Electronic Supplementary Material (ESI) for Soft Matter

Modelling the non-linear viscoelastic behaviour of brain tissue in torsion

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1 Plots of τ and N_z for control tests without any samples between the plates

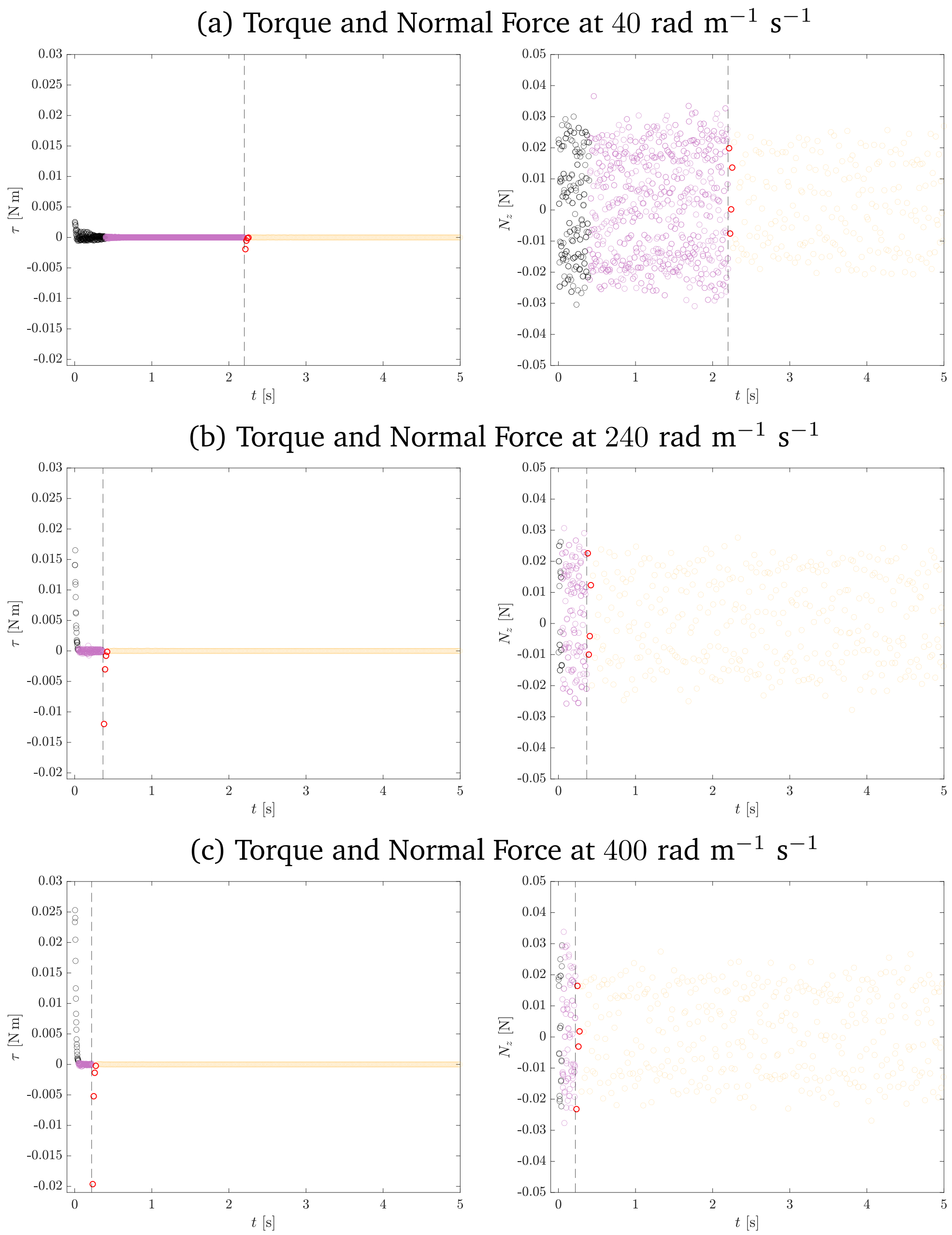

Figure 1: Raw output data from torsion tests performed without any samples between the plates at twist rates of (a) 40, (b) 240, and (c) 400 rad m^{-1} s^{-1}. The black data represent the acceleration phase, purple the constant twist rate phase, red the deceleration phase and orange the hold phase. A dashed line indicates the end of the ramp phase.

