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Dynamics of an internally actuated weakly elastic sphere in general quadratic flow

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1 Particle translating in the general quadratic flow

The components of the disturbance velocity and the pressure fields in the fluid at leading-order are obtained as

$$\begin{aligned} v^{(0)}_{d_q,r} =& \frac{1}{160\xi^5}\Bigg[4\cos\theta(5\gamma_{333}(-5+7\xi^2)+4\xi^2(-5-5\xi^2(-3+\tau)+3\tau)+25\gamma_{333}(5-7\xi^2)\cos 2\theta \\ &+30(-5+7\xi^2)\sin^2\theta((2\gamma_{223}+\gamma_{333})\cos 2\phi-2\gamma_{123}\sin 2\phi))+5(-5+7\xi^2)(4(4\gamma_{122}+\gamma_{133}) \\ &\times\cos 3\phi\sin^3\theta-3\gamma_{133}\cos\phi(\sin\theta+5\sin 3\theta)-3\gamma_{233}(\sin\theta+5\sin 3\theta)\sin\phi+4(4\gamma_{222} \\ &+3\gamma_{233})\sin^3\theta\sin 3\phi)\Bigg], \end{aligned} \tag{1}$$

$$\begin{aligned} v^{(0)}_{d_q,\theta} =& \frac{1}{1920\xi^5}\Bigg[4\sin\theta(45\gamma_{333}(15-7\xi^2)+24\xi^2(-5+5\xi^2(-3+\tau)+3\tau)-5(-45(2\gamma_{223}+\gamma_{333}) \\ &+(-32Q_{12}+21(2\gamma_{223}+\gamma_{333}))\xi^2)\cos 2\phi+10(-45\gamma_{123}+(-8Q_{11}+8Q_{22}+21\gamma_{123})\xi^2)\sin 2\phi \\ &-15(-15+7\xi^2)\cos 2\theta(5\gamma_{333}+3(2\gamma_{223}+\gamma_{333})\cos 2\phi-6\gamma_{123}\sin 2\phi))+10\cos\theta((64Q_{23}\xi^2 \\ &+21\gamma_{133}(15-7\xi^2)+45\gamma_{133}(-15+7\xi^2)\cos 2\theta)\cos\phi-6(4\gamma_{122}+\gamma_{133})(-15+7\xi^2)\cos 3\phi \\ &\times\sin^2\theta+(315\gamma_{233}-(64Q_{13}+147\gamma_{233})\xi^2+45\gamma_{233}(-15+7\xi^2)\cos 2\theta)\sin\phi-6(4\gamma_{222} \\ &+3\gamma_{233})(-15+7\xi^2)\sin^2\theta\sin 3\phi)\Bigg], \end{aligned} \tag{2}$$

$$\begin{aligned} v^{(0)}_{d_q,\phi} =& \frac{1}{192\xi^5}\Bigg[(9\gamma_{233}(-15+7\xi^2)+(-225\gamma_{233}+(-64Q_{13}+105\gamma_{233})\xi^2)\cos 2\theta)\cos\phi-6(4\gamma_{222}+3\gamma_{233}) \\ &\times(-15+7\xi^2)\cos 3\phi\sin^2\theta-48(Q_{11}+Q_{22})\xi^2\sin 2\theta+8(-45\gamma_{123}+(-2Q_{11}+2Q_{22}+21\gamma_{123}) \\ &\times\xi^2)\cos 2\phi\sin 2\theta+(9\gamma_{133}(15-7\xi^2)+(225\gamma_{133}-(64Q_{23}+105\gamma_{133})\xi^2)\cos 2\theta)\sin\phi+4(-45 \\ &\times(2\gamma_{223}+\gamma_{333})+(-8Q_{12}+42\gamma_{223}+21\gamma_{333})\xi^2)\sin 2\theta\sin 2\phi+6(4\gamma_{122}+\gamma_{133})(-15+7\xi^2) \\ &\times\sin^2\theta\sin 3\phi\Bigg] \end{aligned} \tag{3}$$

and

$$\begin{aligned} p^{(0)}_{d_q} =& \frac{1}{64\xi^4}\Bigg[-105\gamma_{133}\cos\phi\sin\theta+4\cos\theta(35\gamma_{333}-8\xi^2(-3+\tau)-175\gamma_{333}\cos 2\theta+210(2\gamma_{223}+\gamma_{333}) \\ &\times\cos 2\phi\sin^2\theta)+35(4(4\gamma_{122}+\gamma_{133})\cos 3\phi\sin^3\theta-3(5\sin 3\theta(\gamma_{133}\cos\phi+\gamma_{233}\sin\phi)+\sin\theta(\gamma_{233} \\ &\times\sin\phi+8\gamma_{123}\sin 2\theta\sin 2\phi))+4(4\gamma_{222}+3\gamma_{233})\sin^3\theta\sin 3\phi)\Bigg], \end{aligned} \tag{4}$$

respectively. Here, γ_{ijk} and Q_{lm} with $i,j,k=1,2,3$ and $l,m=1,2,3$ are the Cartesian components of γ and $\mathbf{Q}$, respectively and $\tau_l=\tau\hat{\mathbf{k}}$. The components of the displacement at $O(\alpha)$ are obtained as

$$u_{q,r}^{(1)}=\frac{1}{256\xi}\left[\frac{1}{(2+\Gamma)(2+3\Gamma)}4\cos\theta(-3\Gamma^2(\xi^3(32+35\gamma_{333}-32\tau)+32(-3+\tau))-8\Gamma(-96+35\gamma_{333}\xi^3 -8(-4+\xi^3)\tau)-4(32(-3+\tau)+\xi^3(-24+35\gamma_{333}+40\tau))+35(2+\Gamma)(2+3\Gamma)\xi^3(5\gamma_{333}\cos 2\theta -6(2\gamma_{223}+\gamma_{333})\cos 2\phi\sin^2\theta))+35\xi^3(-4(4\gamma_{122}+\gamma_{133})\cos 3\phi\sin^3\theta+3\gamma_{233}(\sin\theta+5\sin 3\theta) \times\sin\phi+3\cos\phi(\gamma_{133}\sin\theta+5\gamma_{133}\sin 3\theta+32\gamma_{123}\cos\theta\sin^2\theta\sin\phi)-4(4\gamma_{222}+3\gamma_{233})\sin^3\theta \times\sin 3\phi)\right]-\frac{P_0\xi}{2+3\Gamma}, \tag{5}$$

$$u_{q,\theta}^{(1)}=-\frac{1}{384\xi}\left[\frac{2\sin\theta}{(2+\Gamma)(2+3\Gamma)}(864+1584\Gamma+432\Gamma^2-864\xi^3-1440\Gamma\xi^3-576\Gamma^2\xi^3+1260\gamma_{333}\xi^3 +2520\Gamma\gamma_{333}\xi^3+945\Gamma^2\gamma_{333}\xi^3-288\tau-528\Gamma\tau-144\Gamma^2\tau+1440\xi^3\tau+1824\Gamma\xi^3\tau+576\Gamma^2\xi^3\tau +5(4+8\Gamma+3\Gamma^2)(64Q_{12}+42\gamma_{223}+21\gamma_{333})\xi^3\cos 2\phi-640Q_{11}\xi^3\sin 2\phi+640Q_{22}\xi^3\sin 2\phi -1280Q_{11}\Gamma\xi^3\sin 2\phi+1280Q_{22}\Gamma\xi^3\sin 2\phi-480Q_{11}\Gamma^2\xi^3\sin 2\phi+480Q_{22}\Gamma^2\xi^3\sin 2\phi-840\gamma_{123} \times\xi^3\sin 2\phi-1680\Gamma\gamma_{123}\xi^3\sin 2\phi-630\Gamma^2\gamma_{123}\xi^3\sin 2\phi+105(4+8\Gamma+3\Gamma^2)\xi^3\cos 2\theta(5\gamma_{333} +3(2\gamma_{223}+\gamma_{333})\cos 2\phi-6\gamma_{123}\sin 2\phi))+5\xi^3\cos\theta((128Q_{23}+147\gamma_{133}-315\gamma_{133}\cos 2\theta)\cos\phi +42(4\gamma_{122}+\gamma_{133})\cos 3\phi\sin^2\theta-128Q_{13}\sin\phi+147\gamma_{233}\sin\phi-315\gamma_{233}\cos 2\theta\sin\phi+168\gamma_{222} \times\sin^2\theta\sin 3\phi+126\gamma_{233}\sin^2\theta\sin 3\phi)\right], \tag{6}$$

and

$$u_{q,\phi}^{(1)}=\frac{5}{384}\xi^2\left[63\gamma_{233}\cos\phi-63\gamma_{133}\sin\phi+\cos 2\theta((128Q_{13}+105\gamma_{233})\cos\phi+(128Q_{23}-105\gamma_{133})\sin\phi) +4\sin 2\theta(24(Q_{11}+Q_{22})+(8Q_{11}-8Q_{22}+42\gamma_{123})\cos 2\phi+(16Q_{12}+42\gamma_{223}+21\gamma_{333})\sin 2\phi) +42\sin^2\theta(-((4\gamma_{222}+3\gamma_{233})\cos 3\phi)+(4\gamma_{122}+\gamma_{133})\sin 3\phi)\right]. \tag{7}$$

The expressions of Z_1, Z_2, and Z_3 in point torque at $O(\alpha)$ [eqn (46) in the paper] are obtained as

$$Z_1=\frac{1}{16(2+\Gamma)(2+3\Gamma)}\left[5(4+8\Gamma+3\Gamma^2)(-Q_{22}\gamma_{122}-2Q_{23}\gamma_{123}+Q_{22}\gamma_{133}+Q_{11}(\gamma_{122}+2\gamma_{133}) +2Q_{12}\gamma_{222}+2Q_{12}\gamma_{233})+Q_{13}(60+40\gamma_{223}+40\gamma_{333}+6\Gamma^2(4+5\gamma_{223}+5\gamma_{333})+8\Gamma(12+10\gamma_{223} +10\gamma_{333}-3\tau)-36\tau)\right], \tag{8}$$

$$Z_2=-\frac{1}{16(2+\Gamma)(2+3\Gamma)}\left[5(4+8\Gamma+3\Gamma^2)(2Q_{12}\gamma_{122}+2Q_{13}\gamma_{123}-Q_{11}\gamma_{222}+Q_{22}\gamma_{222}-2Q_{11}\gamma_{233} -Q_{22}\gamma_{233})+Q_{23}(-60+40\gamma_{223}+80\Gamma\gamma_{223}+6\Gamma^2(-4+5\gamma_{223})+24\Gamma(-4+\tau)+36\tau)\right], \tag{9}$$

$$Z_3 = \frac{1}{16(2+\Gamma)(2+3\Gamma)}\Bigg[-10(4+8\Gamma+3\Gamma^2)(Q_{12}\gamma_{123}+Q_{13}\gamma_{133}+Q_{23}\gamma_{233})+Q_{22}(-3\Gamma^2(8+5\gamma_{223} -5\gamma_{333})-8\Gamma(12+5\gamma_{223}-5\gamma_{333}-3\tau)+4(-15-5\gamma_{223}+5\gamma_{333}+9\tau))+Q_{11}(3\Gamma^2(-8+5\gamma_{223} +10\gamma_{333})+8\Gamma(-12+5\gamma_{223}+10\gamma_{333}+3\tau)+4(-15+5\gamma_{223}+10\gamma_{333}+9\tau))\Bigg]. \tag{10}$$

