Supplemental Material to "Long-range order in two-dimensional systems with fluctuating active stresses"

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A. DISPLACEMENT FLUCTUATIONS IN A TWO-DIMENSIONAL CONTINUOUS MEDIUM

We consider the following equation of motion

$$\zeta \dot{\boldsymbol{u}}(\boldsymbol{r},t) = -\int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{r}' \, \underline{\boldsymbol{D}}^{\mathrm{el}}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{u}(\boldsymbol{r}',t) + \boldsymbol{\lambda}(\boldsymbol{r},t) \tag{A1}$$

where $\boldsymbol{u}(\boldsymbol{r},t)$ describes the elastic deformation from position $\boldsymbol{r}$, ζ is a friction coefficient, $\underline{\boldsymbol{D}}^{\mathrm{el}}$ is a dynamical matrix [1] describing elasticity, and $\boldsymbol{\lambda}$ a stochastic term.

Using the Fourier transform in space and time of the displacement field (2)

$$\boldsymbol{u}(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{q}\, e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\, \tilde{\boldsymbol{u}}(\boldsymbol{q},t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{q} \int_{\mathbb{R}} \mathrm{d}\omega\, e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}+\mathrm{i}\omega t}\, \tilde{\boldsymbol{U}}(\boldsymbol{q},\omega), \tag{A2}$$

we write (A1) in Fourier space

$$\mathrm{i}\omega\zeta\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega) = -\tilde{\underline{\boldsymbol{D}}}^{\mathrm{el}}(\boldsymbol{q})\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega) + \tilde{\boldsymbol{\Lambda}}(\boldsymbol{q},\omega). \tag{A3}$$

We assume that the Fourier transform of the dynamical matrix $\tilde{\underline{\boldsymbol{D}}}^{\mathrm{el}}(\boldsymbol{q})$ can be decomposed as follows [2]

$$\tilde{\underline{\boldsymbol{D}}}^{\mathrm{el}}(\boldsymbol{q})\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega) = |\boldsymbol{q}|^2\left[(B+\mu)\tilde{\boldsymbol{U}}_{\parallel}(\boldsymbol{q},\omega) + \mu\tilde{\boldsymbol{U}}_{\perp}(\boldsymbol{q},\omega)\right], \tag{A4}$$

where $\hat{\boldsymbol{q}} = \boldsymbol{q}/|\boldsymbol{q}|$, $\tilde{\boldsymbol{U}}_{\parallel}(\boldsymbol{q},\omega) = (\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)\cdot\hat{\boldsymbol{q}})\,\hat{\boldsymbol{q}}$ and $\tilde{\boldsymbol{U}}_{\perp}(\boldsymbol{q},\omega) = \tilde{\boldsymbol{U}}(\boldsymbol{q},t) - \tilde{\boldsymbol{U}}_{\parallel}(\boldsymbol{q},\omega)$ are respectively the longitudinal and transverse displacements in Fourier space, and B and μ are respectively the bulk and shear moduli. We can thus now write

$$\left\langle \tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)\cdot\tilde{\boldsymbol{U}}(\boldsymbol{q}',\omega')^*\right\rangle = \frac{\left\langle \tilde{\boldsymbol{\Lambda}}_{\parallel}(\boldsymbol{q},\omega)\cdot\tilde{\boldsymbol{\Lambda}}_{\parallel}(\boldsymbol{q}',\omega')^*\right\rangle}{[\mathrm{i}\omega\zeta+|\boldsymbol{q}|^2(B+\mu)][-\mathrm{i}\omega'\zeta+|\boldsymbol{q}'|^2(B+\mu)]} + \frac{\left\langle \tilde{\boldsymbol{\Lambda}}_{\perp}(\boldsymbol{q},\omega)\cdot\tilde{\boldsymbol{\Lambda}}_{\perp}(\boldsymbol{q}',\omega')^*\right\rangle}{[\mathrm{i}\omega\zeta+|\boldsymbol{q}|^2\mu][-\mathrm{i}\omega'\zeta+|\boldsymbol{q}'|^2\mu]}, \tag{A5}$$

where $\tilde{\boldsymbol{\Lambda}}_{\parallel} = (\tilde{\boldsymbol{\Lambda}}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}}$ and $\tilde{\boldsymbol{\Lambda}}_{\perp} = \tilde{\boldsymbol{\Lambda}} - \tilde{\boldsymbol{\Lambda}}_{\parallel}$ are respectively the longitudinal and transverse stochastic terms in Fourier space. We assume that the stochastic term is either the divergence of a tensor field uncorrelated in space, or is a vector field uncorrelated in space, with correlations respectively

$$\langle\boldsymbol{\lambda}(\boldsymbol{r},t)\cdot\boldsymbol{\lambda}(\boldsymbol{r}',t')\rangle = -\sigma^2 a^2\, e^{-|t-t'|/\tau}\,\nabla^2\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{A6a}$$

$$\langle\boldsymbol{\lambda}(\boldsymbol{r},t)\cdot\boldsymbol{\lambda}(\boldsymbol{r}',t')\rangle = f^2 a^2\, e^{-|t-t'|/\tau}\,\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{A6b}$$

where σ is an energy scale, f a force scale, a a coarse-graining length scale, and τ a persistence time. Using the following identities

$$\int_{\mathbb{R}} \mathrm{d}t\, e^{-\mathrm{i}(\omega-\omega')t} = 2\pi\,\delta(\omega-\omega') \tag{A7a}$$

$$\int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}t'\, e^{-\mathrm{i}(\omega t-\omega' t')} e^{-|t-t'|/\tau} \underset{t'=t+s}{=} \int_{\mathbb{R}} \mathrm{d}t\, e^{-\mathrm{i}(\omega-\omega')t}\int_{\mathbb{R}} \mathrm{d}s\, e^{\mathrm{i}\omega' s}e^{-|s|/\tau} = 2\pi\frac{2\tau}{1+\omega^2\tau^2}\,\delta(\omega-\omega'), \tag{A7b}$$

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we write these correlations in Fourier space

$$\left\langle \tilde{\mathbf{\Lambda}}(\boldsymbol{q},\omega)\cdot\tilde{\mathbf{\Lambda}}(\boldsymbol{q}',\omega')^*\right\rangle = (2\pi)^3 \frac{2\sigma^2\tau|\boldsymbol{q}|^2}{1+\omega^2\tau^2}\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}'), \tag{A8a}$$

$$\left\langle \tilde{\mathbf{\Lambda}}(\boldsymbol{q},\omega)\cdot\tilde{\mathbf{\Lambda}}(\boldsymbol{q}',\omega')^*\right\rangle = (2\pi)^3 \frac{2f^2\tau}{1+\omega^2\tau^2}\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}'), \tag{A8b}$$

where we have taken $\boldsymbol{q}=\boldsymbol{q}'$ and $\omega=\omega'$, outside of the Dirac delta functions $\delta(\boldsymbol{q}-\boldsymbol{q}')$ and $\delta(\omega-\omega')$, for simplicity. We further assume by isotropy

$$\left\langle \tilde{\mathbf{\Lambda}}_\parallel(\boldsymbol{q},\omega)\cdot\tilde{\mathbf{\Lambda}}_\parallel(\boldsymbol{q}',\omega')^*\right\rangle = \left\langle \tilde{\mathbf{\Lambda}}_\perp(\boldsymbol{q},\omega)\cdot\tilde{\mathbf{\Lambda}}_\perp(\boldsymbol{q}',\omega')^*\right\rangle = \frac{1}{2}\left\langle \tilde{\mathbf{\Lambda}}(\boldsymbol{q},\omega)\cdot\tilde{\mathbf{\Lambda}}(\boldsymbol{q}',\omega')^*\right\rangle. \tag{A9}$$

