**Abbe test**

This is a statistical test aimed at detecting a trend in a series of (independent and normally distributed) values $x_1$, $x_2$,…$x_n$. The null hypothesis is that there is no such trend, i.e. that all values $x_i$ fluctuate around the same mean value $\mu$ (i.e., they have the same mathematical expectations):

$$H_0: \mu_1 = \mu_2 = \ldots = \mu_n = \mu$$

The null hypothesis is tested against the alternative hypothesis stating that a trend is present:

$$\mu_{i+1} \neq \mu_i \text{ for at least some } i \in 1..n$$

To reject the null hypothesis by error means to conclude that a trend exists when it is actually absent. Such error is called α-type error. To accept the null hypothesis by error means to wrongly decide that there is no trend, while actually a trend exists. Such error is called β-type error and characterises the power of the test. The higher is the power (i.e. the lower is the probability of β-type error), the better.

Let us introduce a new parameter

$$r = \sum_{i=1}^{n-1}(x_{i+1} - x_i)^2 \Big/ 2\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{, where } \bar{x} = \sum_{i=1}^{n} x_i \Big/ n$$

Parameter $r$ can be understood as a ratio of two estimates of the mean intensity variance - obtained by differencing the successive $x_i$ values in the numerator and by the commonly used approach to the uncertainty of the mean for non-transient data in the denominator. The latter approach does not account for a trend in the mean values (mathematical expectations) of $x_i$. Consequently, $r$ values below unity are expected in the presence of a trend, and the stronger it is, the lower is the corresponding value of $r$.

At $n \geq 20$ the statistics of $r$ is approximately normal, centred around unity, provided the null hypothesis postulating the absence of trend holds true. In the table below, critical values of $r$ for $n \leq 60$ are given, corresponding to the condition $probability(r \leq r_{critical})$= 0.05, or 5%. If, for observations $x_1$, $x_2$,…$x_n$, we obtain an $r \leq r_{critical}$, this event would be improbable (less then 5% of probability) provided the null hypothesis holds true. Consequently, this hypothesis is rejected and the alternative hypothesis about the presence of a trend is accepted. The error of wrong rejection of the null hypothesis, i.e. $\alpha$-type error, amounts to 5% in this example.

| $n$ | $r_{critical}$ | $n$ | $r_{critical}$ | $n$ | $r_{critical}$ |
|---|---|---|---|---|---|
| **10** | 0.531 | **27** | 0.695 | **44** | 0.758 |
| **11** | 0.548 | **28** | 0.700 | **45** | 0.760 |
| **12** | 0.564 | **29** | 0.705 | **46** | 0.763 |
| **13** | 0.578 | **30** | 0.709 | **47** | 0.765 |
| **14** | 0.591 | **31** | 0.714 | **48** | 0.777 |
| **15** | 0.603 | **32** | 0.718 | **49** | 0.770 |
| **16** | 0.614 | **33** | 0.722 | **50** | 0.772 |
| **17** | 0.624 | **34** | 0.726 | **51** | 0.774 |
| **18** | 0.633 | **35** | 0.729 | **52** | 0.776 |
| **19** | 0.542 | **36** | 0.733 | **53** | 0.778 |
| **20** | 0.650 | **37** | 0.736 | **54** | 0.780 |
| **21** | 0.657 | **38** | 0.740 | **55** | 0.782 |
| **22** | 0.665 | **39** | 0.743 | **56** | 0.784 |
| **23** | 0.671 | **40** | 0.746 | **57** | 0.785 |
| **24** | 0.678 | **41** | 0.749 | **58** | 0.787 |
| **25** | 0.684 | **42** | 0.752 | **59** | 0.789 |
| **26** | 0.689 | **43** | 0.755 | **60** | 0.791 |

For $n>60$, at $\alpha$=5%, the following approximation applies: $r_{critical} = 1 - 1.645\sqrt{\dfrac{n-2}{(n-1)(n+1)}}$

The power of tests for the presence of trend generally increases with the number of observation $n$ and depends on the specific pattern of the trend (linear or not, which slope if linear). The Abbe test belongs to the most power tests for the presence of trend.

