APPENDIX A: Formal Asymptotic Reduction of the Fluid Momentum and Wall Elasticity Governing Equations

A.1 Fluid momentum conservation equation

The incompressible continuity equation for the fluid for 2D flow is given as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{A1}$$

Considering $\bar{u}=u/U_r$, $\bar{v}=v/V_r$, $\bar{x}=x/L$ and $\bar{y}=y/H$, where U_r and V_r are reference axial and transverse fluid velocities, and employing lubrication approximation ($H \ll L$ and $\alpha=H/L \ll 1$), we obtain, $V_r \sim U_r\alpha$, or, $V_r/U_r \ll 1$.

The steady-state momentum conservation equation for the fluid can be given in the x-direction as,

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+F_x \tag{A2}$$

where, F_x are the body forces acting in x-direction due to applied electric and magnetic forces.

Invoking scaling arguments from above, and using $\bar{p}=p/(\lambda_L+2G)$, $\bar{\tau}_{xx}=\tau_{xx}/\tau_r$, $\bar{\tau}_{xy}=\tau_{xy}/\tau_r$ and $\bar{F}_x=F_x/F_r$ (where $\tau_r\,(=\mu U_r/H)$ and F_r are the reference viscous stresses and body force terms respectively) the terms of Eq. A2 can be written as,

$u\,\partial u/\partial x=\left(U_r^2/L\right)\bar{u}\,\partial\bar{u}/\partial\bar{x}$

$v\,\partial u/\partial y=\left(U_r^2/L\right)\bar{v}\,\partial\bar{u}/\partial\bar{y}$

$\partial p/\partial x=[(\lambda_L+2G)/L]\,\partial\bar{p}/\partial\bar{x}$

$\partial\tau_{xx}/\partial x=\left(\mu U_r/HL\right)\partial\bar{\tau}_{xx}/\partial\bar{x}$

$\partial\tau_{xy}/\partial y=\left(\mu U_r/H^2\right)\partial\bar{\tau}_{xy}/\partial\bar{y}$

$\bar{F}_x=F_x/F_r$

Substituting these terms in Eq. A2 and dividing the entire equation with $\mu U_r/H^2$, we obtain

$$Re\alpha\left(\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}\right)=-\xi_r\frac{\partial\bar{p}}{\partial\bar{x}}+\alpha\frac{\partial\bar{\tau}_{xx}}{\partial\bar{x}}+\frac{\partial\bar{\tau}_{xy}}{\partial\bar{y}}+\frac{H^2F_r}{\mu U_r}\bar{F}_x \tag{A3}$$

where, $Re=\rho U_r H/\mu$. Employing the considerations of creeping flow ($Re \ll 1$) and lubrication approximation ($\alpha \ll 1$), and expanding the force terms according to the discussions in Sec. 2.5, Eq. A3 reduces to,

$$0 = -\xi_r \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{xy}}{\partial \bar{y}} - Ha^2 \bar{u} + Ha\, S + \bar{\kappa}^2 \bar{\psi} \tag{A4}$$

which is identical to Eq. 11.

The steady-state momentum conservation equation for the fluid can be given in the y-direction as,

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + F_y \tag{A5}$$

Using the similar scaling arguments stated above, Eq. A5 evaluates as,

$$Re\alpha^2\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = -\frac{\xi_r \partial \bar{p}}{\alpha\, \partial \bar{y}} + \alpha\frac{\partial \bar{\tau}_{yx}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{yy}}{\partial \bar{y}} + \frac{H^2 F_r}{\mu U_r}\bar{F}_y \tag{A6}$$

or,

$$Re\alpha^3\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = -\xi_r\frac{\partial \bar{p}}{\partial \bar{y}} + \alpha^2\frac{\partial \bar{\tau}_{yx}}{\partial \bar{x}} + \alpha\frac{\partial \bar{\tau}_{yy}}{\partial \bar{y}} + \alpha\frac{H^2 F_r}{\mu U_r}\bar{F}_y \tag{A7}$$

Again, for creeping flow ($Re \ll 1$) and lubrication approximation ($\alpha \ll 1$), Eq. A7 reduces to,

$$\frac{\partial \bar{p}}{\partial \bar{y}} = 0 \tag{A8}$$

Thus, pressure is independent of y.

A.2 Wall elasticity equation

For homogeneous isotropic elastic solid walls exposed to no body forces, the Navier-Lame equation in the x-direction is:

$$G\left(\frac{\partial^2 \delta_x}{\partial x^2} + \frac{\partial^2 \delta_x}{\partial y^2}\right) + (\lambda_L + G)\frac{\partial}{\partial x}\left(\frac{\partial \delta_x}{\partial x} + \frac{\partial \delta_y}{\partial y}\right) = 0 \tag{A9}$$

or,

$$(\lambda_L + 2G)\frac{\partial^2 \delta_x}{\partial x^2} + G\frac{\partial^2 \delta_x}{\partial y^2} + (\lambda_L + G)\frac{\partial^2 \delta_y}{\partial x \partial y} = 0 \tag{A10}$$

Using considerations of slender wall scaling ($D \ll L$ or $\beta = D/L \ll 1$) and thick wall ($D \sim H$ or $\gamma = D/H \approx 1$), and using $\bar{\delta}_x = \delta_x/\delta_r$ and $\bar{\delta}_y = \delta_y/\delta_r$, where δ_r is the reference wall displacement,

$\partial^2\delta_x/\partial x^2 = \delta_r/L^2 \left(\partial^2\bar{\delta}_x/\partial \bar{x}^2\right) = \beta^2\gamma^2\delta_r/D^2 \left(\partial^2\bar{\delta}_x/\partial \bar{x}^2\right)$

$\partial^2\delta_y/\partial y^2 = \gamma^2\delta_r/D^2 \left(\partial^2\bar{\delta}_y/\partial \bar{y}^2\right)$

$\partial^2\delta_y/\partial x\partial y = \gamma\delta_r/DL \left(\partial^2\bar{\delta}_y/\partial \bar{x}\partial \bar{y}\right) = \beta\gamma^2\delta_r/D^2\left(\partial^2\bar{\delta}_y/\partial \bar{x}\partial \bar{y}\right)$

Comparing the terms, and since $\beta = D/L \ll 1$, Eq. A10 reduces to

$$\frac{\partial^2 \bar{\delta}_x}{\partial \bar{y}^2} = 0 \tag{A11}$$

This implies that axial displacement exhibits no transverse curvature. Also, since we observe, $\partial^2 \delta_x / \partial x^2 \sim O(\beta^2) \ll 1$, therefore axial stretching of the solid wall is negligible in comparison to transverse deformation.

For homogeneous isotropic elastic solid walls exposed to no body forces, the Navier-Lame equation in the y-direction is given as:

$$G\left(\frac{\partial^2 \delta_y}{\partial x^2} + \frac{\partial^2 \delta_y}{\partial y^2}\right) + (\lambda_L + G)\frac{\partial}{\partial y}\left(\frac{\partial \delta_x}{\partial x} + \frac{\partial \delta_y}{\partial y}\right) = 0 \tag{A12}$$

or,

$$G\frac{\partial^2 \delta_y}{\partial x^2} + (\lambda_L + 2G)\frac{\partial^2 \delta_y}{\partial y^2} + (\lambda_L + G)\frac{\partial^2 \delta_x}{\partial x \partial y} = 0 \tag{A13}$$

Using similar scaling arguments as above, Eq. A13 reduces to,

$$\frac{\partial^2 \bar{\delta}_y}{\partial \bar{y}^2} = 0 \tag{A14}$$

which is identical to Eq. 22.