2 Particle translating in the quadratic component of elliptical Poiseuille flow

The components of the disturbance velocity and the pressure fields in the fluid at the leading-order are obtained as

$$v^{(0)}_{d_{qe},r} = \frac{\mathscr{A}_1}{\xi}\cos\theta+\frac{1}{\xi^3}\left[\mathscr{A}_2+\mathscr{A}_3\cos 2\theta+\mathscr{A}_4\cos 2\phi\sin^2\theta\right]\cos\theta-\frac{1}{14\xi^5}\Big[5\mathscr{A}_3\cos 3\theta+\cos\theta(3\mathscr{A}_3 +10\mathscr{A}_4\cos 2\phi\sin^2\theta)\Big], \tag{11}$$

$$v^{(0)}_{d_{qe},\theta} = -\frac{\mathscr{A}_1}{2\xi}\sin\theta+\frac{1}{56\xi^3}\left[(56\mathscr{B}_1-\mathscr{A}_4(13+7\cos 2\theta)\cos 2\phi)\sin\theta+7\mathscr{A}_3\sin 3\theta\right] +\frac{1}{112\xi^5}\left[-(6\mathscr{A}_3+5\mathscr{A}_4\cos 2\phi)\sin\theta+15(-2\mathscr{A}_3+\mathscr{A}_4\cos 2\phi)\sin 3\theta\right], \tag{12}$$

$$v^{(0)}_{d_{qe},\phi} = \left[\frac{1}{\xi^3}-\frac{1}{\xi^5}\right]\frac{5}{28}\mathscr{A}_4\sin 2\theta\sin 2\phi\,, \tag{13}$$

and

$$p^{(0)}_{d_{qe}} = \frac{\mathscr{A}_1}{\xi^2}\cos\theta+\frac{1}{2\xi^4}\left[\mathscr{A}_3(5\cos 2\theta-1)+5\mathscr{A}_4\cos 2\phi\sin^2\theta\right]\cos\theta\,, \tag{14}$$

respectively. The expressions for $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$, and $\mathscr{B}_1$ are given in eqn (27) and (28), respectively. The components of the displacement at $O(\alpha)$ are obtained as

$$u^{(1)}_{q_e,r} = \frac{\mathscr{A}_1}{\xi}\cos\theta+\frac{\xi^2}{8}\left[8\mathscr{E}_1-5\mathscr{A}_3\cos 2\theta-5\mathscr{A}_4\cos 2\phi\sin^2\theta\right]\cos\theta-\frac{P_0\xi}{2+3\Gamma}, \tag{15}$$

$$u^{(1)}_{q_e,\theta} = \frac{\mathscr{F}_1}{\xi}\sin\theta+\xi^2\left[\mathscr{F}_2+\frac{5}{112}\mathscr{A}_4(19-7\cos 2\theta)\cos 2\phi+\frac{5}{16}\mathscr{A}_3\csc\theta\sin 3\theta\right]\sin\theta\,, \tag{16}$$

and

$$u^{(1)}_{q_e,\phi} = -\frac{15}{56}\xi^2\mathscr{A}_4\sin 2\theta\sin 2\phi\,. \tag{17}$$

The expressions for $\mathscr{E}_1$, $\mathscr{F}_1$ and $\mathscr{F}_2$ are given in eqn (29) and (30), respectively. The components of disturbance velocity and the pressure fields in the fluid at $O(\alpha)$ are obtained as

$$\begin{aligned}
v^{(1)}_{d_{qe},r} =& \frac{1}{\xi^2}\left[\mathscr{H}_1+\mathscr{H}_2\cos 2\theta+\mathscr{H}_3\cos 2\phi\sin^2\theta\right]+\frac{1}{\xi^4}\left[\mathscr{H}_4+\mathscr{H}_5\cos 2\theta+\mathscr{H}_6\cos 4\theta+(\mathscr{H}_7+\mathscr{H}_8\cos 2\theta)\right.\\
&\left.\times\cos 2\phi\sin^2\theta+\mathscr{H}_9\cos 4\phi\sin^4\theta\right]+\frac{1}{\xi^6}\left[\mathscr{H}_{10}+\mathscr{H}_{11}\cos 2\theta+\mathscr{H}_{12}\cos 4\theta+\mathscr{H}_{13}\cos 6\theta+(\mathscr{H}_{14}\right.\\
&\left.+\mathscr{H}_{15}\cos 2\theta+\mathscr{H}_{16}\cos 4\theta)\cos 2\phi\sin^2\theta+(\mathscr{H}_{17}+\mathscr{H}_{18}\cos 2\theta)\cos 4\phi\sin^4\theta\right]+\frac{1}{924\xi^8}\left[-4\mathscr{H}_{13}\right.\\
&\times(50+105\cos 2\theta+126\cos 4\theta+231\cos 6\theta)-28\mathscr{H}_{16}(35+60\cos 2\theta+33\cos 4\theta)\cos 2\phi\sin^2\theta\\
&\left.+3773\mathscr{H}_9(9+11\cos 2\theta)\cos 4\phi\sin^4\theta\right]+P_0\left[\frac{1}{\xi}(\mathscr{H}_{19}\cos\theta)+\frac{1}{\xi^3}(\mathscr{H}_{20}\cos\theta+\mathscr{H}_{21}\cos\theta\cos 2\theta\right.\\
&\left.+\mathscr{H}_{22}\cos\theta\cos 2\phi\sin^2\theta)+\frac{1}{\xi^5}(\frac{1}{5}\mathscr{H}_{21}\cos\theta-\mathscr{H}_{21}\cos\theta\cos 2\theta-\mathscr{H}_{22}\cos\theta\cos 2\phi\sin^2\theta)\right] \qquad (18)
\end{aligned}$$

$$\begin{aligned}
v^{(1)}_{d_{qe},\theta} =& \frac{1}{\xi^4}\left[\mathscr{I}_1\cos\theta\cos 4\phi\sin^3\theta+(\mathscr{I}_2+\mathscr{I}_3\cos 2\phi)\sin 2\theta+(\mathscr{I}_4+\mathscr{I}_5\cos 2\phi)\sin 4\theta\right]+\frac{1}{\xi^6}\left[\mathscr{I}_6\cos\theta\right.\\
&\times\cos 4\phi\sin^3\theta+\mathscr{I}_7\cos\theta\cos 2\theta\cos 4\phi\sin^3\theta+\mathscr{I}_8\sin 2\theta+\mathscr{I}_9\cos 2\theta\sin 2\theta+\mathscr{I}_{10}\cos 4\theta\sin 2\theta\\
&\left.+\mathscr{I}_{11}\cos 2\phi\sin 2\theta+\mathscr{I}_{12}\cos 2\theta\cos 2\phi\sin 2\theta+\mathscr{I}_{13}\cos 4\theta\cos 2\phi\sin 2\theta\right]+\frac{1}{924\xi^8}\left[-48\mathscr{H}_{13}(19\right.\\
&+12\cos 2\theta+33\cos 4\theta)+4\mathscr{H}_{16}(41-12\cos 2\theta+99\cos 4\theta)\cos 2\phi-539\mathscr{H}_9(7+33\cos 2\theta)\\
&\left.\times\cos 4\phi\sin^2\theta\right]\sin 2\theta+P_0\left[\frac{1}{\xi}\mathscr{I}_{14}\sin\theta+\frac{1}{\xi^3}(\mathscr{I}_{15}\sin\theta+\mathscr{I}_{16}\cos 2\theta\sin\theta+\mathscr{I}_{17}\cos 2\phi\sin\theta\right.\\
&\left.+\mathscr{I}_{18}\cos 2\theta\cos 2\phi\sin\theta)-\frac{1}{65\xi^5}(39\mathscr{I}_{16}(3+5\cos 2\theta)+35\mathscr{I}_{17}(1+3\cos 2\theta)\cos 2\phi)\sin\theta\right], \qquad (19)
\end{aligned}$$

$$\begin{aligned}
v^{(1)}_{d_{qe},\phi} =& \frac{1}{\xi^4}\left[(\mathscr{J}_1+\mathscr{J}_2\cos 2\theta)\sin\theta\sin 2\phi+\mathscr{J}_3\sin^3\theta\sin 4\phi\right]+\frac{1}{\xi^6}\left[(\mathscr{J}_4+\mathscr{J}_5\cos 2\theta+\mathscr{J}_6\cos 4\theta)\sin\theta\right.\\
&\left.\times\sin 2\phi+(\mathscr{J}_7+\mathscr{J}_8\cos 2\theta)\sin^3\theta\sin 4\phi\right]+\frac{1}{\xi^8}\left[-\frac{2}{231}\mathscr{H}_{16}(35+60\cos 2\theta+33\cos 4\theta)\sin\theta\right.\\
&\left.\times\sin 2\phi+\frac{7}{3}\mathscr{H}_9(9+11\cos 2\theta)\sin^3\theta\sin 4\phi\right]+P_0\left[\frac{1}{4\xi^3}-\frac{7}{20\xi^5}\right]\mathscr{J}_9\sin 2\theta\sin 2\phi\,, \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
p^{(1)}_{d_{qe}} =& \frac{\mathscr{K}_1}{\xi^3}+\frac{1}{\xi^5}\left[\mathscr{K}_2+\mathscr{K}_3\cos 2\theta+\mathscr{K}_4\cos 4\theta+(\mathscr{K}_5+\mathscr{K}_6\cos 2\theta)\cos 2\phi\sin^2\theta+\mathscr{K}_7\cos 4\phi\sin^4\theta\right]\\
&+\frac{\mathscr{K}_8}{\xi^7}+P_0\left[\frac{1}{\xi^2}(\mathscr{K}_9\cos\theta)+\frac{1}{\xi^4}(\mathscr{K}_{10}-5\mathscr{K}_{10}\cos 2\theta+\mathscr{K}_{11}\cos 2\phi\sin^2\theta)\cos\theta\right], \qquad (21)
\end{aligned}$$

respectively. Here,

$$\mathscr{K}_1 = 2(\mathscr{H}_1+\mathscr{H}_2\cos 2\theta+\mathscr{H}_3\cos 2\phi\sin^2\theta) \qquad (22)$$

$$\begin{aligned}
\mathscr{K}_8 =& -\frac{1}{294}[-4\mathscr{H}_{13}(50+105\cos 2\theta+126\cos 4\theta+231\cos 6\theta)-28\mathscr{H}_{16}(35+60\cos 2\theta\\
&+33\cos 4\theta)\cos 2\phi\sin^2\theta+3773\mathscr{H}_9(9+11\cos 2\theta)\cos 4\phi\sin^4\theta] \qquad (23)
\end{aligned}$$

The expressions for $\mathscr{H}_n$ (with n =1,2,..,22), $\mathscr{I}_n$ (with n =1,2,..,18), $\mathscr{J}_n$ (with n =1,2,..,9) and $\mathscr{K}_n$ (with n =2,3,..,7 and 9,10,11) are given in eqn (31)-(69).

