\,\mathrm{d}\,y^*$$
$$-\iint_\Omega \sqrt{g}(k-1)(l-1)(m-1)(n-1)C_{kl}C_{mn}$$
$$\cdot\,(\langle k+m-2,0\rangle - \langle k-m,0\rangle)(\langle 0,l+n-2\rangle - \langle 0,l-n\rangle)\,\mathrm{d}\,x^*\,\mathrm{d}\,y^* \tag{79}$$

$$\approx \sum_{i,j,a,b} C_{ij}C_{ab}(i-1)^2(b-1)^2(\delta^0_{i+a-2}+\delta^0_{i-a})(\delta^0_{j+b-2}+\delta^0_{j-b})$$
$$-\sum_{i,j,a,b}(i-1)(j-1)(a-1)(b-1)C_{ij}C_{ab}(\delta^0_{i+a-2}-\delta^0_{i-a})(\delta^0_{j+b-2}-\delta^0_{j-b})$$
$$= 0 \tag{80}$$

6 Surface roughness parameters

The main results of this section are used in Section 2.3 for Equations (14) to (18).

We evaluate the roughness parameters of the following surface profile

$$h^* = -\frac{\epsilon_2}{\pi}\mathbf{w}_x^\intercal\mathbf{C}(r_1,r_2)\mathbf{w}_y - c_m \tag{81}$$

where c_m is taken such that the surface integral of h^* is zero.

6.1 Root mean square

The second order moment can be evaluated by

$$\iint_A h^{*2}\,\mathrm{d}\,x^*\,\mathrm{d}\,y^* = \left(\frac{\epsilon_2}{\pi}\right)^2 \iint_A (w_{xi}C_{ij}w_{yj})(w_{xt}C_{tk}w_{yk})\,\mathrm{d}\,x^*\,\mathrm{d}\,y^* \tag{82}$$

$$= \left(\frac{\epsilon_2}{\pi}\right)^2 C_{ij}C_{tk}\int_0^1 w_{xi}w_{xt}\,\mathrm{d}\,x^* \int_0^1 w_{yj}w_{yk}\,\mathrm{d}\,y^* \tag{83}$$

$$= \left(\frac{\epsilon_2}{\pi}\right)^2 C_{ij}C_{tk}\mathcal{A}_{it}\mathcal{A}_{jk} = \left(\frac{\epsilon_2}{\pi}\right)^2 \mathrm{tr}\,(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\intercal}) \tag{84}$$

where $(i, j = 1, \ldots, 7)$

$$\mathcal{A}_{it} = \int_0^1 \cos 2\pi(i-1)x^* \cos 2\pi(t-1)x^*\,\mathrm{d}\,x^* = \frac{1}{2}\begin{bmatrix} 2 & \\ & \mathbf{I}_{6\times 6}\end{bmatrix} \tag{85}$$

And $\mathrm{Sq}^2 \approx (\epsilon_2/\pi)^2\,\mathrm{tr}\,(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\intercal})$.

6.2 Skewness and kurtosis

Similarly, the third order moment is

$$\iint_A h^{*3}\,\mathrm{d}\,x^*\,\mathrm{d}\,y^* = \left(-\frac{\epsilon_2}{\pi}\right)^3 C_{ab}C_{ij}C_{kl}\underbrace{\int_0^1 w_{xa}w_{xi}w_{xk}\,\mathrm{d}\,x^*}_{\mathcal{A}_{aik}}\mathcal{A}_{bjl} \tag{86}$$

Therefore, the skewness is

$$\mathrm{Ssk} \approx -\,\mathrm{sgn}(\epsilon_2)\frac{C_{ab}C_{ij}C_{kl}\mathcal{A}_{aik}\mathcal{A}_{bjl}}{\left(\mathrm{tr}\,(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\intercal})\right)^{3/2}} \tag{87}$$

The kurtosis is

$$\mathrm{Sku} \approx \frac{C_{ab}C_{ij}C_{km}C_{np}\mathcal{A}_{aikn}\mathcal{A}_{bjmp}}{\left(\mathrm{tr}\,(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\intercal})\right)^{2}} \tag{88}$$

6.3 Pearson's inequality

For simplicity, denote $Y = h^*/\mathrm{Sq}$.