We thus obtain the following space and time Fourier fluctuations

$$\left\langle \tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)\cdot\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)^*\right\rangle = (2\pi)^3\frac{\sigma^2\tau|\boldsymbol{q}|^2\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{[\omega^2\zeta^2+|\boldsymbol{q}|^4(B+\mu)^2][1+\omega^2\tau^2]} + (2\pi)^3\frac{\sigma^2\tau|\boldsymbol{q}|^2\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{[\omega^2\zeta^2+|\boldsymbol{q}|^4\mu^2][1+\omega^2\tau^2]}, \tag{A10a}$$

$$\left\langle \tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)\cdot\tilde{\boldsymbol{U}}(\boldsymbol{q},\omega)^*\right\rangle = (2\pi)^3\frac{f^2\tau\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{[\omega^2\zeta^2+|\boldsymbol{q}|^4(B+\mu)^2][1+\omega^2\tau^2]} + (2\pi)^3\frac{f^2\tau\,\delta(\omega-\omega')\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{[\omega^2\zeta^2+|\boldsymbol{q}|^4\mu^2][1+\omega^2\tau^2]}. \tag{A10b}$$

Using the following identities

$$\int_{\mathbb{R}} \mathrm{d}\omega \frac{1}{a^2+\omega^2b^2} = \frac{\pi}{ab}, \tag{A11a}$$

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}\omega \int_{\mathbb{R}} \mathrm{d}\omega'\, e^{\mathrm{i}(\omega-\omega')t}\,\frac{\delta(\omega-\omega')}{(a^2+\omega^2b^2)(c^2+\omega^2d^2)} &= \int_{\mathbb{R}} \mathrm{d}\omega\,\frac{1}{b^2c^2-a^2d^2}\left[\frac{b^2}{a^2+\omega^2b^2}-\frac{d^2}{c^2+\omega^2d^2}\right]\\ &= \frac{1}{b^2c^2-a^2d^2}\left[\frac{\pi b}{a}-\frac{\pi d}{c}\right]\\ &= \frac{\pi}{ac[bc+ad]},\end{aligned} \tag{A11b}$$

we obtain the equal-time Fourier fluctuations

$$\langle\tilde{\boldsymbol{u}}(\boldsymbol{q},t)\cdot\tilde{\boldsymbol{u}}(\boldsymbol{q},t)^*\rangle = 2\pi\frac{\pi\sigma^2\tau|\boldsymbol{q}|^2\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{|\boldsymbol{q}|^2(B+\mu)[\zeta+|\boldsymbol{q}|^2(B+\mu)\tau]} + 2\pi\frac{\pi\sigma^2\tau|\boldsymbol{q}|^2\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{|\boldsymbol{q}|^2\mu[\zeta+|\boldsymbol{q}|^2\mu\tau]}, \tag{A12a}$$

$$\langle\tilde{\boldsymbol{u}}(\boldsymbol{q},t)\cdot\tilde{\boldsymbol{u}}(\boldsymbol{q},t)^*\rangle = 2\pi\frac{\pi f^2\tau\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{|\boldsymbol{q}|^2(B+\mu)[\zeta+|\boldsymbol{q}|^2(B+\mu)\tau]} + 2\pi\frac{\pi f^2\tau\,a^2\delta(\boldsymbol{q}-\boldsymbol{q}')}{|\boldsymbol{q}|^2\mu[\zeta+|\boldsymbol{q}|^2\mu\tau]}, \tag{A12b}$$

which correspond to (8) given the longitudinal and transversal correlation length scales $\xi_\parallel = \sqrt{(B+\mu)\tau/\zeta}$ and $\xi_\perp = \sqrt{\mu\tau/\zeta}$.

B. DISPLACEMENT FLUCTUATIONS IN A TWO-DIMENSIONAL TRIANGULAR LATTICE

We consider a periodic triangular lattice in two dimensions, see Fig. B1, where $\boldsymbol{e}_\alpha = a(\cos((\alpha-1)\pi/3),\ \sin((\alpha-1)\pi/3))$. We denote N the total number of sites – of the order of $\sqrt{N}$ in each dimension –, $\boldsymbol{r}_i^0$ the reference lattice position of particle $i\in\{1,\ldots,N\}$, and $\boldsymbol{u}_i = \boldsymbol{r}_i - \boldsymbol{r}_i^0$ its displacement from this lattice position. We define in this lattice the discrete divergence operator [3] which to a quantity $\boldsymbol{v}_{i\to\alpha}$ defined on links $i\to\alpha$ associates

$$\nabla\cdot\underline{\boldsymbol{v}}_i = \sum_{\alpha=1}^{3}[\boldsymbol{v}_{i\to\alpha} - \boldsymbol{v}_{(i-\alpha)\to\alpha}] \tag{B13}$$

defined on sites i, and the Laplacian operator ∇^2 which to a quantity $\boldsymbol{w}_i$ defined on sites i associates

$$\nabla^2\boldsymbol{w}_i = \sum_{\alpha=1}^{3}[\boldsymbol{w}_{i+\alpha} + \boldsymbol{w}_{i-\alpha} - 2\boldsymbol{w}_i] \tag{B14}$$

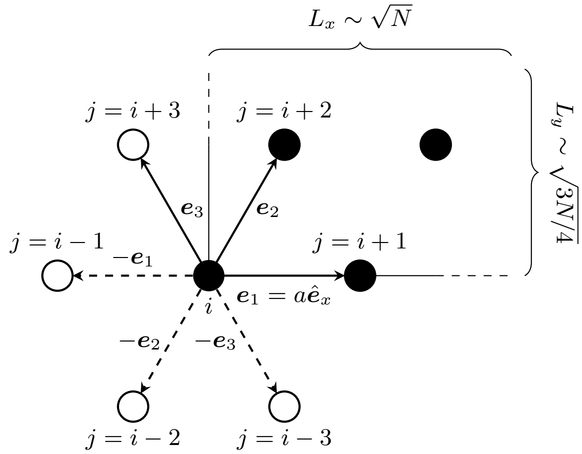

FIG. B1. Notations within the triangular lattice.

also defined on sites i.