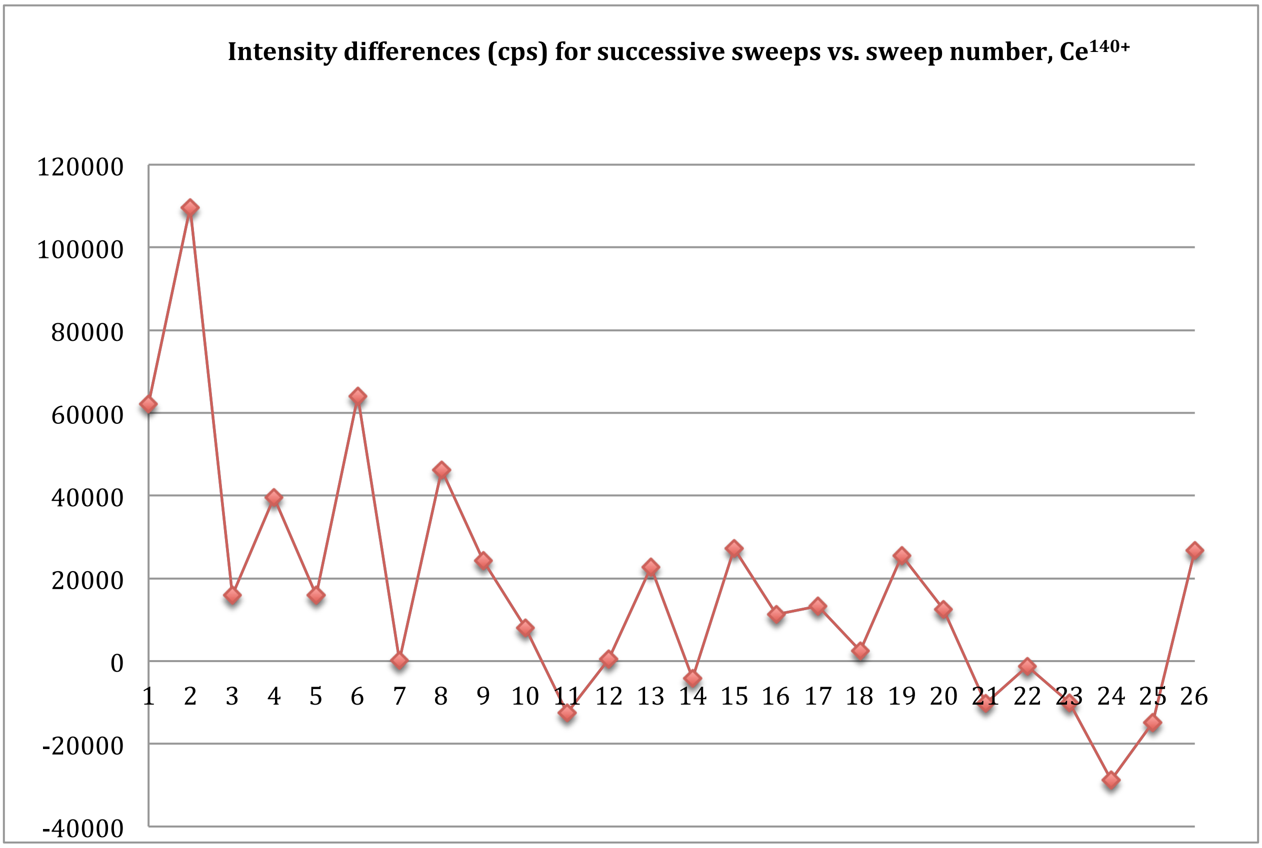
(1) Yu. V. Linnik, *Method of least squares and principles of the theory of observations*. Pergamon Press, New York-Oxford-London-Paris, 1961, 360 p.
(2) S.B. Lemeshko, Measurement Techniques. 2006, **49**, 962-969.