2 Filtering procedure

The Savitzky–Golay filter parameters were selected to accurately capture the ramp phase, the peak of the relaxation curves and the rapid decay that follows, while also lying near the "midpoint" of the experimental data, as shown in Fig. 5 in the main paper. To maintain consistency, we used the same parameter values for both the torque and normal force, selecting a polynomial order of 5 and a window length of 31 for the 240 and 400 $\mathrm{rad\,m^{-1}\,s^{-1}}$ data. For the 40 $\mathrm{rad\,m^{-1}\,s^{-1}}$ data—which included a larger number of datapoints in the ramp phase (1000) compared to the 240 $\mathrm{rad\,m^{-1}\,s^{-1}}$ data (175) and 400 $\mathrm{rad\,m^{-1}\,s^{-1}}$ data (100)—we maintained a polynomial order of 5 but used a larger window length of 61.

3 Components of $\boldsymbol{T}$ for an arbitrary twist history $\boldsymbol{\phi}$

$$
\begin{aligned}
T_{rr}(r,t) = &-\mu_0\left[\lambda+1+2\gamma(\lambda-1)\right]\left(\frac{\lambda^3-1}{6\lambda^2}\right)-\frac{\mu_0 r^2}{6}\left[\lambda-1+2\gamma(\lambda+1)\right]\lambda\phi(t)^2\\
&+\frac{1}{6\lambda^2}\sum_{i=1}^{n}\int_0^t\frac{1}{\tau_i}\mu_i\mathrm{e}^{-(t-s)/\tau_i}\left[\left\{\lambda-1+2\gamma(\lambda+1)\right\}r^2R_0^2\lambda^2\phi(s)^2+\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(\lambda^3-1\right)\right]\mathrm{d}s-p(r,t),
\end{aligned}
\tag{S1}
$$

$$
\begin{aligned}
T_{\theta\theta}(r,t) = &-\mu_0\left[\lambda+1+2\gamma(\lambda-1)\right]\left(\frac{\lambda^3-1}{6\lambda^2}\right)+\frac{\mu_0 r^2}{3}\left[1+2\gamma(2\lambda-1)\right]\lambda\phi(t)^2\\
&-\frac{1}{6\lambda^2}\sum_{i=1}^{n}\int_0^t\frac{1}{\tau_i}\mu_i\mathrm{e}^{-(t-s)/\tau_i}\Big[\left\{\lambda+2+2\gamma(\lambda-2)\right\}r^4\lambda^3\phi(s)^4-2\left\{\lambda+2+2\gamma(\lambda-2)\right\}r^4\lambda^3\phi(s)^3\phi(t)\\
&+\left\{\lambda+2+2\gamma(\lambda-2)\right\}r^4\lambda^3\phi(s)^2\phi(t)^2+\left\{2\lambda^4+3\lambda^3-6\lambda+2+2\gamma\left(2\lambda^4-3\lambda^3+2\lambda-2\right)\right\}r^2\phi(s)^2\\
&-2\left\{\lambda+1+2\gamma(\lambda-1)\right\}r^2\left(\lambda^3+2\right)\phi(s)\phi(t)-2\left\{\lambda+1+2\gamma(\lambda-1)\right\}r^2\lambda\left(\lambda^3-1\right)\phi(t)^2\\
&+\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(\lambda^3-1\right)\Big]\mathrm{d}s-p(r,t),
\end{aligned}
\tag{S2}
$$

$$
\begin{aligned}
T_{zz}(r,t) = &\ \mu_0\left[\lambda+1+2\gamma(\lambda-1)\right]\left(\frac{\lambda^3-1}{3\lambda^2}\right)-\frac{\mu_0 r^2}{6}\left[\lambda+2+2\gamma(\lambda-2)\right]\lambda\phi(t)^2\\
&+\frac{1}{6\lambda^2}\sum_{i=1}^{n}\int_0^t\frac{1}{\tau_i}\mu_i\mathrm{e}^{-(t-s)/\tau_i}\left[\left\{\lambda+2+2\gamma(\lambda-2)\right\}r^2\lambda^3\phi(s)^2-2\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(\lambda^3-1\right)\right]\mathrm{d}s-p(r,t),
\end{aligned}
\tag{S3}
$$