$$\mathrm{Rsk} = \frac{1}{A^*}\iint_\Omega Y^3 \,\mathrm{d}\,A^* - \frac{1}{A^*\mathrm{Sq}}\underbrace{\iint_\Omega h^* \,\mathrm{d}\,A^*}_{=0} = \frac{1}{A^*}\iint_\Omega \left(Y\left(Y^2-1\right)\right)\mathrm{d}\,A^* \tag{89}$$

$$\le \text{(Cauchy-Schwarz)}$$

$$\underbrace{\sqrt{\left(\frac{1}{A^*\mathrm{Sq}^2}\iint_\Omega h^{*2}\,\mathrm{d}\,A^*\right)}}_{=1}\cdot\sqrt{\frac{1}{A^*}\iint_\Omega \left(Y^2-1\right)^2\mathrm{d}\,A^*} \tag{90}$$

$$= \sqrt{\frac{1}{A^*\mathrm{Sq}^4}\iint_\Omega h^{*4}\,\mathrm{d}\,A^* - \frac{2}{A^*\mathrm{Sq}^2}\iint_\Omega h^{*2}\,\mathrm{d}\,A^* + \frac{1}{A^*}\iint_\Omega \mathrm{d}\,A^*} \tag{91}$$

$$= \sqrt{\mathrm{Rku}-1} \tag{92}$$

6.4 Autocorrelation function

Define the autocorrelation function as

$$\mathrm{acf}(\Delta x^*, \Delta y^*) = \frac{1}{A\mathrm{Sq}^2}\iint_\Omega \sqrt{g}h^*(x^*,y^*)h^*(x^*+\Delta x^*, y^*+\Delta y^*)\mathrm{d}x^*\mathrm{d}y^* \tag{93}$$

The constant c_m does not affect the acf distribution since

$$\iint_\Omega (f(x^*,y^*)-c_m)(f(x^*+\Delta x^*, y^*+\Delta y^*)-c_m)\mathrm{d}x^*\mathrm{d}y^* \tag{94}$$

$$\begin{aligned}= &\iint_\Omega f(x^*,y^*)f(x^*+\Delta x^*, y^*+\Delta y^*)\mathrm{d}x^*\mathrm{d}y^* + c_m^2 \\ &- c_m\iint_\Omega f(x^*,y^*)\mathrm{d}x^*\mathrm{d}y^* \\ &- c_m\iint_{\Omega\oplus[\Delta x^*\times\Delta y^*]} f(x^*+\Delta x^*, y^*+\Delta y^*)\,\mathrm{d}(x^*+\Delta x^*)\,\mathrm{d}(y^*+\Delta y^*)\end{aligned} \tag{95}$$

$$= \iint_\Omega f(x^*,y^*)f(x^*+\Delta x^*, y^*+\Delta y^*)\mathrm{d}x^*\mathrm{d}y^* + \text{constant} \tag{96}$$

We neglect the constant and compute

$$\begin{aligned}
\mathrm{acf} &\approx \frac{\sum_{i,j,a,b} C_{ij}C_{ab}}{\mathrm{tr}(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\mathsf{T}})}\int_0^1 w_{x^*i}w_{(x^*+\Delta x^*)a}\mathrm{d}x^*\int_0^1 w_{y^*j}w_{(y^*+\Delta y^*)b}\mathrm{d}y^* && (97)\\
&= \frac{\sum_{i,j,a,b} C_{ij}C_{ab}}{\mathrm{tr}(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\mathsf{T}})}\int_0^1 \cos 2\pi(i-1)x^*\cos 2\pi(a-1)(x^*+\Delta x^*)\mathrm{d}x^*\\
&\quad\cdot\int_0^1 \cos 2\pi(j-1)y^*\cos 2\pi(b-1)(y^*+\Delta y^*)\mathrm{d}y^* && (98)\\
&\approx \frac{\sum_{i,j,a,b} C_{ij}C_{ab}\mathcal{A}_{ia}\mathcal{A}_{jb}}{\mathrm{tr}(\mathcal{A}\cdot\mathbf{C}\cdot\mathcal{A}\cdot\mathbf{C}^{\mathsf{T}})}\cos 2\pi(a-1)\Delta x^*\cos 2\pi(b-1)\Delta y^* && (99)
\end{aligned}$$

7 Nonlinearity

The main results of this section are for future work.

If we keep the ϵ_2-term for nonlinearity analysis. Equation reduces to

$$\begin{aligned}
0 =\; & -(h_{xx}+h_{yy})\\
& + \sum_j 2j\epsilon_{2j}\left((2j-1)(n^3)^{2j-2}(n^1n^3_x+n^2n^3_y)+(n^3)^{2j-1}(n^1_x+n^2_y)\right)\\
& + \epsilon_2(n^3)^2(h_{xx}+h_{yy}) - 2\epsilon_2\left[n^1(n^1h_{xx}+n^2h_{xy})+n^2(n^1h_{xy}+n^2h_{yy})\right]
\end{aligned} \tag{100}$$

Nondimensionalize the PDE and let $L = -\frac{\epsilon_2}{\pi}\mathbf{w}_x^{\mathsf{T}}\mathbf{C}\mathbf{w}_y$ denote the linear solution. Consider the solution h_n^* of Equation (100) as a perturbation η around L such that

$$h_n^* = L + \eta \tag{101}$$

Cancelling out the linear part, the PDE reduces to (denote $\mathbf{n}_\parallel = \begin{bmatrix} n^1 & n^2\end{bmatrix}^{\mathsf{T}}$)

$$0 = \left[(n^3)^2\mathbf{I} - 2\mathbf{n}_\parallel\mathbf{n}_\parallel\right] : \nabla\nabla L + \left[\left((n^3)^2 - \frac{1}{\epsilon_2}\right)\mathbf{I} - 2\mathbf{n}_\parallel\mathbf{n}_\parallel\right] : \nabla\nabla\eta \tag{102}$$