The components of the displacement at $O(\alpha^2)$ are obtained as

$$
\begin{aligned}
u^{(2)}_{q_e,r} =& \frac{1}{270336(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}\,\xi \Big[152064(14+17\Gamma)(5+2\Gamma(4+\Gamma))(-336\\
&+\Gamma(-796-532\Gamma-97\Gamma^2+24(1+\Gamma)(2+3\Gamma)\xi^2))+3(-704(14+17\Gamma)(44576+\Gamma(133000\\
&+\Gamma(148348+\Gamma(86978+9\Gamma(3279+523\Gamma)))))+(2+3\Gamma)(-5408928+\Gamma(45241520\\
&+\Gamma(203174800+\Gamma(265312840+\Gamma(133872038+22501155\Gamma)))))\xi^2+47520(1+\Gamma)(2+\Gamma)\\
&\times(1+2\Gamma)(2+3\Gamma)^2(14+19\Gamma)\xi^4)(\psi_x^4+\psi_y^4)+2(-704(14+17\Gamma)(158368+\Gamma(476440\\
&+\Gamma(506164+3\Gamma(81298+9\Gamma(2099+223\Gamma)))))-(2+3\Gamma)(21730464+\Gamma(49882960\\
&+\Gamma(30285680+\Gamma(8254520+\Gamma(13803226+7705725\Gamma)))))\xi^2+142560(1+\Gamma)(2+\Gamma)\\
&\times(1+2\Gamma)(2+3\Gamma)^2(14+19\Gamma)\xi^4)\psi_x^2\psi_y^2+792(-8(14+17\Gamma)(53536+\Gamma(188024+\Gamma(240828\\
&+\Gamma(139922+\Gamma(37591+3984\Gamma)))))+3(2+3\Gamma)(-23072+\Gamma(29840+\Gamma(270784+\Gamma(380432\\
&+\Gamma(187858+26243\Gamma)))))\xi^2+540(1+\Gamma)(2+\Gamma)(1+2\Gamma)(2+3\Gamma)^2(14+19\Gamma)\xi^4)(\psi_x^2+\psi_y^2)\\
&+(2+3\Gamma)(12(12672(14+17\Gamma)(5+2\Gamma(4+\Gamma))(-112+\Gamma(-214-97\Gamma+72(1+\Gamma)\xi^2))\\
&+(-528(14+17\Gamma)(8400+\Gamma(25196+\Gamma(25916+\Gamma(10919+1569\Gamma))))+(-3004960\\
&+\Gamma(39545840+\Gamma(164058192+\Gamma(210758056+\Gamma(106120078+17932119\Gamma)))))\xi^2+26400(1\\
&+\Gamma)(2+\Gamma)(1+2\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^4)(\psi_x^4+\psi_y^4)+2(-176(14+17\Gamma)(14000+\Gamma(40388\\
&+\Gamma(40148+9\Gamma(1773+223\Gamma))))+(-4024160+\Gamma(-5046960+\Gamma(7610992+\Gamma(12918456\\
&+(3637978-649731\Gamma)\Gamma))))\xi^2+26400(1+\Gamma)(2+\Gamma)(1+2\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^4)\psi_x^2\psi_y^2\\
&+88(-18(14+17\Gamma)(6160+\Gamma(19172+\Gamma(20872+83\Gamma(111+16\Gamma))))+(-115360\\
&+\Gamma(415760+\Gamma(2270304+\Gamma(3009952+\Gamma(1480306+215943\Gamma)))))\xi^2+900(1+\Gamma)(2+\Gamma)\\
&\times(1+2\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^4)(\psi_x^2+\psi_y^2))\cos 2\theta+105(2+\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^2\\
&\times((-10732-21264\Gamma-10589\Gamma^2+5280(1+\Gamma)(1+2\Gamma)\xi^2)\psi_x^4+2(-14372-27504\Gamma-12799\Gamma^2\\
&+5280(1+\Gamma)(1+2\Gamma)\xi^2)\psi_x^2\psi_y^2+(-10732-21264\Gamma-10589\Gamma^2+5280(1+\Gamma)(1+2\Gamma)\xi^2)\psi_y^4\\
&+88(-412-804\Gamma-389\Gamma^2+180(1+\Gamma)(1+2\Gamma)\xi^2)(\psi_x^2+\psi_y^2))\cos 4\theta-80(2+\Gamma)(\psi_x^2-\psi_y^2)\\
&\times(66(12(14+17\Gamma)(168+\Gamma(470+337\Gamma))-(2+3\Gamma)(28840+\Gamma(75932+\Gamma(67178\\
&+21451\Gamma)))\xi^2+900(1+\Gamma)(1+2\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^4)+(-88(14+17\Gamma)(616+\Gamma(4010\\
&+3\Gamma(1943+735\Gamma)))+(2+3\Gamma)(-503160+\Gamma(-207644+\Gamma(723654+279563\Gamma)))\xi^2\\
&+19800(1+\Gamma)(1+2\Gamma)(2+3\Gamma)(14+19\Gamma)\xi^4)(\psi_x^2+\psi_y^2)+21(2+3\Gamma)(14+19\Gamma)\xi^2(22(-412\\
&-804\Gamma-389\Gamma^2+180(1+\Gamma)(1+2\Gamma)\xi^2)+(-2396-4824\Gamma-2473\Gamma^2+1320(1+\Gamma)(1\\
&+2\Gamma)\xi^2)(\psi_x^2+\psi_y^2))\cos 2\theta)\cos 2\phi\sin^2\theta+34440(2+\Gamma)^2(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)\xi^2\\
&\times(\psi_x-\psi_y)^2(\psi_x+\psi_y)^2\cos 4\phi\sin^4\theta)\Big]-P_0\Big[\frac{3}{2(2+\Gamma)(2+3\Gamma)^2\xi}((4+8\Gamma+3\Gamma^2\\
&+2(-1+\Gamma^2)\xi^3)+(2+\Gamma)(2+3\Gamma)(\psi_x^2+\psi_y^2))\cos\theta\Big], \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
u^{(2)}_{q_e,\theta} =& \frac{1}{135168}\xi\left[17220\xi^2(\psi_x^2-\psi_y^2)^2\cos\theta\cos 4\phi\sin^3\theta + \frac{1}{(2+\Gamma)^2(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)}\right.\\
&\times\left[-76032(14+17\Gamma)(5+2\Gamma(4+\Gamma))(-112-214\Gamma-97\Gamma^2+24(1+\Gamma)(7+5\Gamma)\xi^2)\right.\\
&-9(-352(14+17\Gamma)(8400+\Gamma(25196+\Gamma(25916+\Gamma(10919+1569\Gamma))))+(85468320\\
&+\Gamma(363516752+\Gamma(588408176+\Gamma(450819128+3\Gamma(54400798+7423935\Gamma)))))\xi^2\\
&+3520(1+\Gamma)(2+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)(\psi_x^4+\psi_y^4)-2(-1056(14+17\Gamma)\\
&\times(14000+\Gamma(40388+\Gamma(40148+9\Gamma(1773+223\Gamma))))+(214221280+\Gamma(811856368\\
&+\Gamma(1159899984+\Gamma(763309352+\Gamma(220127246+19036695\Gamma)))))\xi^2+31680(1+\Gamma)\\
&\times(2+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)\psi_x^2\psi_y^2-264(-36(14+17\Gamma)(6160+\Gamma(19172\\
&+\Gamma(20872+83\Gamma(111+16\Gamma))))+(4682720+\Gamma(19309712+\Gamma(30458016+\Gamma(22718368\\
&+\Gamma(7874554+978495\Gamma)))))\xi^2+360(1+\Gamma)(2+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)(\psi_x^2\\
&+\psi_y^2)+5(2+\Gamma)(4(\psi_x^2-\psi_y^2)(66(-12(14+17\Gamma)(168+\Gamma(470+337\Gamma))-(2+3\Gamma)(78680\\
&+\Gamma(220852+\Gamma(198118+55841\Gamma)))\xi^2+72(1+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)\\
&+(88(14+17\Gamma)(616+\Gamma(4010+3\Gamma(1943+735\Gamma)))-3(2+3\Gamma)(2465512+\Gamma(6890292\\
&+\Gamma(6085022+1643967\Gamma)))\xi^2+1584(1+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)(\psi_x^2+\psi_y^2))\cos 2\phi\\
&+21(2+3\Gamma)(14+19\Gamma)\xi^2\cos 2\theta((10732+21264\Gamma+10589\Gamma^2-2112(1+\Gamma)(9+7\Gamma)\xi^2)\\
&\times(\psi_x^4+\psi_y^4)+2(14372+27504\Gamma+12799\Gamma^2-2112(1+\Gamma)(9+7\Gamma)\xi^2)\psi_x^2\psi_y^2-88(-412\\
&-804\Gamma-389\Gamma^2+72(1+\Gamma)(9+7\Gamma)\xi^2)(\psi_x^2+\psi_y^2)-4(\psi_x^2-\psi_y^2)(22(-412-804\Gamma-389\Gamma^2\\
&+72(1+\Gamma)(9+7\Gamma)\xi^2)+(-2396-4824\Gamma-2473\Gamma^2+528(1+\Gamma)(9+7\Gamma)\xi^2)(\psi_x^2+\psi_y^2))\\
&\left.\left.\times\cos 2\phi))\right]\sin 2\theta\right]+P_0\left[\frac{3}{4(2+\Gamma)(2+3\Gamma)^2\xi}((6+11\Gamma+3\Gamma^2+4(1+\Gamma)(3+2\Gamma)\xi^3)\right.\\
&\left.+(3+\Gamma)(2+3\Gamma)(\psi_x^2+\psi_y^2))\sin\theta\right], \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
u^{(2)}_{q_e,\phi} =& \frac{1}{33792(2+\Gamma)(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)}5\xi(\psi_x^2-\psi_y^2)\sin\theta\left[(132(12(14+17\Gamma)(168+\Gamma(470\right.\\
&+337\Gamma))+(2+3\Gamma)(37912+\Gamma(90848+\Gamma(65216+13015\Gamma)))\xi^2+180(1+\Gamma)(2+3\Gamma)(9+7\Gamma)\\
&\times(14+19\Gamma)\xi^4)+(-176(14+17\Gamma)(616+\Gamma(4010+3\Gamma(1943+735\Gamma)))+(2+3\Gamma)(8353688\\
&+\Gamma(21352972+\Gamma(17033794+4151585\Gamma)))\xi^2+7920(1+\Gamma)(2+3\Gamma)(9+7\Gamma)(14+19\Gamma)\xi^4)\\
&\times(\psi_x^2+\psi_y^2)+7(2+3\Gamma)(14+19\Gamma)\xi^2(132(4-42\Gamma-67\Gamma^2+36(1+\Gamma)(9+7\Gamma)\xi^2)+(51332\\
&+85848\Gamma+28111\Gamma^2+1584(1+\Gamma)(9+7\Gamma)\xi^2)(\psi_x^2+\psi_y^2))\cos 2\theta)\sin 2\phi-861(2+\Gamma)\\
&\left.\times(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)\xi^2(\psi_x^2-\psi_y^2)\sin^2\theta\sin 4\phi\right]. \qquad (26)
\end{aligned}
$$