$$
\begin{aligned}
T_{\theta z}(r,t) = &\ \frac{\mu_0 r}{2}\left[\lambda+1+2\gamma(\lambda-1)\right]\lambda\phi_0(t)\\
&+\frac{r}{6\lambda^2}\sum_{i=1}^{n}\int_0^t\frac{1}{\tau_i}\mu_i\mathrm{e}^{-(t-s)/\tau_i}\Big[\left\{3\lambda-2-2\gamma(\lambda-2)\right\}r^2\lambda^3\phi(s)^3+\left\{\lambda+2+2\gamma(\lambda-2)\right\}r^2\lambda^3\phi(s)^2\phi(t)\\
&-\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(\lambda^3-1\right)\phi(s)-2\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(\lambda^3-1\right)\phi(t)\Big]\mathrm{d}s,
\end{aligned}
\tag{S4}
$$

$$
T_{r\theta}(r,t)=0,\qquad T_{rz}(r,t)=0,
\tag{S5}
$$

where

$$
\begin{aligned}
p(r,t) = &-\mu_0\left[\lambda+1+2\gamma(\lambda-1)\right]\left(\frac{\lambda^3-1}{6\lambda^2}\right)-\frac{\mu_0}{12}\left[\left\{5\lambda-2+2\gamma(5\lambda+2)\right\}r^2-3\left(1+2\gamma\right)R_0^2\right]\lambda\phi(t)^2\\
&+\frac{1}{24\lambda^3}\sum_{i=1}^{n}\int_0^t\frac{1}{\tau_i}\mu_i\mathrm{e}^{-(t-s)/\tau_i}\Big[\left\{\lambda+2+2\gamma(\lambda-2)\right\}\left(r^4\lambda^2-R_0^4\right)\lambda^2\phi(s)^4\\
&-2\left\{\lambda+2+2\gamma(\lambda-2)\right\}\left(r^4\lambda^2-R_0^4\right)\lambda^2\phi(s)^3\phi(t)+\left\{\lambda+2+2\gamma(\lambda-2)\right\}\left(r^4\lambda^2-R_0^4\right)\lambda^2\phi(s)^2\phi(t)^2\\
&-\left\{\left[2\lambda^4-12\lambda^3-4\lambda-4+4\gamma\left(\lambda^4+6\lambda^3-2\lambda+2\right)\right]r^2\lambda+2\left[\lambda^4+4\lambda^3+2\lambda+2+2\gamma\left(\lambda^4-4\lambda^3+2\lambda-2\right)\right]R_0^2\right\}\lambda\phi(s)^2\\
&-4\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(r^2\lambda-R_0^2\right)\lambda\left(\lambda^3+2\right)\phi(s)\phi(t)-4\left\{\lambda+1+2\gamma(\lambda-1)\right\}\left(r^2\lambda-R_0^2\right)\left(\lambda^3-1\right)\phi(t)^2\\
&-4\left\{\lambda+1+2\gamma(\lambda-1)\right\}\lambda\left(\lambda^3-1\right)\Big]\mathrm{d}s.
\end{aligned}
\tag{S6}
$$

4 Expressions for τ and N_z for an arbitrary twist history ϕ

$$\tau(t) = \frac{\pi\mu_0 R_0^4}{4} A(\lambda,\gamma)\phi(t) - \frac{\pi R_0^4}{36\lambda^4}\sum_{i=1}^{n}\int_0^t \frac{1}{\tau_i}\mu_i \mathrm{e}^{-(t-s)/\tau_i}\big[2R_0^3\lambda^3 B(\lambda,\gamma)\phi(s)^3 - 2R_0^3\lambda^3 B(\lambda,\gamma)\phi(s)^2\phi(t)$$
$$+ 3\lambda\left(\lambda^3+2\right)A(\lambda,\gamma)\phi(s) + 6\lambda\left(\lambda^3-1\right)A(\lambda,\gamma)\phi(t)\big]\,\mathrm{d}s, \tag{S7}$$

