Supporting Information for

4D Printing of Fully Programmable Sheets of Digital Metamaterials

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Calculating the Gaussian curvature from a metric

As stated by *Gauss's Theorema Egregium*, the Gaussian curvature is invariant under isometries and, hence, can be calculated directly from the metric tensor.

For a general metric tensor, $\boldsymbol{a}(u,v) \equiv \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$, the calculation is cumbersome and can be expressed using the general Brioschi formula

$$K = \frac{1}{2}(\det \boldsymbol{a})^{-2}\left[\det\begin{pmatrix} 2F_{uv} - E_{vv} - G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & 2E & 2F \\ G_v & 2F & 2G \end{pmatrix} - \det\begin{pmatrix} 0 & E_v & G_u \\ E_v & 2E & 2F \\ G_u & 2F & 2G \end{pmatrix}\right]$$

, where subscript indices denote derivation with respect to the corresponding coordinates.

Since our gels swell uniformly from flat configurations, we can greatly simplify this expression by choosing to work with a coordinate system of the flat gel. Usually, these are either the Cartesian or the polar coordinates, depending on the symmetries of the desired swelling field.

For Cartesian coordinates (x and y), $\bar{\boldsymbol{a}} \equiv \omega_0^2(x,y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the programmed Gaussian curvature reads: $\bar{K} = -\frac{(\partial_x^2 + \partial_y^2)\omega_0^2}{2\omega_0^2}$.

In cases with polar symmetry, it is easier to work with the polar coordinates r and φ, hence $\bar{\boldsymbol{a}} \equiv \omega_0^2(r)\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, and the programmed Gaussian curvature reads: $\bar{K} = \frac{r\omega_0'^2 - \omega_0(\omega_0' + r\omega_0'')}{r\omega_0^4}$.

The same expressions are used to calculate the needed differential swelling field, ω_0 for a desired shape with Gaussian curvature, $\bar{K}$, by solving the differential equations above.

The reduction of the growth rule to metric and curvature

Introducing a general swelling field, $\omega(z)$, to a voxel-sized, ℓ_0, will result in the swelling of the midline, ω_0, and curvature κ. For $-\frac{t}{2} \leq z \leq \frac{t}{2}$ we get that $\bar{\ell}(z) = \ell_0\,\omega(z)$ and that $\ell(z) = \omega_0\ell_0(1 + +\kappa\, z)$.

The energy, of that voxel, is

$$E(z) \propto \left(\bar{\ell}(z) - \ell(z)\right)^2 \propto \left(\omega(z) - \omega_0(1+\kappa\, z)\right)^2 = \omega^2(z) - 2\omega(z)\omega_0(1+\kappa\, z) + \omega_0^2(1+\kappa\, z)^2$$

and we minimize $E \equiv \int E(z)dz$, therefore

$$0 = \partial_{\omega_0}E \propto \int dz[-2\omega(z)(1+\kappa\, z) + 2\omega_0(1+\kappa\, z)^2]$$

hence

$$\int dz\,\omega(z)(1+\kappa\, z) = \int dz\,\omega_0(1+\kappa\, z)^2 = \omega_0(\,t + \frac{\kappa^2}{12}t^3)$$

Also,

$$0 = \partial_\kappa E \propto \int dz[-2\omega(z)\omega_0\, z + 2\omega_0^2(z + \kappa z^2)]$$

hence

$$\int dz\, \omega(z)\, z = \int dz\, \omega_0(z + \kappa z^2) = \omega_0 \frac{\kappa}{12} t^3$$

then

$$\int dz\, \omega(z) \; = \omega_0 \left(t + \frac{\kappa^2}{12} t^3 \right) - \omega_0 \frac{\kappa^2}{12} t^3 = \omega_0 t$$

And finally

$$\omega_0 = \frac{1}{t} \int dz\, \omega(z) = \langle \omega \rangle$$
$$\kappa = \frac{12}{\omega_0 t^3} \int dz\, \omega(z)\, z = \frac{12}{\omega_0 t^2} \langle \omega\, z \rangle$$

In our case $\omega(z) = \begin{cases} \omega_\mathrm{t} & z > 0 \\ \omega_\mathrm{b} & z \le 0 \end{cases}$ and so $\begin{cases} \omega_0 = \frac{\omega_\mathrm{t} + \omega_\mathrm{b}}{2} \\ \kappa = \frac{12}{\omega_0 t^3} \frac{t^2}{8} (\omega_\mathrm{t} - \omega_\mathrm{b}) = \frac{3}{t} \frac{\omega_\mathrm{t} - \omega_\mathrm{b}}{\omega_\mathrm{t} + \omega_\mathrm{b}} \end{cases}$

The latter coincides with Timoshenko's result of $\kappa = \frac{3\epsilon}{2t}$ (for $\epsilon = \frac{\omega_\mathrm{t} - \omega_\mathrm{b}}{(\omega_\mathrm{t} + \omega_\mathrm{b})/2}$)[1].

Calculating ω_0 and Δ for a spherical cap

We are looking for a metric $\bar{\boldsymbol{a}}$ with an associated Gaussian curvature $\bar{K} \equiv R^{-2}$, where R is the desired radius of the spherical cap. Utilizing the inherent symmetry of the problem, we can work in polar coordinates and assume ω_0 has no azimuthal dependence, hence, $\bar{\boldsymbol{a}} \equiv \omega_0^2(r) \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$.

Using the Brioschi formula for orthogonal metric (see section above), $\bar{K} = \frac{r\omega_0'^2 - \omega_0(\omega_0' + r\omega_0'')}{r\omega_0^4}$, solving this differential equation yields $\omega_0(r) = \frac{R\, c_1 \operatorname{sech}\, (c_1(\,\log r + R\, c_2))}{r}$, where c_1 and c_2 are constants of integration.

To get a spherical cap and to avoid a sharp cusp at $r \to 0$, we choose $c_1 = 1$, and continue to determine c_2 such as $\omega_0(r = 0) = 1.25$ (the maximal swelling possible).

Then, the maximal radius of the printed disc is given by $\omega_0(r_{\max}) = 1$ (the minimal swelling ratio possible).

References

1 S. Timoshenko, *J Opt Soc Am*, 1925, **11**, 233.