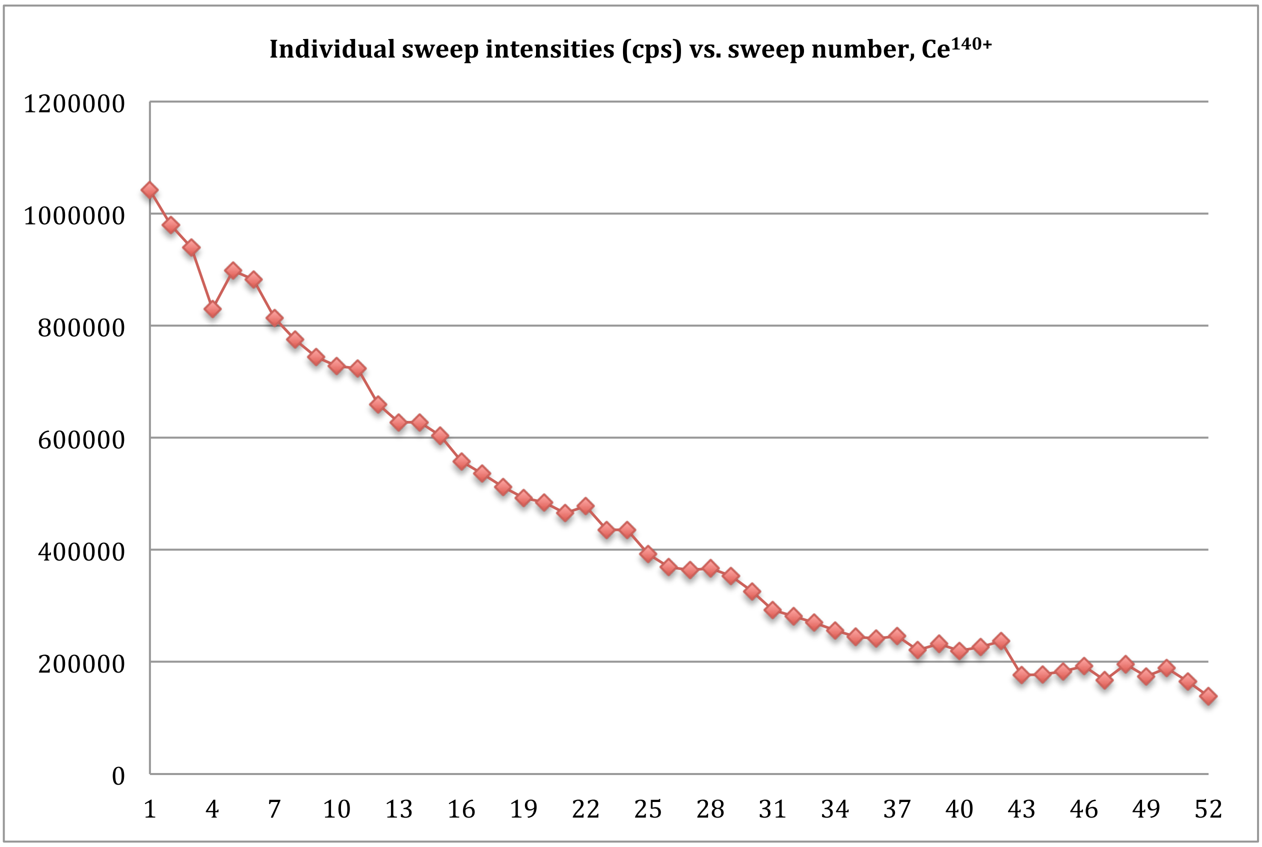
$Ce^{140+}$ signal, spot ablation of the standard glass NIST612 at a repetition rate of 15 Hz, beam size of 75 mkm and on-sample energy density of 4.5 J/cm$^2$