The expressions of $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{E}_n$, $\mathscr{F}_n$, $\mathscr{H}_n$, $\mathscr{I}_n$, $\mathscr{J}_n$, and $\mathscr{K}_n$ in velocity, pressure, displacement and deformation till $O(\alpha)$ are obtained as

$$
\mathscr{A}_1 = \frac{3+\psi_x^2+\psi_y^2}{2}, \mathscr{A}_2 = -\frac{4+\psi_x^2+\psi_y^2}{8}, \mathscr{A}_3 = -\frac{7(\psi_x^2+\psi_y^2)}{8}, \mathscr{A}_4 = \frac{7(\psi_x^2-\psi_y^2)}{4}, \qquad (27)
$$

$$\mathscr{B}_1 = -\frac{1}{4} - \frac{11(\psi_x^2+\psi_y^2)}{64}\,, \tag{28}$$

$$\mathscr{E}_1 = -\frac{3[32(-1+\Gamma^2)+(-44+40\Gamma+39\Gamma^2)(\psi_x^2+\psi_y^2)]}{64(2+\Gamma)(2+3\Gamma)}, \mathscr{F}_1 = -\frac{3+\Gamma}{4+2\Gamma}\mathscr{A}_1\,, \tag{29}$$

$$\mathscr{F}_2 = \frac{192(3+5\Gamma+2\Gamma^2)+(932+1160\Gamma+363\Gamma^2)(\psi_x^2+\psi_y^2)}{128(2+\Gamma)(2+3\Gamma)}\,, \tag{30}$$

$$\begin{aligned}\mathscr{H}_1 =& \frac{1}{384(2+\Gamma)(2+3\Gamma)}\left[216(5+2\Gamma(4+\Gamma))+9(100+\Gamma(88+7\Gamma))(\psi_x^2+\psi_y^2)\right.\\ &\left.+4[-3(10+\Gamma(44+21\Gamma))\psi_x^4+4(35+34\Gamma+6\Gamma^2)\psi_x^2\psi_y^2-3(10+\Gamma(44+21\Gamma))\psi_y^4]\right]\,,\end{aligned} \tag{31}$$

$$\mathscr{H}_2 = 3\mathscr{H}_1,\ \mathscr{H}_3 = \frac{5(\psi_x^2-\psi_y^2)\left(9+8(\psi_x^2+\psi_y^2)\right)}{192}\,, \tag{32}$$

$$\begin{aligned}\mathscr{H}_4 =& \frac{1}{135168(2+\Gamma)(2+3\Gamma)}\left[-76032(5+2\Gamma(4+\Gamma))+3(-13100+\Gamma(102632+62643\Gamma))\psi_x^4\right.\\ &-2(347500+\Gamma(308504+43221\Gamma))\psi_x^2\psi_y^2+3(-13100+\Gamma(102632+62643\Gamma))\psi_y^4-396(1520\\ &\left.+\Gamma(1784+281\Gamma))(\psi_x^2+\psi_y^2)\right]\,,\end{aligned} \tag{33}$$

$$\begin{aligned}\mathscr{H}_5 =& \frac{1}{11264(2+\Gamma)(2+3\Gamma)}\left[-19008(5+2\Gamma(4+\Gamma))+(-4540+\Gamma(69064+40551\Gamma))\psi_x^4\right.\\ &-6(23580+\Gamma(21112+3033\Gamma))\psi_x^2\psi_y^2+(-4540+\Gamma(69064+40551\Gamma))\psi_y^4-132(1000\\ &\left.+\Gamma(1128+167\Gamma))(\psi_x^2+\psi_y^2)\right]\,,\end{aligned} \tag{34}$$

$$\begin{aligned}\mathscr{H}_6 =& \frac{1}{45056(2+\Gamma)(2+3\Gamma)}\left[175((-604+\Gamma(904+735\Gamma))\psi_x^4-2(1844+\Gamma(1576+195\Gamma))\psi_x^2\psi_y^2\right.\\ &\left.+(-604+\Gamma(904+735\Gamma))\psi_y^4-132(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right]\,,\end{aligned} \tag{35}$$

$$\begin{aligned}\mathscr{H}_7 =& \frac{1}{8448(2+\Gamma)(2+3\Gamma)}\left[5(\psi_x^2-\psi_y^2)(99(384+\Gamma(568+113\Gamma))-2(-4846+\Gamma(10108+7503\Gamma))\right.\\ &\left.\times(\psi_x^2+\psi_y^2))\right]\,,\end{aligned} \tag{36}$$

$$\mathscr{H}_8 = \frac{175(\psi_x^2-\psi_y^2)(33(4+\Gamma)(4+5\Gamma)-2(-74+\Gamma(116+93\Gamma))(\psi_x^2+\psi_y^2))}{2816(2+\Gamma)(2+3\Gamma)}\,, \tag{37}$$

$$\begin{aligned}\mathscr{H}_9 =& \frac{525(\psi_x^2-\psi_y^2)^2}{5632}\,,\ \mathscr{H}_{10} = \frac{1}{540672(2+\Gamma)(2+3\Gamma)}\left[5(-3(2756+\Gamma(81544+44835\Gamma))\psi_x^4\right.\\ &+2(174652+\Gamma(121208+2685\Gamma))\psi_x^2\psi_y^2-3(2756+\Gamma(81544+44835\Gamma))\psi_y^4+14256(4+\Gamma)\\ &\left.\times(4+5\Gamma)(\psi_x^2+\psi_y^2))\right]\,,\end{aligned} \tag{38}$$

$$\begin{aligned}\mathscr{H}_{11} =& \frac{1}{360448(2+\Gamma)(2+3\Gamma)}\left[25(-(1252+\Gamma(70088+38955\Gamma))\psi_x^4+2(52148+\Gamma(36712\right.\\ &\left.+1095\Gamma))\psi_x^2\psi_y^2-(1252+\Gamma(70088+38955\Gamma))\psi_y^4+4224(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right]\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\mathscr{H}_{12} =& \frac{1}{180224(2+\Gamma)(2+3\Gamma)}\left[175((652-\Gamma(7144+4263\Gamma))\psi_x^4+2(6788+\Gamma(5128+339\Gamma))\psi_x^2\psi_y^2\right.\\ &\left.+(652-\Gamma(7144+4263\Gamma))\psi_y^4+528(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right]\,,\end{aligned} \tag{40}$$

$$\mathscr{H}_{13} = -\frac{8575(3\psi_x^4+2\psi_x^2\psi_y^2+3\psi_y^4)}{32768}\,, \tag{41}$$

$$\mathscr{H}_{14} = \frac{1}{67584(2+\Gamma)(2+3\Gamma)} \left[125(\psi_x^2-\psi_y^2)(-792(4+\Gamma)(4+5\Gamma)+(6052+\Gamma(24776+11667\Gamma)) \times(\psi_x^2+\psi_y^2))\right], \tag{42}$$

$$\mathscr{H}_{15} = \frac{525(\psi_x^2-\psi_y^2)\left(-22(4+\Gamma)(4+5\Gamma)+(228+\Gamma(808+369\Gamma))(\psi_x^2+\psi_y^2)\right)}{5632(2+\Gamma)(2+3\Gamma)}, \tag{43}$$

$$\mathscr{H}_{16} = \frac{8575(\psi_x^4-\psi_y^4)}{2048}, \mathscr{H}_{17} = -\frac{151}{4}\mathscr{H}_9, \mathscr{H}_{18} = -\frac{539}{12}\mathscr{H}_9, \mathscr{H}_{19} = -\frac{3(1+\psi_x^2+\psi_y^2)}{2(2+3\Gamma)}, \tag{44}$$

$$\mathscr{H}_{20} = \frac{12+5\psi_x^2+5\psi_y^2}{16+24\Gamma}, \mathscr{H}_{21} = \frac{35(\psi_x^2+\psi_y^2)}{8(2+3\Gamma)}, \mathscr{H}_{22} = \frac{35(-\psi_x^2+\psi_y^2)}{4(2+3\Gamma)}, \tag{45}$$