We introduce the following overdamped equation of motion

$$\zeta \dot{\boldsymbol{u}}_i(t) - \gamma \nabla^2 \dot{\boldsymbol{u}}_i = k \nabla^2 \boldsymbol{u}_i + \boldsymbol{\lambda}_i(t) \tag{B15}$$

where ζ is a friction coefficient, γ a pair friction coefficient, k a spring constant, and $\boldsymbol{\lambda}_i(t)$ is the stochastic active term.

We define the discrete space Fourier transform

$$\tilde{\boldsymbol{v}}_{\boldsymbol{q}} = \sum_{i=1}^{N} e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_i^0}\, \boldsymbol{v}_i, \tag{B16a}$$

$$\boldsymbol{v}_i = \frac{1}{N} \sum_{\substack{m,n=0 \\ \boldsymbol{q}=\frac{2\pi}{\sqrt{N}a}(m,n)}}^{N-1} e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_i^0}\, \tilde{\boldsymbol{v}}_{\boldsymbol{q}}, \tag{B16b}$$

with the following orthogonality relation

$$\sum_{i=1}^{N} e^{\mathrm{i}(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{r}_i^0} = N\, \delta_{\boldsymbol{q},\boldsymbol{q}'}. \tag{B17}$$

We write (B15) in Fourier space

$$\mathrm{i}\omega\zeta \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) + \mathrm{i}\omega Q(\boldsymbol{q})^2 \gamma \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) = -Q(\boldsymbol{q})^2 k \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) + \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}}(\omega) \tag{B18}$$

where we have introduced the kernel

$$Q(\boldsymbol{q})^2 = 2 \sum_{\alpha=1}^{3} (1 - \cos(\boldsymbol{q}\cdot\boldsymbol{e}_\alpha)) \underset{|\boldsymbol{q}|\to 0}{=} \frac{3}{2} |\boldsymbol{q}|^2 a^2. \tag{B19}$$

We thus obtain the following space and time Fourier fluctuations

$$\left\langle \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{U}}_{\boldsymbol{q}'}(\omega')^* \right\rangle = \frac{\left\langle \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}'}(\omega') \right\rangle}{[i\omega(\zeta + Q(\boldsymbol{q})^2\gamma) + Q(\boldsymbol{q})^2 k][-i\omega'(\zeta + Q(\boldsymbol{q}')^2\gamma) + Q(\boldsymbol{q}')^2 k]}. \tag{B20}$$

B.1. Active stresses

We first provide the derivation for a fluctuating stress with finite-time correlations, for simplicity in the absence of pair friction. We have

$$\boldsymbol{\lambda}_i(t) = \nabla \cdot \underline{\boldsymbol{\sigma}}_i, \tag{B21a}$$
$$\langle \boldsymbol{\sigma}_{i\to\alpha}(t) \cdot \boldsymbol{\sigma}_{j\to\beta}(t') \rangle = \sigma^2 a^{-2} e^{-|t-t'|/\tau}\, \delta_{ij}\delta_{\alpha\beta}\delta(t-t'), \tag{B21b}$$

which read in Fourier space

$$\left\langle \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}'}(\omega')^* \right\rangle = 2\pi \frac{2\sigma^2 a^{-2}\tau Q(\boldsymbol{q})^2}{1+\omega^2\tau^2} N\delta_{\boldsymbol{q},\boldsymbol{q}'}\, \delta(\omega-\omega'). \tag{B22}$$

such that the space and time Fourier fluctuations are

$$\left\langle \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{U}}_{\boldsymbol{q}'}(\omega')^* \right\rangle = 2\pi \frac{2\sigma^2 a^{-2}\tau Q(\boldsymbol{q})^2}{[\omega^2\zeta^2 + Q(\boldsymbol{q})^4 k^2][1+\omega^2\tau^2]} N\delta_{\boldsymbol{q},\boldsymbol{q}'}\, \delta(\omega-\omega'). \tag{B23}$$

We use (A11) and write the equal-time Fourier fluctuations

$$\langle \tilde{\boldsymbol{u}}_{\boldsymbol{q}}(t) \cdot \tilde{\boldsymbol{u}}_{\boldsymbol{q}'}(t)^* \rangle = \frac{\sigma^2 a^{-2}\tau}{k(\zeta + Q(\boldsymbol{q})^2 k\tau)} N\delta_{\boldsymbol{q},\boldsymbol{q}'} = \frac{\sigma^2\tau^2}{\zeta^2} \frac{\xi^{-2}}{1+Q(\boldsymbol{q})^2(\xi/a)^2} N\delta_{\boldsymbol{q},\boldsymbol{q}'}, \tag{B24}$$

where we have introduced the correlation length $\xi = a\sqrt{k\tau/\zeta}$. We obtain the same large-wavelength scaling as (B28): the finite-time correlation of the stress only affects the small-wavelength fluctuations.

B.2. Fluctuation-dissipation and the thermal limit

It is important to compare and contrast our results with those expected in thermal equilibrium. Given the dissipation term on the l.h.s. of (B15), the second fluctuation-dissipation theorem [4–6] dictates that the stochastic term $\boldsymbol{\lambda}_i(t)$ should have the following decomposition and correlations

$$\boldsymbol{\lambda}_i(t) = \boldsymbol{\eta}_i(t) + \nabla \cdot \underline{\boldsymbol{\sigma}}_i, \tag{B25a}$$
$$\langle \boldsymbol{\eta}_i(t) \cdot \boldsymbol{\eta}_j(t') \rangle = 2k_{\mathrm{B}}T_1\zeta\, \delta_{ij}\, \delta(t-t'), \tag{B25b}$$
$$\langle \boldsymbol{\sigma}_{i\to\alpha}(t) \cdot \boldsymbol{\sigma}_{j\to\beta}(t') \rangle = 2k_{\mathrm{B}}T_2\gamma\, \delta_{ij}\delta_{\alpha\beta}\, \delta(t-t'), \tag{B25c}$$
$$\langle \boldsymbol{\eta}_j(t) \cdot \boldsymbol{\sigma}_{j\to\beta}(t') \rangle = 0. \tag{B25d}$$

with $T = T_1 = T_2$ the bath temperature.