Let $\lambda = 2\pi i$, the director fields are functions of $\mathrm{e}^{\lambda x^*}$ and $\mathrm{e}^{\lambda y^*}$. They can be written in a compact form $\sum_j c_j\,\mathrm{e}^{-\lambda(a_jx^*+b_jy^*)}$. Therefore

$$\mathbf{R} = (n^3)^2\mathbf{I} - 2\mathbf{n}_\parallel\mathbf{n}_\parallel = \begin{bmatrix} \sum_i c_{1i}\,\mathrm{e}^{-\lambda(a_{1i}x^*+b_{1i}y^*)} & \sum_i c_{2i}\,\mathrm{e}^{-\lambda(a_{2i}x^*+b_{2i}y^*)}\\ \sum_i c_{2i}\,\mathrm{e}^{-\lambda(a_{2i}x^*+b_{2i}y^*)} & \sum_i c_{1i}\,\mathrm{e}^{-\lambda(a_{1i}x^*+b_{1i}y^*)}\end{bmatrix} \tag{103}$$

Calculating the Fourier transform on both sides of Equation (102):

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix} : \nabla\nabla L \, \mathrm{e}^{-\lambda(\omega_x x^* + \omega_y y^*)} \, \mathrm{d}\, x^* \, \mathrm{d}\, y^* \tag{104}$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} R_1 - \frac{1}{\epsilon_2} & R_2 \\ R_2 & R_1 - \frac{1}{\epsilon_2} \end{bmatrix} : \nabla\nabla \eta \, \mathrm{e}^{-\lambda(\omega_x x^* + \omega_y y^*)} \, \mathrm{d}\, x^* \, \mathrm{d}\, y^* \tag{105}$$

which yields

$$\begin{aligned} 0 = & - \sum_i c_{1i} \lambda^2 (a_{1i} + \omega_x)^2 \left(\widehat{L}(\omega_x + a_{1i}, \omega_y + b_{1i}) + \widehat{\eta}(\omega_x + a_{1i}, \omega_y + b_{1i}) \right) \\ & - 2 \sum_i c_{2i} \lambda^2 (a_{2i} + \omega_x)(b_{2i} + \omega_y) \left(\widehat{L}(\omega_x + a_{2i}, \omega_y + b_{2i}) + \widehat{\eta}(\omega_x + a_{2i}, \omega_y + b_{2i}) \right) \\ & - \sum_i c_{3i} \lambda^2 (b_{3i} + \omega_y)^2 \left(\widehat{L}(\omega_x + a_{3i}, \omega_y + b_{3i}) + \widehat{\eta}(\omega_x + a_{3i}, \omega_y + b_{3i}) \right) \\ & + \frac{1}{\epsilon_2} \lambda^2 (\omega_x + \omega_y)^2 \widehat{\eta}(\omega_x, \omega_y) \end{aligned} \tag{106}$$

If $\epsilon_2 \to 0$, we have

$$0 \approx \frac{1}{\epsilon_2} \lambda^2 (\omega_x + \omega_y)^2 \widehat{\eta}(\omega_x, \omega_y) \tag{107}$$

hence

$$0 \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\epsilon_2} \lambda^2 (\omega_x + \omega_y)^2 \widehat{\eta}(\omega_x, \omega_y) \, \mathrm{e}^{+\lambda(\omega_x x^* + \omega_y y^*)} \, \mathrm{d}\, \omega_x \, \mathrm{d}\, \omega_y = -\frac{1}{\epsilon_2} \nabla^2 \eta \tag{108}$$

The method can be expanded to higher order models. For a 6th-order mode, the highest order of director field is $\epsilon_6 (n^3)^4 (n^1)^2$, which is

$$(n^3)^4 (n^1)^2 = (\cos 2\pi x^* \cos 2\pi y^*)^4 (\sin 2\pi x^* \cos 2\pi y^*)^2 \tag{109}$$

$$\begin{aligned} &= c(\mathrm{e}^{i2\pi x^*} + \mathrm{e}^{-i2\pi x^*})^4 (\mathrm{e}^{i2\pi y^*} + \mathrm{e}^{-i2\pi y^*})^4 \\ &\quad \cdot (\mathrm{e}^{i2\pi x^*} - \mathrm{e}^{-i2\pi x^*})^2 (\mathrm{e}^{i2\pi y^*} + \mathrm{e}^{-i2\pi y^*})^2 \end{aligned} \tag{110}$$

$$= c \, \mathrm{e}^{i2\pi \cdot 12 x^*} + \text{lower-order terms} \tag{111}$$

The highest frequency shift we should observe is 12.