$$\mathscr{I}_1 = \frac{-2}{5}\mathscr{H}_9, \tag{46}$$

$$\mathscr{I}_2 = \frac{1}{33792(2+\Gamma)(2+3\Gamma)} \left[-38016(5+2\Gamma(4+\Gamma))+9(1340+3\Gamma(3944+2051\Gamma))\psi_x^4-2(76940 +\Gamma(71512+11373\Gamma))\psi_x^2\psi_y^2+9(1340+3\Gamma(3944+2051\Gamma))\psi_y^4-396(480+\Gamma(472+53\Gamma)) \times(\psi_x^2+\psi_y^2)\right], \tag{47}$$

$$\mathscr{I}_3 = \frac{5(\psi_x^2-\psi_y^2)(99\Gamma(8+7\Gamma)+4(199+794\Gamma+372\Gamma^2)(\psi_x^2+\psi_y^2))}{8448(2+\Gamma)(2+3\Gamma)}, \tag{48}$$

$$\mathscr{I}_4 = \frac{1}{22528(2+\Gamma)(2+3\Gamma)} \left[35((-604+\Gamma(904+735\Gamma))\psi_x^4-2(1844+\Gamma(1576+195\Gamma))\psi_x^2\psi_y^2 +(-604+\Gamma(904+735\Gamma))\psi_y^4-132(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right], \tag{49}$$

$$\mathscr{I}_5 = \frac{35(\psi_x^2-\psi_y^2)(-33(4+\Gamma)(4+5\Gamma)+2(-74+\Gamma(116+93\Gamma))(\psi_x^2+\psi_y^2))}{5632(2+\Gamma)(2+3\Gamma)}, \tag{50}$$

$$\mathscr{I}_6 = \frac{211}{15}\mathscr{H}_9, \mathscr{I}_7 = \frac{77}{3}\mathscr{H}_9, \tag{51}$$

$$\mathscr{I}_8 = \frac{1}{135168(2+\Gamma)(2+3\Gamma)} \left[5(-9(4596+\Gamma(14824+6615\Gamma))\psi_x^4+2(25636+(584-9285\Gamma)\Gamma) \times\psi_x^2\psi_y^2-9(4596+\Gamma(14824+6615\Gamma))\psi_y^4+3168(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right], \tag{52}$$

$$\mathscr{I}_9 = \frac{1}{11264(2+\Gamma)(2+3\Gamma)} \left[35((788-\Gamma(2648+1785\Gamma))\psi_x^4+2(3548+\Gamma(2872+285\Gamma))\psi_x^2\psi_y^2 +(788-\Gamma(2648+1785\Gamma))\psi_y^4+264(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right], \tag{53}$$

$$\mathscr{I}_{10} = \frac{8}{7}\mathscr{H}_{13}, \tag{54}$$

$$\mathscr{I}_{11} = -\frac{5(\psi_x^2-\psi_y^2)(264(4+\Gamma)(4+5\Gamma)+(6196+\Gamma(8168+2271\Gamma))(\psi_x^2+\psi_y^2))}{11264(2+\Gamma)(2+3\Gamma)}, \tag{55}$$

$$\mathscr{I}_{12} = \frac{35(\psi_x^2-\psi_y^2)(66(4+\Gamma)(4+5\Gamma)-(-284+\Gamma(488+381\Gamma))(\psi_x^2+\psi_y^2))}{2816(2+\Gamma)(2+3\Gamma)}, \tag{56}$$

$$\mathscr{I}_{13} = -\frac{2}{7}\mathscr{H}_{16}, \mathscr{I}_{14} = \frac{3(1+\psi_x^2+\psi_y^2)}{4(2+3\Gamma)}, \mathscr{I}_{15} = \frac{24+45\psi_x^2+45\psi_y^2}{32(2+3\Gamma)}, \tag{57}$$

$$\mathscr{I}_{16} = \frac{35(\psi_x^2+\psi_y^2)}{32(2+3\Gamma)}, \mathscr{I}_{17} = \frac{65(\psi_x^2-\psi_y^2)}{32(2+3\Gamma)}, \mathscr{I}_{18} = \frac{7}{13}\mathscr{I}_{17}, \tag{58}$$

$$\mathscr{J}_1 = \frac{5(\psi_x^2 - \psi_y^2)(198(40+\Gamma(56+9\Gamma)) - (-1340+\Gamma(6824+4341\Gamma))(\psi_x^2+\psi_y^2))}{8448(2+\Gamma)(2+3\Gamma)}, \tag{59}$$

$$\mathscr{J}_2 = \frac{5(\psi_x^2 - \psi_y^2)(198(72+\Gamma(104+19\Gamma)) - (-3284+\Gamma(9272+6447\Gamma))(\psi_x^2+\psi_y^2))}{8448(2+\Gamma)(2+3\Gamma)}, \tag{60}$$

$$\mathscr{J}_3 = \frac{2}{5}\mathscr{H}_9, \tag{61}$$

$$\mathscr{J}_4 = \frac{25(\psi_x^2 - \psi_y^2)(-396(4+\Gamma)(4+5\Gamma) + (716+\Gamma(7768+4101\Gamma))(\psi_x^2+\psi_y^2))}{16896(2+\Gamma)(2+3\Gamma)}, \tag{62}$$

$$\mathscr{J}_5 = \frac{35(\psi_x^2 - \psi_y^2)(-99(4+\Gamma)(4+5\Gamma) + (344+\Gamma(2272+1149\Gamma))(\psi_x^2+\psi_y^2))}{4224(2+\Gamma)(2+3\Gamma)}, \tag{63}$$

$$\mathscr{J}_6 = \frac{12}{49}\mathscr{H}_{16},\ \mathscr{J}_7 = -\frac{266}{15}\mathscr{H}_9,\ \mathscr{J}_8 = -22\mathscr{H}_9,\ \mathscr{J}_9 = \frac{25(-\psi_x^2+\psi_y^2)}{4(2+3\Gamma)}, \tag{64}$$

$$\begin{aligned}\mathscr{K}_2 = &\frac{1}{22528(2+\Gamma)(2+3\Gamma)}\left[63((-604+\Gamma(904+735\Gamma))\psi_x^4 - 2(1844+\Gamma(1576+195\Gamma))\psi_x^2\psi_y^2\right.\\ &\left.+(-604+\Gamma(904+735\Gamma))\psi_y^4 - 132(4+\Gamma)(4+5\Gamma)(\psi_x^2+\psi_y^2))\right],\end{aligned} \tag{65}$$

$$\mathscr{K}_3 = \frac{20}{9}\mathscr{K}_2,\ \mathscr{K}_4 = \frac{35}{9}\mathscr{K}_2, \tag{66}$$

$$\mathscr{K}_5 = \frac{175(\psi_x^2 - \psi_y^2)(33(4+\Gamma)(4+5\Gamma) - 2(-74+\Gamma(116+93\Gamma))(\psi_x^2+\psi_y^2))}{1408(2+\Gamma)(2+3\Gamma)}, \tag{67}$$

$$\mathscr{K}_6 = \frac{7}{5}\mathscr{K}_5,\ \mathscr{K}_7 = \frac{14}{5}\mathscr{H}_9,\ \mathscr{K}_9 = -\frac{3(1+\psi_x^2+\psi_y^2)}{2(2+3\Gamma)},\ \mathscr{K}_{10} = -\frac{35(\psi_x^2+\psi_y^2)}{16(2+3\Gamma)}, \tag{68}$$

$$\mathscr{K}_{11} = -\frac{175(\psi_x^2-\psi_y^2)}{8(2+3\Gamma)}. \tag{69}$$

The expressions of $\mathscr{L}_n$ in the surface deformation at $O(\alpha^2)$ are obtained as

$$\begin{aligned}\mathscr{L}_1 = &-\frac{1}{2162688(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}[304128(14+17\Gamma)(7840\\ &+\Gamma(33504+\Gamma(53806+\Gamma(40067+\Gamma(13704+1739\Gamma))))) + 3(3828969536+\Gamma(19657251968\\ &+\Gamma(40731330192+\Gamma(43445669056+\Gamma(25023158572+9\Gamma(813558984+94192115\Gamma))))))\psi_x^4\\ &+2(8793555008+\Gamma(39493764224+\Gamma(66758503056+\Gamma(50939754688+\Gamma(14952492076\\ &-3\Gamma(246275624+268856295\Gamma))))))\psi_x^2\psi_y^2 + 3(3828969536+\Gamma(19657251968\\ &+\Gamma(40731330192+\Gamma(43445669056+\Gamma(25023158572+9\Gamma(813558984+94192115\Gamma))))))\psi_y^4\\ &+6336(5509504+\Gamma(27914608+\Gamma(56856960+\Gamma(59270984+\Gamma(33048488\\ &+9208023\Gamma+992088\Gamma^2)))))(\psi_x^2+\psi_y^2)],\end{aligned} \tag{70}$$

$$\mathscr{L}_2 = \frac{1}{1441792(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}[202752(14+17\Gamma)(5600+\Gamma(27360+\Gamma(50066 +\Gamma(42109+\Gamma(15864+2101\Gamma))))) - (2634134720+\Gamma(10174509440+\Gamma(21446075568 +\Gamma(34750482496+\Gamma(35446691908+3\Gamma(5917712968+1100401563\Gamma))))))\psi_x^4 + 6(-598261440 +\Gamma(970209920+\Gamma(10885313744+\Gamma(21189560768+21\Gamma(831256684+308613192\Gamma +42016347\Gamma^2)))))\psi_x^2\psi_y^2 - (2634134720+\Gamma(10174509440+\Gamma(21446075568+\Gamma(34750482496 +\Gamma(35446691908+3\Gamma(5917712968+1100401563\Gamma))))))\psi_y^4 + 5632(2481920+\Gamma(15231920 +\Gamma(36584256+\Gamma(43640392+\Gamma(26832856+7843683\Gamma+812898\Gamma^2)))))(\psi_x^2+\psi_y^2)] , \tag{71}$$

$$\mathscr{L}_3 = -\frac{5}{720896(2+\Gamma)(2+3\Gamma)(14+17\Gamma)}\left[(-477848+\Gamma(3199164+\Gamma(6842534+2783505\Gamma)))\psi_x^4 +2(-2774408+\Gamma(-4182636+\Gamma(-457246+691995\Gamma)))\psi_x^2\psi_y^2 + (-477848+\Gamma(3199164 +\Gamma(6842534+2783505\Gamma)))\psi_y^4 + 4928(-208+\Gamma(-306+\Gamma(109+240\Gamma)))(\psi_x^2+\psi_y^2)\right] , \tag{72}$$