The fluctuations of the driving noise deriving from (B25) are

$$\left\langle \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{\Lambda}}_{\boldsymbol{q}'}(\omega')^* \right\rangle = 4\pi k_{\mathrm{B}}T_1\zeta N\delta_{\boldsymbol{q},\boldsymbol{q}'}\delta(\omega-\omega') + 4\pi k_{\mathrm{B}}T_2\gamma Q(\boldsymbol{q})^2 N\delta_{\boldsymbol{q},\boldsymbol{q}'}\delta(\omega-\omega'). \tag{B26}$$

We refine our expression of the displacement fluctuations

$$\left\langle \tilde{\boldsymbol{U}}_{\boldsymbol{q}}(\omega) \cdot \tilde{\boldsymbol{U}}_{\boldsymbol{q}'}(\omega')^* \right\rangle = \frac{4\pi k_{\mathrm{B}}T_1\zeta N\delta_{\boldsymbol{q},\boldsymbol{q}'}\delta(\omega-\omega')}{\omega^2(\zeta+Q(\boldsymbol{q})^2\gamma)^2 + Q(\boldsymbol{q})^4k^2} + \frac{4\pi k_{\mathrm{B}}T_2\gamma Q(\boldsymbol{q})^2 N\delta_{\boldsymbol{q},\boldsymbol{q}'}\delta(\omega-\omega')}{\omega^2(\zeta+Q(\boldsymbol{q})^2\gamma)^2 + Q(\boldsymbol{q})^4k^2}, \tag{B27}$$

and take the inverse time Fourier transform using (A11) to compute the equal-time Fourier fluctuations

$$\langle \tilde{\boldsymbol{u}}_{\boldsymbol{q}}(t) \cdot \tilde{\boldsymbol{u}}_{\boldsymbol{q}'}(t)^* \rangle = \frac{k_{\mathrm{B}}T_1\zeta}{(\zeta+Q(\boldsymbol{q})^2\gamma)Q(\boldsymbol{q})^2 k} N\delta_{\boldsymbol{q},\boldsymbol{q}'} + \frac{k_{\mathrm{B}}T_2\gamma}{(\zeta+Q(\boldsymbol{q})^2\gamma)k} N\delta_{\boldsymbol{q},\boldsymbol{q}'}. \tag{B28}$$

We stress here that $\langle |\tilde{\boldsymbol{u}}_{\boldsymbol{q}}(t)|^2 \rangle$ remains finite for $|\boldsymbol{q}| \to 0$ if and only if $T_1 = 0$ and $\zeta \neq 0$. Therefore, it is the competition between a fluctuating stress ($T_2 \neq 0$) and a particle-wise drag force ($\zeta \neq 0$) in the absence of a particle-wise fluctuating force ($T_1 = 0$) which damps large-wavelength fluctuations. Moreover, the effect of an additional viscosity-like term only affects the small-wavelength behaviour.

C. LARGE-DISTANCE SCALING OF THE TRANSLATIONAL ORDER CORRELATION FUNCTION

We consider the local translational order parameter (12)

$$\psi_{\boldsymbol{q}_0,i} = e^{\mathrm{i}\boldsymbol{q}_0\cdot(\boldsymbol{r}_i-\boldsymbol{r}_0)} \tag{C29}$$

where $\boldsymbol{q}_0$ is a reciprocal vector of the lattice, $\boldsymbol{r}_i = \boldsymbol{r}_i^0 + \boldsymbol{u}_i$ the position of particle i which is displaced of $\boldsymbol{u}_i$ from its lattice position $\boldsymbol{r}_i^0$. We write the fluctuations

$$\left\langle \psi_{\boldsymbol{q}_0,i}\psi^*_{\boldsymbol{q}_0,j}\right\rangle = \left\langle e^{\mathrm{i}\boldsymbol{q}_0(\boldsymbol{r}_i-\boldsymbol{r}_j)}\right\rangle = \left\langle e^{\mathrm{i}\boldsymbol{q}_0\cdot(\boldsymbol{r}_i^0-\boldsymbol{r}_j^0)}e^{\mathrm{i}\boldsymbol{q}_0\cdot(\boldsymbol{u}_i-\boldsymbol{u}_j)}\right\rangle. \tag{C30}$$

We note that for two lattice points $\boldsymbol{r}_i^0$ and $\boldsymbol{r}_j^0$ then $\boldsymbol{q}_0\cdot(\boldsymbol{r}_i^0-\boldsymbol{r}_j^0)$ is 0 modulo 2π therefore $e^{\mathrm{i}\boldsymbol{q}_0\cdot(\boldsymbol{r}_i^0-\boldsymbol{r}_j^0)} = 1$. We also define for a random variable X its characteristic function $t\mapsto\langle\exp(\mathrm{i}tX)\rangle$. For a normally distributed random variable X with zero mean $\langle X\rangle = 0$ then

$$\langle\exp(\mathrm{i}tX)\rangle = \exp\left(-\frac{1}{2}t^2\left\langle X^2\right\rangle\right). \tag{C31}$$

We assume for two distant particles i and j that $\boldsymbol{q}_0\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)$ is normally distributed – from the linearity of (B15) and the Gaussian nature of driving noises – and assume from isotropy

$$\left\langle(\boldsymbol{q}_0\cdot(\boldsymbol{u}_i-\boldsymbol{u}_j))^2\right\rangle = \frac{1}{2}|\boldsymbol{q}_0|^2\left\langle|\boldsymbol{u}_i-\boldsymbol{u}_j|^2\right\rangle = |\boldsymbol{q}_0|^2\left\langle|\boldsymbol{u}_i|^2\right\rangle - |\boldsymbol{q}_0|^2\left\langle\boldsymbol{u}_i\cdot\boldsymbol{u}_j\right\rangle \tag{C32}$$

where we have used $\left\langle|\boldsymbol{u}_i|^2\right\rangle = \left\langle|\boldsymbol{u}_j|^2\right\rangle$ for all i and j in steady state.

As a matter of simplification we use the continuous-space spectrum of displacements (8) to compute the correlations $\left\langle|\boldsymbol{u}_i|^2\right\rangle$ and $\langle\boldsymbol{u}_i\cdot\boldsymbol{u}_j\rangle$. We then write

$$\left\langle|\boldsymbol{u}_i|^2\right\rangle = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}'\ \langle\tilde{\boldsymbol{u}}(\boldsymbol{q},t)\cdot\tilde{\boldsymbol{u}}(\boldsymbol{q}',t)^*\rangle, \tag{C33a}$$

$$\langle\boldsymbol{u}_i\cdot\boldsymbol{u}_j\rangle = \langle\boldsymbol{u}_j\cdot\boldsymbol{u}_i\rangle = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}'\ \cos\left(\boldsymbol{q}\cdot\boldsymbol{r}_i^0-\boldsymbol{q}'\cdot\boldsymbol{r}_j^0\right)\langle\tilde{\boldsymbol{u}}(\boldsymbol{q},t)\cdot\tilde{\boldsymbol{u}}(\boldsymbol{q}',t)^*\rangle. \tag{C33b}$$

such that using (C32)

$$\left\langle(\boldsymbol{q}_0\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i))^2\right\rangle = \frac{|\boldsymbol{q}_0|^2}{(2\pi)^4}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}'\ \left[1-\cos\left(\boldsymbol{q}\cdot\boldsymbol{r}_i^0-\boldsymbol{q}\cdot\boldsymbol{r}_j^0\right)\right]\langle\tilde{\boldsymbol{u}}(\boldsymbol{q},t)\cdot\tilde{\boldsymbol{u}}(\boldsymbol{q}',t)^*\rangle. \tag{C34}$$