$$N_z(t) = \pi\mu_0 R_0^2\left(\frac{\lambda^3-1}{2\lambda^2}\right)B(\lambda,\gamma) - \frac{\pi\mu_0 R_0^4}{8}B(\lambda,\gamma)\phi(t)^2 - \frac{\pi R_0^2}{72\lambda^4}\sum_{i=1}^{n}\int_0^t \frac{1}{\tau_i}\mu_i \mathrm{e}^{-(t-s)/\tau_i}\big[2R_0^4\lambda^3 B(\lambda,\gamma)\phi(s)^4$$
$$- 4R_0^4\lambda^3 B(\lambda,\gamma)\phi(s)^3\phi(t) + 2R_0^4\lambda^3 B(\lambda,\gamma)\phi(s)^2\phi(t)^2 + 3R_0^2\left(\lambda^4-2\lambda^3+2\lambda+2+2\gamma C(\lambda)\right)\phi(s)^2$$
$$- 6R_0^2\lambda\left(\lambda^3+2\right)A(\lambda,\gamma)\phi(s)\phi(t) - 6R_0\lambda^{3/2}\left(\lambda^3-1\right)A(\lambda,\gamma)\phi(t) + 36\lambda^2\left(\lambda^3-1\right)A(\lambda,\gamma)\big]\,\mathrm{d}s. \tag{S8}$$

5 Expressions for τ and N_z during the ramp phase ($0 \le t \le t^\star$)

$$\tau^{\mathrm{ramp}}(t) = \frac{\pi\mu_\infty R_0^4 t}{4t^\star}A(\lambda,\gamma)\phi_0 + \frac{\pi R_0^4}{12\lambda^3 t^\star}A(\lambda,\gamma)\sum_{i=1}^{n}\mu_i\big[2\left(\lambda^3-1\right)t\mathrm{e}^{-t/\tau_i} + \left(\lambda^3+2\right)\tau_i\left(1-\mathrm{e}^{-t/\tau_i}\right)\big]\phi_0$$
$$+ \frac{\pi R_0^6}{18\lambda t^{\star 3}}B(\lambda,\gamma)\sum_{i=1}^{n}\mu_i\tau_i\mathrm{e}^{-t/\tau_i}\big[-2\tau_i\,(t+3\tau_i) + \mathrm{e}^{t/\tau_i}\left(t^2-4t\tau_i+6\tau_i^2\right)\big]\phi_0^3, \tag{S9}$$

$$N_z^{\mathrm{ramp}}(t) = \pi\mu(t)R_0^2\left(\frac{\lambda^3-1}{2\lambda^2}\right)A(\lambda,\gamma) - \frac{\pi\mu_\infty R_0^4 t^2}{8t^{\star 2}}B(\lambda,\gamma)\phi_0^2 - \frac{\pi R_0^4}{12\lambda^4 t^{\star 2}}\sum_{i=1}^{n}\mu_i\mathrm{e}^{-t/\tau_i}\big[\left(1-2\gamma\right)\{\mathrm{e}^{t/\tau_i}\left(3\lambda^3\tau_i t - C(\lambda)\tau_i^2\right)$$
$$- D(\lambda)t^2 + E(\lambda)\tau_i t + C(\lambda)\tau_i^2\} + 2\lambda\left(\lambda^3+2\right)\tau_i^2\mathrm{e}^{t/\tau_i} + 2\lambda\left(\lambda^3-1\right)t^2 - 2\lambda\left(\lambda^3+2\right)\tau_i t - 2\lambda\left(\lambda^3+2\right)\tau_i^2\big]\phi_0^2$$
$$- \frac{\pi R_0^6}{18\lambda t^{\star 4}}B(\lambda,\gamma)\sum_{i=1}^{n}\mu_i\tau_i^2\mathrm{e}^{-t/\tau_i}\big[\mathrm{e}^{t/\tau_i}\left(t^2-6\tau_i t+12\tau_i^2\right) - t^2 - 6\tau_i t - 12\tau_i^2\big]\phi_0^4. \tag{S10}$$