$$\mathscr{L}_4 = -\frac{5}{28}\mathscr{H}_{13}, \mathscr{L}_5 = \frac{1}{270336(2+\Gamma)(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)}\left[5(\psi_x^2-\psi_y^2)(1056(61600 +\Gamma(246476+\Gamma(360128+\Gamma(219889+43092\Gamma)))) + (145457200+\Gamma(459208832 +\Gamma(473450696+\Gamma(169213408+13223499\Gamma))))(\psi_x^2+\psi_y^2))\right] , \tag{73}$$

$$\mathscr{L}_6 = -\frac{1}{22528(2+\Gamma)(2+3\Gamma)(14+17\Gamma)}\left[35(\psi_x^2-\psi_y^2)(88(-208+\Gamma(-306+\Gamma(109+240\Gamma))) +3(-10888+\Gamma(-6812+3\Gamma(5054+3081\Gamma)))(\psi_x^2+\psi_y^2))\right] , \tag{74}$$

$$\mathscr{L}_7 = \frac{5}{28}\mathscr{H}_{16}, \mathscr{L}_8 = -\frac{2819}{1680}\mathscr{H}_9, \mathscr{L}_9 = \frac{385}{48}\mathscr{H}_9, \mathscr{L}_{10} = \frac{1}{(2+3\Gamma)^2}, \mathscr{L}_{11} = -\frac{105(\psi_x^2+\psi_y^2)}{128(2+3\Gamma)}, \tag{75}$$

$$\mathscr{L}_{12} = -\frac{3(64(5+8\Gamma+2\Gamma^2)+3(220+184\Gamma+21\Gamma^2)(\psi_x^2+\psi_y^2))}{128(2+\Gamma)(2+3\Gamma)^2}, \mathscr{L}_{13} = \frac{105(\psi_x^2-\psi_y^2)}{32(2+3\Gamma)} . \tag{76}$$

The expressions of $\mathscr{X}_n$ in the point force till $O(\alpha^2)$ are obtained as

$$\mathscr{X}_1 = \frac{9(-140-360\Gamma-227\Gamma^2+6\Gamma^3+\Gamma^4)}{8(2+\Gamma)(2+3\Gamma)^2(14+19\Gamma)}, \mathscr{X}_2 = \frac{3(-2520-4244\Gamma+234\Gamma^2+1893\Gamma^3+47\Gamma^4)}{32(2+\Gamma)(2+3\Gamma)^2(14+19\Gamma)}, \tag{77}$$

$$\mathscr{X}_3 = -\frac{3047408+14243936\Gamma+27918120\Gamma^2+29116864\Gamma^3+15872575\Gamma^4+3404742\Gamma^5}{448(2+\Gamma)(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}, \tag{78}$$

$$\mathscr{X}_4 = -\frac{142225832+608291292\Gamma+980502694\Gamma^2+715141569\Gamma^3+202539888\Gamma^4}{44352(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}, \tag{79}$$

$$\mathscr{X}_5 = \frac{-26025272-48717452\Gamma+21169226\Gamma^2+75027291\Gamma^3+29247957\Gamma^4}{14784(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}, \tag{80}$$

$$\mathscr{X}_6 = \frac{-4220272+14434336\Gamma+92547880\Gamma^2+140403704\Gamma^3+82322565\Gamma^4+16118532\Gamma^5}{2016(2+\Gamma)(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}. \tag{81}$$

3 Particle translating in the quadratic component of Plane Poiseuille flow

The expressions of A_n, B_n, E_n, F_n, H_n, I_n, J_n, K_n, and L_n in velocity, pressure, displacement, deformation and point force at different orders in α are obtained as

$$A_1 = \frac{3+\psi^2}{2}, A_2 = -\frac{4+\psi^2}{8}, A_3 = -\frac{7\psi^2}{8}, A_4 = \frac{7\psi^2}{4}, B_1 = \left(-\frac{1}{4} - \frac{11\psi^2}{64}\right), \tag{82}$$

$$E_1 = -\frac{3(32(-1+\Gamma^2)+(-2+3\Gamma)(22+13\Gamma)\psi^2)}{64(2+\Gamma)(2+3\Gamma)}, F_1 = -\frac{(3+\Gamma)(3+\psi^2)}{4(2+\Gamma)}, \tag{83}$$

$$F_2 = \frac{192(1+\Gamma)(3+2\Gamma)+(932+\Gamma(1160+363\Gamma))\psi^2}{128(2+\Gamma)(2+3\Gamma)}, \tag{84}$$

$$H_1 = \frac{72(5+2\Gamma(4+\Gamma))+3(100+\Gamma(88+7\Gamma))\psi^2-4(10+\Gamma(44+21\Gamma))\psi^4}{128(2+\Gamma)(2+3\Gamma)}, \tag{85}$$

$$H_2 = 3H_1, H_3 = \frac{5(9\psi^2+8\psi^4)}{192}, \tag{86}$$

$$H_4 = \frac{1}{45056(2+\Gamma)(2+3\Gamma)}\left[-25344(5+2\Gamma(4+\Gamma))-132(1520+\Gamma(1784+281\Gamma))\psi^2+(-13100 +\Gamma(102632+62643\Gamma))\psi^4\right], \tag{87}$$

$$H_5 = \frac{1}{11264(2+\Gamma)(2+3\Gamma)}\left[-19008(5+2\Gamma(4+\Gamma))-132(1000+\Gamma(1128+167\Gamma))\psi^2+(-4540 +\Gamma(69064+40551\Gamma))\psi^4\right], \tag{88}$$

$$H_6 = \frac{175\left(-132(4+\Gamma)(4+5\Gamma)\psi^2+(-604+\Gamma(904+735\Gamma))\psi^4\right)}{45056(2+\Gamma)(2+3\Gamma)}, \tag{89}$$

$$H_7 = \frac{5\psi^2\left(99(384+\Gamma(568+113\Gamma))-2(-4846+\Gamma(10108+7503\Gamma))\psi^2\right)}{8448(2+\Gamma)(2+3\Gamma)}, \tag{90}$$

$$H_8 = \frac{175\psi^2(33(4+\Gamma)(4+5\Gamma)-2(-74+\Gamma(116+93\Gamma))\psi^2)}{2816(2+\Gamma)(2+3\Gamma)}, \tag{91}$$

$$H_9 = \frac{525\psi^4}{5632}, H_{10} = \frac{5\left(14256(4+\Gamma)(4+5\Gamma)\psi^2-3(2756+\Gamma(81544+44835\Gamma))\psi^4\right)}{540672(2+\Gamma)(2+3\Gamma)}, \tag{92}$$

$$H_{11} = \frac{25(4224(4+\Gamma)(4+5\Gamma)\psi^2-(1252+\Gamma(70088+38955\Gamma))\psi^4)}{360448(2+\Gamma)(2+3\Gamma)}, \tag{93}$$

$$H_{12} = \frac{175(528(4+\Gamma)(4+5\Gamma)\psi^2+(652-\Gamma(7144+4263\Gamma))\psi^4)}{180224(2+\Gamma)(2+3\Gamma)}, \tag{94}$$

$$H_{13} = -\frac{539}{64}H_9, H_{14} = \frac{125\psi^2(-792(4+\Gamma)(4+5\Gamma)+(6052+\Gamma(24776+11667\Gamma))\psi^2)}{67584(2+\Gamma)(2+3\Gamma)}, \tag{95}$$

$$H_{15} = \frac{525\psi^2(-22(4+\Gamma)(4+5\Gamma)+(228+\Gamma(808+369\Gamma))\psi^2)}{5632(2+\Gamma)(2+3\Gamma)}, \tag{96}$$

$$H_{16} = \frac{539}{12}H_9, H_{17} = -\frac{151}{4}H_9, H_{18} = -\frac{539}{12}H_9, H_{19} = -\frac{3(1+\psi^2)}{2(2+3\Gamma)}, \tag{97}$$

$$H_{20} = \frac{12+5\psi^2}{16+24\Gamma}, H_{21} = \frac{35\psi^2}{16+24\Gamma}, H_{22} = -\frac{35\psi^2}{8+12\Gamma}, \tag{98}$$

$$I_1 = -\frac{2}{5}H_9, I_2 = \frac{1}{11264(2+\Gamma)(2+3\Gamma)}\left[3\left(-4224(5+2\Gamma(4+\Gamma)) - 44(480+\Gamma(472 +53\Gamma))\psi^2 + (1340+3\Gamma(3944+2051\Gamma))\psi^4\right)\right], \tag{99}$$

$$I_3 = \frac{5\psi^2(99\Gamma(8+7\Gamma)+4(199+794\Gamma+372\Gamma^2)\psi^2)}{8448(2+\Gamma)(2+3\Gamma)}, \tag{100}$$

$$I_4 = \frac{35(-132(4+\Gamma)(4+5\Gamma)\psi^2 + (-604+\Gamma(904+735\Gamma))\psi^4)}{22528(2+\Gamma)(2+3\Gamma)}, \tag{101}$$

$$I_5 = \frac{35\psi^2(-33(4+\Gamma)(4+5\Gamma)+2(-74+\Gamma(116+93\Gamma))\psi^2)}{5632(2+\Gamma)(2+3\Gamma)}, I_6 = \frac{211}{15}H_9, \tag{102}$$

$$I_7 = \frac{77}{3}H_9, I_8 = \frac{5(3168(4+\Gamma)(4+5\Gamma)\psi^2 - 9(4596+\Gamma(14824+6615\Gamma))\psi^4)}{135168(2+\Gamma)(2+3\Gamma)}, \tag{103}$$

$$I_9 = \frac{35(264(4+\Gamma)(4+5\Gamma)\psi^2 + (788-\Gamma(2648+1785\Gamma))\psi^4)}{11264(2+\Gamma)(2+3\Gamma)}, I_{10} = -\frac{77}{8}H_9, \tag{104}$$

$$I_{11} = -\frac{5\psi^2(264(4+\Gamma)(4+5\Gamma)+(6196+\Gamma(8168+2271\Gamma))\psi^2)}{11264(2+\Gamma)(2+3\Gamma)}, \tag{105}$$

$$I_{12} = \frac{35\psi^2(66(4+\Gamma)(4+5\Gamma)+(284-\Gamma(488+381\Gamma))\psi^2)}{2816(2+\Gamma)(2+3\Gamma)}, I_{13} = -\frac{77}{6}H_9, \tag{106}$$

$$I_{14} = \frac{3(1+\psi^2)}{4(2+3\Gamma)}, I_{15} = \frac{24+45\psi^2}{64+96\Gamma}, I_{16} = \frac{35\psi^2}{64+96\Gamma}, I_{17} = \frac{65\psi^2}{64+96\Gamma}, I_{18} = \frac{7}{13}I_{17}, \tag{107}$$