We note that (8) are symmetric by rotation of $\boldsymbol{q}$. With

$$\boldsymbol{\Delta} = \boldsymbol{r}_i^0-\boldsymbol{r}_j^0,\ r = |\boldsymbol{\Delta}|, \tag{C35}$$

we define for all wave vector $\boldsymbol{q}$ the angle ϕ such that $\boldsymbol{q}\cdot\boldsymbol{\Delta} = q\,r\,\cos\phi$ where $q = |\boldsymbol{q}|$. Therefore for any function $f(|\boldsymbol{q}|)$ we have the following identity

$$\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}\cos\left(\boldsymbol{q}\cdot\boldsymbol{\Delta}\right)\ f(|\boldsymbol{q}|) = \int_{\mathbb{R}}\mathrm{d}q\int_0^{2\pi}\mathrm{d}\phi\, q\cos\left(q\,r\,\cos\phi\right)\ f(q) = \int_{\mathbb{R}}\mathrm{d}q\,2\pi q\,J_0(q\,r)\,f(q), \tag{C36a}$$

$$\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{q}\,f(|\boldsymbol{q}|) = \int_{\mathbb{R}}\mathrm{d}q\,2\pi q\,f(q), \tag{C36b}$$

where J_0 is the 0-th Bessel function of the first kind.

C.1. Fluctuating stress

Using (8a), (C33), and (C36), we write

$$\left\langle|\boldsymbol{u}_i|^2\right\rangle = \frac{\sigma^2\tau^2a^2}{4\pi\zeta^2}\left[\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q}{\xi_\parallel^2(1+q^2\xi_\parallel^2)} + \int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q}{\xi_\perp^2(1+q^2\xi_\perp^2)}\right], \tag{C37a}$$

$$\langle\boldsymbol{u}_i\cdot\boldsymbol{u}_j\rangle = \frac{\sigma^2\tau^2a^2}{4\pi\zeta^2}\left[\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q\,J_0(qr)}{\xi_\parallel^2(1+q^2\xi_\parallel^2)} + \int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q\,J_0(qr)}{\xi_\perp^2(1+q^2\xi_\perp^2)}\right], \tag{C37b}$$

where the upper cutoff in the integral translates the fact that the continuous description applies to scales greater than the coarse-graining scale a, and the lower cutoff corresponds to the system size L. At fixed r we first note that

$$J_0(qr) \underset{q\to 0}{=} 1 - \frac{1}{\Gamma(2)}\left(\frac{qr}{2}\right)^2 \tag{C38}$$

with Γ the Euler gamma function. It follows that all integrands in (C37) converge to finite values for $q \to 0$ therefore all integrals converge for $L \to \infty$ and we thus take this limit for the derivation below. To compute (C37a) we use the following identity

$$\int_0^{2\pi/a} \mathrm{d}q\, \frac{q}{1+q^2\xi^2} = \frac{1}{2\xi^2}\log\left(1 + (2\pi)^2\frac{\xi^2}{a^2}\right) \tag{C39}$$

and write

$$\left\langle |\boldsymbol{u}_i|^2 \right\rangle \underset{L\to\infty}{=} \frac{\sigma^2\tau^2a^2}{8\pi\zeta^2}\left[\frac{1}{\xi_\parallel^4}\log\left(1+(2\pi)^2(\xi_\parallel/a)^2\right) + \frac{1}{\xi_\perp^4}\log\left(1+(2\pi)^2(\xi_\perp/a)^2\right)\right] \equiv \left\langle u^2\right\rangle_\infty^\sigma. \tag{C40}$$

To compute (C37b) we use the following inequality

$$|J_0(qr)| \leq \frac{1}{\sqrt{qr}} \tag{C41}$$

to bound the following integral

$$\left|\int_0^{2\pi/a} \mathrm{d}q\, \frac{q\, J_0(qr)}{1+q^2\xi^2}\right| \leq \frac{1}{\sqrt{r}}\int_0^{2\pi/a}\mathrm{d}q\,\frac{\sqrt{q}}{1+q^2\xi^2} \xrightarrow[r\to\infty]{} 0 \tag{C42}$$

where the latter limit derives from the fact that the integral is finite, which implies

$$\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j\rangle \xrightarrow[\substack{L\to\infty\\ r\to\infty}]{} 0. \tag{C43}$$

Therefore, using (C30), (C31), (C32), (C40), and (C43), we obtain

$$\left\langle \psi_{\boldsymbol{q}_0,i}\psi^*_{\boldsymbol{q}_0,j}\right\rangle \underset{\substack{L\to\infty\\ |\boldsymbol{r}_i^0-\boldsymbol{r}_j^0|\to\infty}}{=} \exp\left(-\frac{1}{2}|\boldsymbol{q}_0|^2\left\langle u^2\right\rangle_\infty^\sigma\right). \tag{C44}$$

C.2. Fluctuating force

Using (8b), (C33), (C34), and (C36), we write

$$\left\langle |\boldsymbol{u}_i|^2\right\rangle = \frac{f^2\tau^2a^2}{4\pi\zeta^2}\left[\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q}{q^2\xi_\parallel^2(1+q^2\xi_\parallel^2)} + \int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q}{q^2\xi_\perp^2(1+q^2\xi_\perp^2)}\right], \tag{C45a}$$

$$\left\langle (\boldsymbol{q}_0\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i))^2\right\rangle = \frac{f^2\tau^2a^2|\boldsymbol{q}_0|^2}{4\pi\zeta^2}\left[\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q(1-J_0(qr))}{q^2\xi_\parallel^2(1+q^2\xi_\parallel^2)} + \int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q(1-J_0(qr))}{q^2\xi_\perp^2(1+q^2\xi_\perp^2)}\right], \tag{C45b}$$

where the upper cutoff in the integral translates the fact that the continuous description applies to scales greater than the coarse-graining scale a, and the lower cutoff corresponds to the system size L. To compute (C45a) we note that $q[q^2(1+q^2\xi^2)]^{-1} \sim q^{-1}$ for $q \to 0$ and $\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\, q^{-1}$ diverges for $L \to \infty$, therefore we can use the equivalence of the integrals

$$\int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{q}{q^2(1+q^2\xi^2)} \underset{L\to\infty}{=} \int_{2\pi/L}^{2\pi/a}\mathrm{d}q\,\frac{1}{q} = \log(L/a), \tag{C46}$$

and write

$$\left\langle |\boldsymbol{u}_i|^2 \right\rangle \underset{L\to\infty}{=} \frac{f^2\tau^2a^2}{4\pi\zeta^2}\left[\frac{1}{\xi_\parallel^2}+\frac{1}{\xi_\perp^2}\right]\log(L/a) \equiv C^f \log(L/a). \tag{C47}$$