**ABBE TEST DETECTS A TREND IN THE SERIES OF INTENSITY DIFFERENCES**

| Original intensity data and their differencing | | | | | Abbe test for the series of intensity differences | |
|---|---|---|---|---|---|---|
| time | sweep number | intensity (cps) | number of intensity difference | intensity differences $I_{2i-1}-I_{2i}$ | difference of difference $(I_{2i-1}-I_{2i})-(I_{2i-1}-I_{2i})_{mean}$ | square of difference of difference |
| 81.7 | 1 | 1041891 | 1 | 62144 | 2022026430 | 2261448256 |
| 84.2 | 2 | 979747 | 2 | 109699 | 8560252979 | 8796446863 |
| 86.6 | 3 | 939667 | 3 | 15909 | 1607164 | 563652887 |
| 89.0 | 4 | 829968 | 4 | 39651 | 505064279 | 566187153 |
| 91.4 | 5 | 898389 | 5 | 15856 | 1745179 | 2316045703 |
| 93.8 | 6 | 882480 | 6 | 63981 | 2190638773 | 4065329630 |
| 96.2 | 7 | 814392 | 7 | 221 | 287494993 | 2113051278 |
| 98.6 | 8 | 774741 | 8 | 46189 | 841711242 | 476926909 |
| 101.0 | 9 | 743821 | 9 | 24351 | 51460593 | 268609564 |
| 103.4 | 10 | 727965 | 10 | 7961 | 84929285 | 420415295 |
| 105.9 | 11 | 723885 | 11 | -12543 | 883262934 | 170007372 |
| 108.3 | 12 | 659904 | 12 | 496 | 278257506 | 494024246 |
| 110.7 | 13 | 627699 | 13 | 22723 | 30753723 | 721387413 |
| 113.1 | 14 | 627477 | 14 | -4136 | 454246199 | 984371016 |
| 115.5 | 15 | 604149 | 15 | 27239 | 101236503 | 254594932 |
| 117.9 | 16 | 557960 | 16 | 11283 | 34743904 | 4340321 |
| 120.3 | 17 | 536829 | 17 | 13366 | 14524120 | 117447680 |
| 122.7 | 18 | 512479 | 18 | 2529 | 214575048 | 527192454 |
| 125.1 | 19 | 492396 | 19 | 25489 | 69094190 | 166720149 |
| 127.5 | 20 | 484435 | 20 | 12577 | 21157463 | 516682731 |
| 130.0 | 21 | 465383 | 21 | -10153 | 746949693 | 78334115 |
| 132.4 | 22 | 477925 | 22 | -1303 | 341500207 | 76714058 |
| 134.8 | 23 | 435748 | 23 | -10061 | 741929367 | 349142073 |
| 137.2 | 24 | 435252 | 24 | -28747 | 2108988445 | 192691706 |
| 139.6 | 25 | 392008 | 25 | -14865 | 1026714142 | 1731614463 |
| 142.0 | 26 | 369285 | 26 | 26747 | 91590479 | |
| 144.4 | 27 | 363413 | sum | **17177** | **21706454838** | **28233378267** |
| 146.8 | 28 | 367549 | | | | |
| 149.2 | 29 | 353005 | | | | |
| 151.6 | 30 | 325767 | | | | |
| 154.1 | 31 | 292795 | | | | |
| 156.5 | 32 | 281512 | | | | |
| 158.9 | 33 | 270354 | | | | |
| 161.3 | 34 | 256988 | | | | |
| 163.7 | 35 | 244592 | | | | |
| 166.1 | 36 | 242063 | | | | |
| 168.5 | 37 | 246461 | | | | |
| 170.9 | 38 | 220971 | | | | |
| 173.3 | 39 | 232571 | | | | |
| 175.7 | 40 | 219994 | | | | |
| 178.2 | 41 | 226976 | | | | |
| 180.6 | 42 | 237129 | | | | |
| 183.0 | 43 | 176046 | | | | |
| 185.4 | 44 | 177349 | | | | |
| 187.8 | 45 | 182934 | | | | |
| 190.2 | 46 | 192995 | | | | |
| 192.6 | 47 | 167632 | | | | |
| 195.0 | 48 | 196379 | | | | |
| 197.4 | 49 | 174735 | | | | |
| 199.8 | 50 | 189600 | | | | |
| 202.3 | 51 | 165173 | | | | |
| 204.7 | 52 | 138425 | | | | |

| | |
|---|---|
| *r* calculated | **0.650** |
| *r* critical | **0.689** |
| A trend is present if *r* calculated is below this value (95% cnf. level) | |