$$J_1 = \frac{5\psi^2(198(40+\Gamma(56+9\Gamma))+(1340-\Gamma(6824+4341\Gamma))\psi^2)}{8448(2+\Gamma)(2+3\Gamma)}, \tag{108}$$

$$J_2 = \frac{5\psi^2(198(72+\Gamma(104+19\Gamma))+(3284-\Gamma(9272+6447\Gamma))\psi^2)}{8448(2+\Gamma)(2+3\Gamma)}, \tag{109}$$

$$J_3 = \frac{2}{5}H_9, J_4 = \frac{25\psi^2(-396(4+\Gamma)(4+5\Gamma)+(716+\Gamma(7768+4101\Gamma))\psi^2)}{16896(2+\Gamma)(2+3\Gamma)}, \tag{110}$$

$$J_5 = \frac{35\psi^2(-99(4+\Gamma)(4+5\Gamma)+(344+\Gamma(2272+1149\Gamma))\psi^2)}{4224(2+\Gamma)(2+3\Gamma)}, J_6 = 11H_9, \tag{111}$$

$$J_7 = -\frac{266}{15}H_9, J_8 = -22H_9, J_9 = -\frac{25\psi^2}{8+12\Gamma}, \tag{112}$$

$$K_1 = -\frac{(1+3\cos 2\theta)}{64(2+\Gamma)(2+3\Gamma)}\left[-72(5+2\Gamma(4+\Gamma)) - 3(100+\Gamma(88+7\Gamma))\psi^2 + 4(10 +\Gamma(44+21\Gamma))\psi^4\right] + \frac{5}{96}\left[\psi^2(9+8\psi^2)\cos 2\phi \sin^2\theta\right], \tag{113}$$

$$K_2 = \frac{63(-132(4+\Gamma)(4+5\Gamma)\psi^2 + (-604+\Gamma(904+735\Gamma))\psi^4)}{22528(2+\Gamma)(2+3\Gamma)}, \tag{114}$$

$$K_3 = \frac{20}{9}K_2, K_4 = \frac{35}{9}K_2, \tag{115}$$

$$K_5 = \frac{175\psi^2(33(4+\Gamma)(4+5\Gamma)-2(-74+\Gamma(116+93\Gamma))\psi^2)}{1408(2+\Gamma)(2+3\Gamma)}, \tag{116}$$

$$K_6 = \frac{7}{5}K_5, K_7 = \frac{14}{5}H_9, \tag{117}$$

$$K_8 = \frac{174\psi^4}{49152}\left[-3(50+105\cos 2\theta+126\cos 4\theta+231\cos 6\theta)+112(35+60\cos 2\theta + 33\cos 4\theta)\cos 2\phi \sin^2\theta - 336(9+11\cos 2\theta)\cos 4\phi \sin^4\theta\right] \tag{118}$$

$$K_9 = -\frac{3(1+\psi^2)}{2(2+3\Gamma)},\, K_{10} = -\frac{35\psi^2}{32+48\Gamma},\, K_{11} = 10K_{10}\,, \tag{119}$$

$$L_1 = -\frac{1}{2162688(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}\left[304128(14+17\Gamma)(7840+\Gamma(33504 +\Gamma(53806+\Gamma(40067+\Gamma(13704+1739\Gamma))))) + 6336(5509504+\Gamma(27914608 +\Gamma(56856960+\Gamma(59270984+\Gamma(33048488+9208023\Gamma+992088\Gamma^2)))))\psi^2 +3(3828969536+\Gamma(19657251968+\Gamma(40731330192+\Gamma(43445669056+\Gamma(25023158572 +9\Gamma(813558984+94192115\Gamma))))))\psi^4\right]\,, \tag{120}$$

$$L_2 = \frac{1}{1441792(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}\left[202752(14+17\Gamma)(5600+\Gamma(27360 +\Gamma(50066+\Gamma(42109+\Gamma(15864+2101\Gamma))))) + 5632(2481920+\Gamma(15231920 +\Gamma(36584256+\Gamma(43640392+\Gamma(26832856+7843683\Gamma+812898\Gamma^2)))))\psi^2 -(2634134720+\Gamma(10174509440+\Gamma(21446075568+\Gamma(34750482496+\Gamma(35446691908 +3\Gamma(5917712968+1100401563\Gamma))))))\psi^4\right]\,, \tag{121}$$

$$L_3 = -\frac{5}{720896(2+\Gamma)(2+3\Gamma)(14+17\Gamma)}\left[4928(-208+\Gamma(-306+\Gamma(109+240\Gamma)))\psi^2 +(-477848+\Gamma(3199164+\Gamma(6842534+2783505\Gamma)))\psi^4\right]\,, \tag{122}$$

$$L_4 = \frac{385}{256}H_9,\, L_5 = \frac{1}{270336(2+\Gamma)(2+3\Gamma)(14+17\Gamma)(14+19\Gamma)}\left[5\psi^2(1056(61600+\Gamma(246476 +\Gamma(360128+\Gamma(219889+43092\Gamma)))) + (145457200+\Gamma(459208832+\Gamma(473450696 +\Gamma(169213408+13223499\Gamma))))\psi^2)\right]\,, \tag{123}$$

$$L_6 = \frac{1}{22528(2+\Gamma)(2+3\Gamma)(14+17\Gamma)}\left[35\psi^2(-88(-208+\Gamma(-306+\Gamma(109+240\Gamma)))+ 3(10888+\Gamma(6812-3\Gamma(5054+3081\Gamma)))\psi^2)\right]\,, \tag{124}$$

$$L_7 = \frac{385}{48}H_9\,, L_8 = -\frac{2819}{1680}H_9\,, L_9 = \frac{385}{48}H_9,\, L_{10} = \frac{1}{(2+3\Gamma)^2},\, L_{11} = -\frac{105\psi^2}{128(2+3\Gamma)}\,, \tag{125}$$

$$L_{12} = -\frac{960+384\Gamma(4+\Gamma)+9(22+3\Gamma)(10+7\Gamma)\psi^2}{128(2+\Gamma)(2+3\Gamma)^2},\, L_{13} = \frac{105\psi^2}{64+96\Gamma}\,. \tag{126}$$

4 Particle translating in the quadratic component of Hagen-Poiseuille flow

The expressions of $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{E}_n$, $\tilde{F}_n$, $\tilde{H}_n$, $\tilde{I}_n$, $\tilde{J}_n$, $\tilde{K}_n$, and $\tilde{L}_n$ in velocity, pressure, displacement, deformation and point force at different orders in α are obtained as

$$\tilde{A}_1 = \frac{3+2\psi^2}{2}\,, \tilde{A}_2 = -\frac{2+\psi^2}{4}\,, \tilde{A}_3 = -\frac{7\psi^2}{4}\,, \tilde{A}_4 = 0,\, \tilde{B}_1 = -\frac{1}{4} - \frac{11\psi^2}{32}, \tag{127}$$

$$\tilde{E}_1 = -\frac{3(32(-1+\Gamma^2)+2(-2+3\Gamma)(22+13\Gamma)\psi^2)}{64(2+\Gamma)(2+3\Gamma)}\,, \tag{128}$$

$$\tilde{F}_1 = -\frac{(3+\Gamma)(3+2\psi^2)}{4(2+\Gamma)}, \tilde{F}_2 = \frac{96(1+\Gamma)(3+2\Gamma)+(932+\Gamma(1160+363\Gamma))\psi^2}{64(2+\Gamma)(2+3\Gamma)}, \tag{129}$$

$$\tilde{H}_1 = \frac{108(5+2\Gamma(4+\Gamma))+9(100+\Gamma(88+7\Gamma))\psi^2-4(-40+\Gamma(64+51\Gamma))\psi^4}{192(2+\Gamma)(2+3\Gamma)}, \tag{130}$$

$$\tilde{H}_2 = 3\tilde{H}_1, \tilde{H}_3 = 0, \tilde{H}_4 = \frac{1}{16896(2+\Gamma)(2+3\Gamma)}\left[-9504(5+2\Gamma(4+\Gamma))-99(1520+\Gamma(1784+281\Gamma))\psi^2+(-96700+\Gamma(-152+36177\Gamma))\psi^4\right], \tag{131}$$

$$\tilde{H}_5 = \frac{1}{1408(2+\Gamma)(2+3\Gamma)}\left[-2376(5+2\Gamma(4+\Gamma))-33(1000+\Gamma(1128+167\Gamma))\psi^2+(-18820+\Gamma(1432+7863\Gamma))\psi^4\right], \tag{132}$$

$$\tilde{H}_6 = \frac{525\psi^2(-11(4+\Gamma)(4+5\Gamma)+(-204+\Gamma(-56+45\Gamma))\psi^2)}{5632(2+\Gamma)(2+3\Gamma)}, \tag{133}$$

$$\tilde{H}_7 = \tilde{H}_8 = \tilde{H}_9 = 0, \tag{134}$$

$$\tilde{H}_{10} = \frac{17820(4+\Gamma)(4+5\Gamma)\psi^2-5(-41596+\Gamma(30856+32955\Gamma))\psi^4}{67584(2+\Gamma)(2+3\Gamma)}, \tag{135}$$

$$\tilde{H}_{11} = \frac{26400(4+\Gamma)(4+5\Gamma)\psi^2-25(-12724+\Gamma(8344+9465\Gamma))\psi^4}{45056(2+\Gamma)(2+3\Gamma)}, \tag{136}$$

$$\tilde{H}_{12} = \frac{525\psi^2(44(4+\Gamma)(4+5\Gamma)+(620-3\Gamma(56+109\Gamma))\psi^2)}{22528(2+\Gamma)(2+3\Gamma)}, \tilde{H}_{13} = -\frac{8575\psi^4}{4096}, \tag{137}$$

$$\tilde{H}_{14} = \tilde{H}_{15} = \tilde{H}_{16} = \tilde{H}_{17} = \tilde{H}_{18} = 0, \tilde{H}_{19} = -\frac{3+6\psi^2}{4+6\Gamma}, \tilde{H}_{20} = \frac{6+5\psi^2}{8+12\Gamma}, \tag{138}$$