It follows from (C38) that both integrands in (C45b) converge in the limit $q\to 0$ therefore both integrals converge for $L\to\infty$ and we thus take this limit for the derivation below. We focus on the following integral

$$\int_0^{2\pi/a} \mathrm{d}q\, \frac{q(1-J_0(qr))}{q^2(1+q^2\xi^2)} = \int_0^{2\pi/r} \mathrm{d}q\, \frac{q(1-J_0(qr))}{q^2(1+q^2\xi^2)} + \int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{q}{q^2(1+q^2\xi^2)} - \int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{q\,J_0(qr)}{q^2(1+q^2\xi^2)}. \tag{C48}$$

This decomposition is such that the first term on the r.h.s. cancels in the limit $r\to\infty$. For the second term we note once again that $q[q^2(1+q^2\xi^2)]^{-1}\sim q^{-1}$ for $q\to 0$ and $\int_{2\pi/r}^{2\pi/a}\mathrm{d}q\,q^{-1}$ diverges for $r\to\infty$, therefore we can use the equivalence on the integrals

$$\int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{q}{q^2(1+q^2\xi^2)} \underset{r\to\infty}{=} \int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{1}{q} = \log(r/a). \tag{C49}$$

Finally for the third term, using (C41) we bound the integral in the following way

$$\left|\int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{qJ_0(qr)}{q^2(1+q^2\xi^2)}\right| \leq \frac{1}{\sqrt{r}}\int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{1}{q^{3/2}(1+q^2\xi^2)}, \tag{C50}$$

where we note once again that $[q^{3/2}(1+q^2\xi^2)]^{-1}\sim q^{-3/2}$ for $q\to 0$ and $\int_{2\pi/r}^{2\pi/a}\mathrm{d}q\,q^{-3/2}$ diverges for $r\to\infty$, therefore we can use the equivalence on the integrals

$$\frac{1}{\sqrt{r}}\int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{1}{q^{3/2}(1+q^2\xi^2)} \underset{r\to\infty}{=} \frac{1}{\sqrt{r}}\int_{2\pi/r}^{2\pi/a} \mathrm{d}q\, \frac{1}{q^{3/2}} = \sqrt{\frac{2}{\pi}}\left[1-\sqrt{\frac{a}{r}}\right], \tag{C51}$$

which is finite in the $r\to\infty$ limit. Therefore only the second term in the r.h.s. of (C48) diverges, and we have the following equivalence

$$\int_0^{2\pi/a} \mathrm{d}q\, \frac{q(1-J_0(qr))}{q^2(1+q^2\xi^2)} \underset{r\to\infty}{=} \log(r/a). \tag{C52}$$

which eventually leads to

$$\left\langle (\boldsymbol{q}_0\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i))^2 \right\rangle \underset{\substack{L\to\infty\\ |\boldsymbol{r}_i^0-\boldsymbol{r}_j^0|\to\infty}}{=} \frac{f^2\tau^2a^2|\boldsymbol{q}_0|^2}{4\pi\zeta^2}\left[\frac{1}{\xi_\parallel^2}+\frac{1}{\xi_\perp^2}\right]\log(r/a) = |\boldsymbol{q}_0|^2\,C^f\,\log(r/a). \tag{C53}$$

Therefore, using (C30), (C31), (C47), and (C53), we obtain

$$\left\langle \psi_{\boldsymbol{q}_0,i}\psi^*_{\boldsymbol{q}_0,j} \right\rangle \underset{\substack{L\to\infty\\ |\boldsymbol{r}_i^0-\boldsymbol{r}_j^0|\to\infty}}{=} (r/a)^{-\frac{1}{2}|\boldsymbol{q}_0|^2 C^f}. \tag{C54}$$

D. LARGE-WAVELENGTH SCALING OF THE STRUCTURE FACTOR

We consider the structure factor (16)

$$S(\boldsymbol{q}) = \frac{1}{N}\sum_{i,j=1}^N \left\langle e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)} \right\rangle = \frac{1}{N}\sum_{i,j=1}^N e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j^0-\boldsymbol{r}_i^0)}\left\langle e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i)} \right\rangle. \tag{D55}$$

For normally distributed driving noise then, due to the linearity of (B15), displacements are also normally distributed. Using (C31) we write

$$\langle \exp\left(\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i)\right)\rangle = \exp\left(-\frac{1}{2}\left\langle (\boldsymbol{q}\cdot(\boldsymbol{u}_j-\boldsymbol{u}_i))^2\right\rangle\right) = \exp\left(-\frac{1}{4}|\boldsymbol{q}|^2\left\langle |\boldsymbol{u}_j-\boldsymbol{u}_i|^2\right\rangle\right) \tag{D56}$$

where the second equality derives from isotropy. It is possible to Taylor expand the latter exponential if

$$|\boldsymbol{q}|^2 \left\langle |\boldsymbol{u}_j - \boldsymbol{u}_i|^2 \right\rangle \ll 1. \tag{D57}$$

In the limit $L \to \infty$ we consider wavevectors which scale as

$$|\boldsymbol{q}| \sim \frac{2\pi}{L}. \tag{D58}$$

Therefore, considering $\left\langle |\boldsymbol{u}_i - \boldsymbol{u}_j|^2 \right\rangle \leq 2 \left\langle |\boldsymbol{u}_i|^2 \right\rangle$ in steady state, and the following scalings

$$\left\langle |\boldsymbol{u}_i|^2 \right\rangle \underset{L\to\infty}{=} \left\langle u^2 \right\rangle_\infty^\sigma, \tag{D59a}$$

$$\left\langle |\boldsymbol{u}_i|^2 \right\rangle \underset{L\to\infty}{=} C^f \log(L/a), \tag{D59b}$$

for fluctuating stress (C40) and fluctuating force (C47) respectively, we have that condition (D57) is satisfied for large enough $L \gg a$. We then write

$$\begin{aligned} S(\boldsymbol{q}) &\underset{\substack{L\to\infty \\ |\boldsymbol{q}|\sim 2\pi/L}}{=} \frac{1}{N} \sum_{i,j=1}^{N} e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j^0 - \boldsymbol{r}_i^0)} \left(1 - \frac{1}{4} |\boldsymbol{q}|^2 \left\langle |\boldsymbol{u}_j - \boldsymbol{u}_i|^2 \right\rangle \right) \\ &= \frac{1}{N} \sum_{i,j=1}^{N} e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j^0 - \boldsymbol{r}_i^0)} \left(1 - \frac{1}{2} |\boldsymbol{q}|^2 \left\langle |\boldsymbol{u}_i|^2 \right\rangle + \frac{1}{2} |\boldsymbol{q}|^2 \left\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \right\rangle \right), \end{aligned} \tag{D60}$$

where the first two terms cancel due to the orthogonality relation (B17). Using (B16) we thus obtain

$$S(\boldsymbol{q}) \underset{\substack{L\to\infty \\ |\boldsymbol{q}|\sim 2\pi/L}}{=} \frac{1}{2N} |\boldsymbol{q}|^2 \left\langle |\tilde{\boldsymbol{u}}_{\boldsymbol{q}}|^2 \right\rangle. \tag{D61}$$

E. DYNAMICAL TRANSITION IN ACTIVE BROWNIAN PARTICLES (ABP) AND PAIR ACTIVE BROWNIAN PARTICLES (PABP)

We analyse the melting transition in both our particle models – active Brownian particles (ABP) and pair active Brownian particles (pABP) – from a dynamical point of view, using two-time correlation functions borrowed from glass physics. First, the mean square displacement MSD(t) (27) [7] which corresponds to the variance of displacements over particles as a function of time, and the self-intermediate scattering function [8]

$$F_s(\boldsymbol{k}, t) = \left\langle \left| \frac{1}{N} \sum_{i=1}^{N} \exp\left(\mathrm{i}\boldsymbol{k} \cdot \left([\boldsymbol{r}_i(t) - \overline{\boldsymbol{r}}(t)] - [\boldsymbol{r}_i(0) - \overline{\boldsymbol{r}}(0)] \right) \right) \right| \right\rangle, \tag{E62}$$

with $\overline{\boldsymbol{r}}(t) = (1/N) \sum_{i=1}^{N} \boldsymbol{r}_i(t)$ the position of the centre of mass at time t and $\boldsymbol{k} = 2\pi(1/D, 0)$, which may be interpreted as the proportion of particles whose displacements at time t are less than $2\pi/|\boldsymbol{k}| = D$ in norm, assuming isotropy. We show both functions in Fig. E1.

In the ABP model, the melting transition of the soft crystal lies between $f = 0.07$ and 0.08, with a bounded MSD (Fig. E1(a)) and no or few defects below that value, and a defect-rich liquid above that value. Melting is also evident in the more rapid decay of the self-intermediate scattering function above the transition (Fig. E1(c)).

In the pABP model, there is an abrupt transition from solid to liquid between $f = 0.112$ and 0.115, evident in the change between a state with very low plateau values of the MSD (Fig. E1(b)) and a correspondingly high plateau value of the self-intermediate scattering function (Fig. E1(d)) on the one hand, and a rapidly diffusing liquid state where correlations are lost on the other hand.

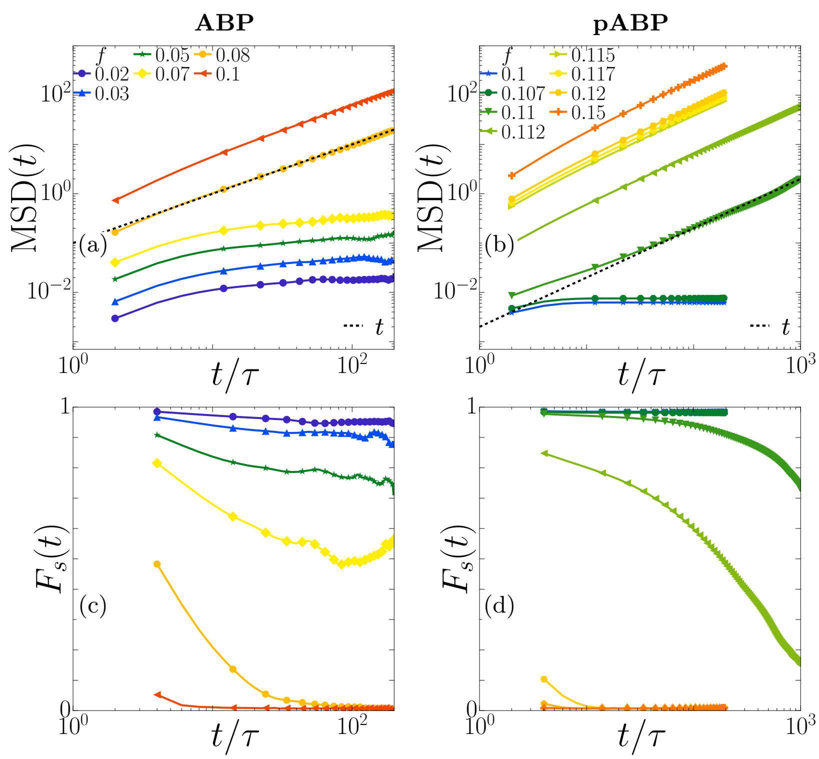

FIG. E1. (a, b) Mean sqaured displacements as functions of time MSD(t) (27) in steady state for different stochastic force amplitudes f. Dashed lines are linear functions of time as guides to the eye. (c, d) Self-intermediate scattering function $F_s(t)$ (E62) in steady state for different stochastic force amplitudes f. (a, c) Active Brownian particles (ABP) and (b, d) pair active Brownian particles (pABP), with $N = 16384$ particles and persistence time $\tau_p = 25$. Identical markers between (a) and (c) and between (b) and (d) correspond to identical data sets.

We confirm that pair active Brownian particles systems at $f = 0.11$ (solid phase) and $f = 0.112$ (phase-separated) have reached steady by reproducing the time evolution of the hexatic order parameter in these systems, starting from an initial ordered configuration, in Fig. E2. We have used the data starting roughly from the half of this time lapse (highlighted with a vertical dashed line) to compute the mean squared displacement and self-intermediate scattering function of Fig. E1. We also provide videos highlighting the argument of the local hexatic order parameter (see *e.g.* Fig. F1(a-c)) at $f = 0.11$ (`supp_video_1.mp4`) and $f = 0.112$ (`supp_video_2.mp4`).

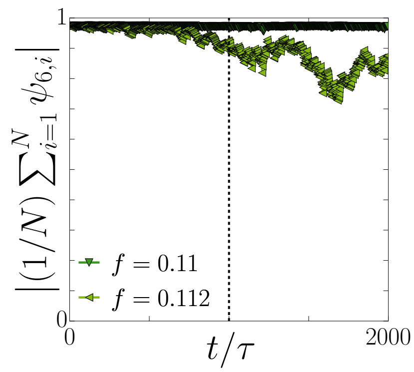

FIG. E2. Time lapse of the ensemble-averaged hexatic order parameter $\left|(1/N)\sum_{i=1}^{N}\psi_{6,i}\right|$ for pair active Brownian particles systems from an initial ordered configuration. Vertical dashed line represents half of the time lapse.

F. DEFECTS IN PAIR ACTIVE BROWNIAN PARTICLES (PABP)

We reproduce in Fig. F1(a-d) the snapshots of Fig. 3 and add the corresponding visualisation of the coordination number z_i of particles (Fig. F1(e-f).