$$\tilde{H}_{21} = \frac{35\psi^2}{8+12\Gamma}, \tilde{H}_{22} = 0, \tag{139}$$

$$\tilde{I}_1 = 0, \tilde{I}_2 = \frac{1}{4224(2+\Gamma)(2+3\Gamma)}\left[-4752(5+2\Gamma(4+\Gamma))-99(480+\Gamma(472+53\Gamma))\psi^2+(-16220+\Gamma(8744+11001\Gamma))\psi^4\right], \tilde{I}_3 = 0, \tag{140}$$

$$\tilde{I}_4 = \frac{105\psi^2(-11(4+\Gamma)(4+5\Gamma)+(-204+\Gamma(-56+45\Gamma))\psi^2)}{2816(2+\Gamma)(2+3\Gamma)}, \tilde{I}_5 = 0, \tilde{I}_6 = 0, \tag{141}$$

$$\tilde{I}_7 = 0, \tilde{I}_8 = \frac{3960(4+\Gamma)(4+5\Gamma)\psi^2-5(3932+\Gamma(33208+17205\Gamma))\psi^4}{16896(2+\Gamma)(2+3\Gamma)}, \tag{142}$$

$$\tilde{I}_9 = \frac{35\psi^2(66(4+\Gamma)(4+5\Gamma)+(1084+(56-375\Gamma)\Gamma)\psi^2)}{1408(2+\Gamma)(2+3\Gamma)}, \tilde{I}_{10} = -\frac{1225\psi^4}{512}, \tag{143}$$

$$\tilde{I}_{11} = \tilde{I}_{12} = \tilde{I}_{13} = 0, \tilde{I}_{14} = \frac{3+6\psi^2}{8+12\Gamma}, \tilde{I}_{15} = \frac{12+45\psi^2}{32+48\Gamma}, \tilde{I}_{16} = \frac{35\psi^2}{32+48\Gamma}, \tag{144}$$

$$\tilde{I}_{17} = \tilde{I}_{18} = 0, \tag{145}$$

$$\tilde{J}_1 = \tilde{J}_2 = \tilde{J}_3 = \tilde{J}_4 = \tilde{J}_5 = \tilde{J}_6 = \tilde{J}_7 = \tilde{J}_8 = \tilde{J}_9 = 0, \tag{146}$$

$$\tilde{K}_1 = -\frac{(1+3\cos 2\theta)}{96(2+\Gamma)(2+3\Gamma)}\left[-108(5+2\Gamma(4+\Gamma))-9(100+\Gamma(88+7\Gamma))\psi^2+4(-40 +\Gamma(64+51\Gamma))\psi^4\right], \tag{147}$$

$$\tilde{K}_2 = \frac{189\psi^2(-11(4+\Gamma)(4+5\Gamma)+(-204+\Gamma(-56+45\Gamma))\psi^2)}{2816(2+\Gamma)(2+3\Gamma)}, \tilde{K}_3 = \frac{20}{9}\tilde{K}_2, \tilde{K}_4 = \frac{35}{9}\tilde{K}_2, \tag{148}$$

$$\tilde{K}_5 = \tilde{K}_6 = \tilde{K}_7 = 0, \tilde{K}_8 = -\frac{175\psi^4}{6144}\left[50+105\cos 2\theta+126\cos 4\theta+231\cos 6\theta\right], \tag{149}$$

$$\tilde{K}_9 = -\frac{3+6\psi^2}{4+6\Gamma}, \tilde{K}_{10} = -\frac{35\psi^2}{16+24\Gamma}, \tilde{K}_{11} = 0, \tag{150}$$

$$\tilde{L}_1 = -\frac{1}{270336(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}[38016(14+17\Gamma)(7840+\Gamma(33504 +\Gamma(53806+\Gamma(40067+\Gamma(13704+1739\Gamma)))))+1584(5509504+\Gamma(27914608 +\Gamma(56856960+\Gamma(59270984+\Gamma(33048488+9208023\Gamma+992088\Gamma^2)))))\psi^2 +(5070115904+\Gamma(24616380032+\Gamma(47238123408+\Gamma(45319190464+\Gamma(22505491948 +5306816424\Gamma+434154555\Gamma^2)))))\psi^4\right], \tag{151}$$

$$\tilde{L}_2 = \frac{1}{180224(2+\Gamma)^2(2+3\Gamma)^2(14+17\Gamma)(14+19\Gamma)}[25344(14+17\Gamma)(5600+\Gamma(27360 +\Gamma(50066+\Gamma(42109+\Gamma(15864+2101\Gamma)))))+1408(2481920+\Gamma(15231920 +\Gamma(36584256+\Gamma(43640392+\Gamma(26832856+7843683\Gamma+812898\Gamma^2)))))\psi^2 +(-1107229760+\Gamma(-1815969920+\Gamma(2802466416+\Gamma(7204549952+\Gamma(4230619796 +3(140791016-54514569\Gamma)\Gamma)))))\psi^4\right], \tag{152}$$

$$\tilde{L}_3 = -\frac{1}{90112(2+\Gamma)(2+3\Gamma)(14+17\Gamma)}\left[6160(-208+\Gamma(-306+\Gamma(109+240\Gamma)))\psi^2 +35(-116152+\Gamma(-35124+\Gamma(228046+124125\Gamma)))\psi^4\right], \tag{153}$$

$$\tilde{L}_4 = \frac{6125\psi^4}{16384}, \tilde{L}_5 = \tilde{L}_6 = \tilde{L}_7 = \tilde{L}_8 = \tilde{L}_9 = 0, \tilde{L}_{10} = \frac{1}{(2+3\Gamma)^2}, \tilde{L}_{11} = -\frac{105\psi^2}{64(2+3\Gamma)}, \tag{154}$$

$$\tilde{L}_{12} = -\frac{96(5+2\Gamma(4+\Gamma))+9(22+3\Gamma)(10+7\Gamma)\psi^2}{64(2+\Gamma)(2+3\Gamma)^2}, \tilde{L}_{13} = 0. \tag{155}$$

5 Passive particle translating in the quadratic component of elliptical Poiseuille flow

The velocity boundary conditions at different orders in α, given in eqn (32)-(35) in the paper, are modified. At leading-order, the modified boundary conditions are obtained as

$$\mathbf{v}^{(0)} = \mathbf{0} \text{ as } \xi = 1, \tag{156}$$

$$\to \mathbf{v}_q - (V_0^{(0)})_p\hat{\mathbf{k}} \text{ as } \xi \to \infty. \tag{157}$$

Here, $\mathbf{v}^{(0)} = \mathbf{v}_d^{(0)} + \mathbf{v}_q - (V_0^{(0)})_p\hat{\mathbf{k}}$. At $O(\alpha)$, it is

$$\mathbf{v}^{(1)} + f^{(1)}\frac{\partial \mathbf{v}^{(0)}}{\partial \xi} - (V_0^{(1)})_p\hat{\mathbf{k}} = \mathbf{0} \tag{158}$$

and at $O(\alpha^2)$, it is

$$\mathbf{v}^{(2)} + f^{(1)}\frac{\partial \mathbf{v}^{(1)}}{\partial \xi} + f^{(2)}\frac{\partial \mathbf{v}^{(0)}}{\partial \xi} + \frac{\left[f^{(1)}\right]^2}{2}\frac{\partial^2 \mathbf{v}^{(0)}}{\partial \xi^2} - (V_0^{(2)})_p\hat{\mathbf{k}} = \mathbf{0}\,. \tag{159}$$

The velocity fields $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are the disturbance fields and decay as $\xi \to \infty$.
The expressions for $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$ and $\mathscr{N}_i$ (with i =1,2,...,13) in surface deformation until $O(\alpha^2)$ are obtained as

$$\mathscr{M}_1 = -\frac{5(-10+17\Gamma)(\psi_x^2+\psi_y^2)}{64(2+3\Gamma)}, \mathscr{M}_2 = -\frac{7}{8}(\psi_x^2+\psi_y^2), \mathscr{M}_3 = \frac{7}{4}(\psi_x^2-\psi_y^2)\,, \tag{160}$$

$$\begin{aligned}\mathscr{N}_1 =& \frac{1}{2162688(2+3\Gamma)^2(14+19\Gamma)}\left[5(-3(4237912+\Gamma(14866340+\Gamma(16607002+5674079\Gamma)))(\psi_x^4\right.\\ &\left.+\psi_y^4)+2(-3094616+\Gamma(3665756+3\Gamma(9068578+7462219\Gamma)))\psi_x^2\psi_y^2)\right]\,,\end{aligned} \tag{161}$$

$$\begin{aligned}\mathscr{N}_2 =& -\frac{1}{4325376(2+3\Gamma)^2(14+19\Gamma)}[25((11844280+\Gamma(33765044+3\Gamma(14910102+9465065\Gamma)))\\ &\times(\psi_x^4+\psi_y^4)+2(10045672+\Gamma(9708092-3\Gamma(5099982+4530781\Gamma)))\psi_x^2\psi_y^2)]\,,\end{aligned} \tag{162}$$

$$\mathscr{N}_3 = \frac{25((2918-84327\Gamma)(\psi_x^4+\psi_y^4)+2(52130-10509\Gamma)\psi_x^2\psi_y^2)}{2162688(2+3\Gamma)}, \mathscr{N}_4 = \frac{6125(3(\psi_x^4+\psi_y^4)+2\psi_x^2\psi_y^2)}{131072}, \tag{163}$$

$$\mathscr{N}_5 = \frac{-25(-294700+\Gamma(-381584+7627\Gamma))(\psi_x^4-\psi_y^4)}{90112(2+3\Gamma)(14+19\Gamma)}, \mathscr{N}_6 = -\frac{35(-2846+3651\Gamma)(\psi_x^4-\psi_y^4)}{67584(2+3\Gamma)}, \tag{164}$$

$$\mathscr{N}_7 = \frac{6125(\psi_x^4-\psi_y^4)}{8192}, \mathscr{N}_8 = -\frac{14095(\psi_x^2-\psi_y^2)^2}{90112}, \mathscr{N}_9 = \frac{6125(\psi_x^2-\psi_y^2)^2}{8192}, \tag{165}$$

$$\mathscr{N}_{10} = \frac{1}{(2+3\Gamma)^2}, \mathscr{N}_{11} = -\frac{105(\psi_x^2+\psi_y^2)}{128(2+3\Gamma)}, \mathscr{N}_{12} = \frac{15(-34+13\Gamma)(\psi_x^2+\psi_y^2)}{128(2+3\Gamma)^2}, \tag{166}$$

$$\mathscr{N}_{13} = \frac{105(\psi_x^2-\psi_y^2)}{64+96\Gamma}\,. \tag{167}$$