We highlight structural order with three descriptors: the local orientational order parameter $\psi_{6,i}$ (11) (Fig. F1(a-b)) which characterises the local orientation of the lattice, the local translational order parameter $\psi_{\boldsymbol{q}_0,i}$ (Fig. F1(c-d)) which characterises the translational regularity of the lattice, and the coordination z_i of particles (Fig. F1(e-f)), *i.e.* the number of neighbours of each particle. We compute these neighbours with a Delaunay triangulation from the particles' positions – this is done with library `CGAL` [9]. We stress that in a regular triangular lattice, $z_i = 6$. Coordination numbers $z_i \neq 6$ (typically $z_i = 5$ or 7) characterise topological defects known as *disclinations* which locally break translational order. However orientational order is not broken if these defects are bound. In this latter case they correspond to a *dislocation*.

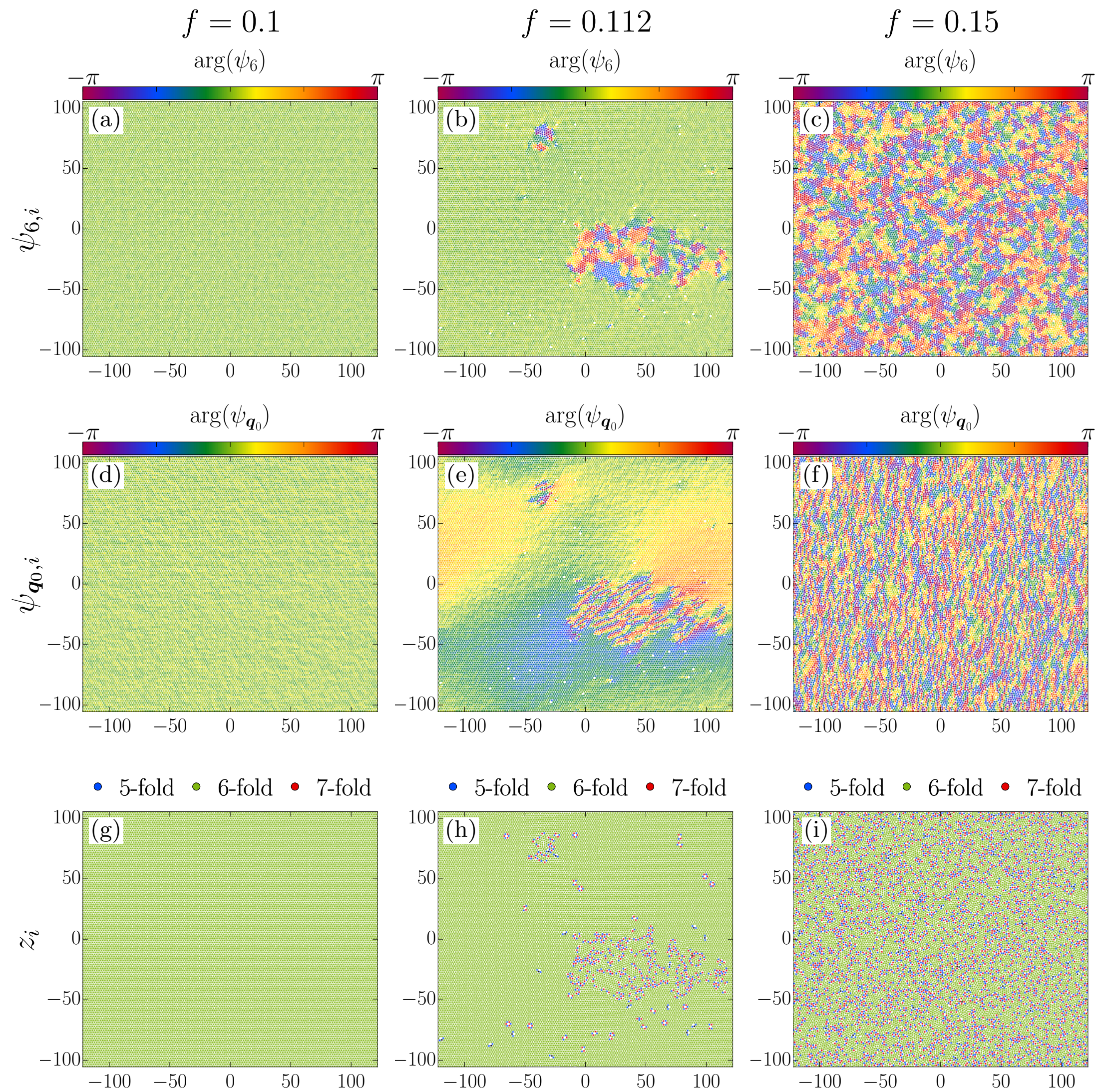

FIG. F1. Order and disorder for pair active Brownian particles (pABP). Visualisation of the argument of the local hexatic order parameter $\arg(\psi_{6,i})$ (11) (a-c), the argument of the local translational order parameter $\arg(\psi_{\boldsymbol{q}_0,i})$ (12) (d-f), and the coordination number z_i of particles (g-i). We used $N = 16384$ particles and persistence time $\tau = 25$. Stochastic force amplitude is (a, d, g) $f = 0.1$, (b, e, h) $f = 0.112$, (c, f, i) $f = 0.15$.

G. UNSCALED STRUCTURE FACTOR

We reproduce in Fig. G1 the unscaled data of Fig. 6.

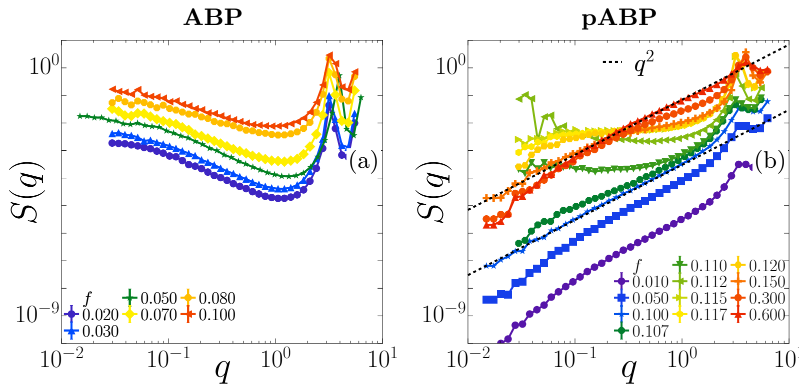

FIG. G1. Structure factor $S(q)$ (16) defined as cylindrical average of $S(\boldsymbol{q})$ over wave-vectors $\boldsymbol{q} = (2\pi m/L_x, 2\pi n/L_y)$ which satisfy $|\boldsymbol{q}| \in [q - \delta q/2, q + \delta q/2]$ with $\delta q = 10^{-2}$. We plot the structure factor for different stochastic force amplitudes f in (a) for the particle model with particle-wise stochastic force (ABP), and in (b) for the particle model with pair-wise stochastic forces (pABP). We used $N = 16384$ or 65536 particles, and persistence $\tau = 25$.

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