Intensity differences (cps) for successive sweeps vs. sweep number, Ce140+
120000
100000
80000
60000
40000
20000
0
-20000
-40000
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

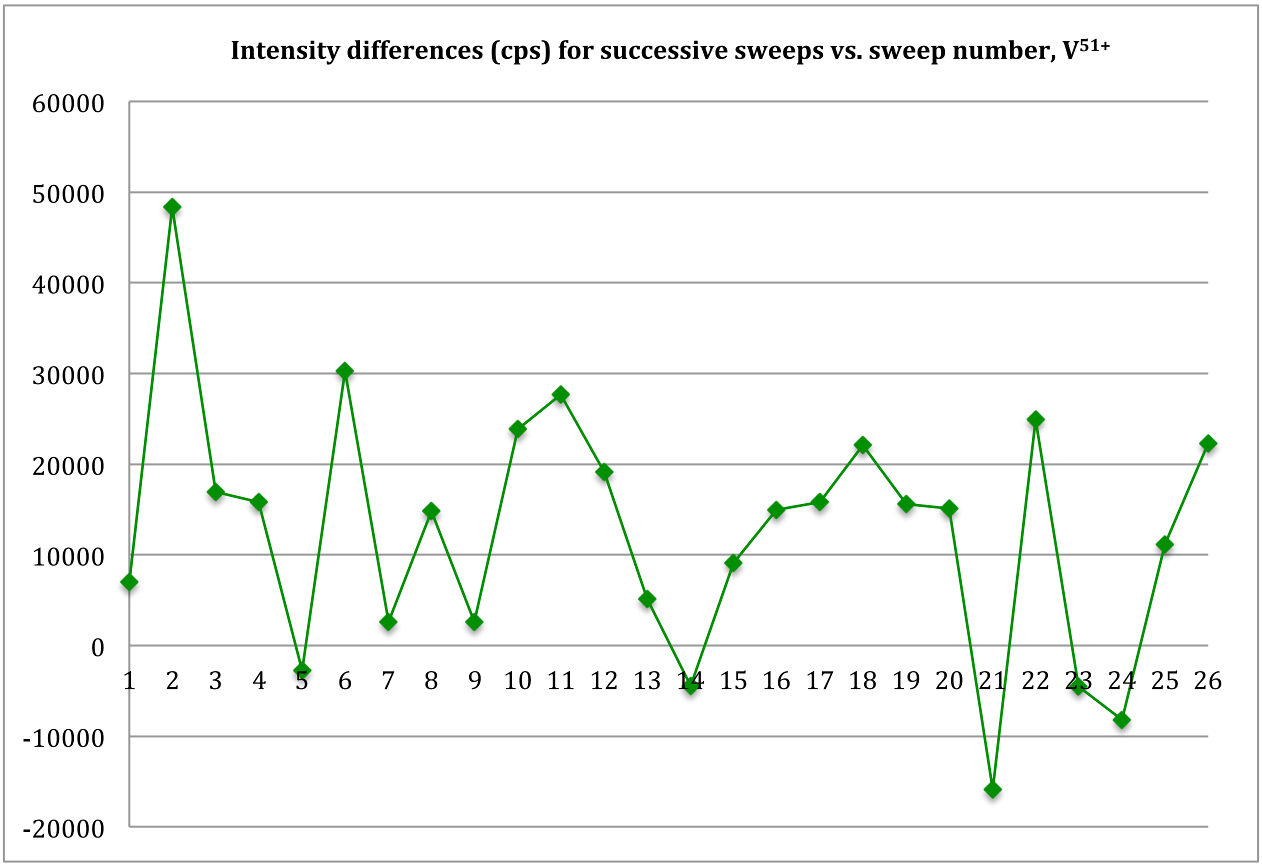

Individual sweep intensities (cps) vs. sweep number, Ce140+
1200000
1000000
800000
600000
400000
200000
0
1 4 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52

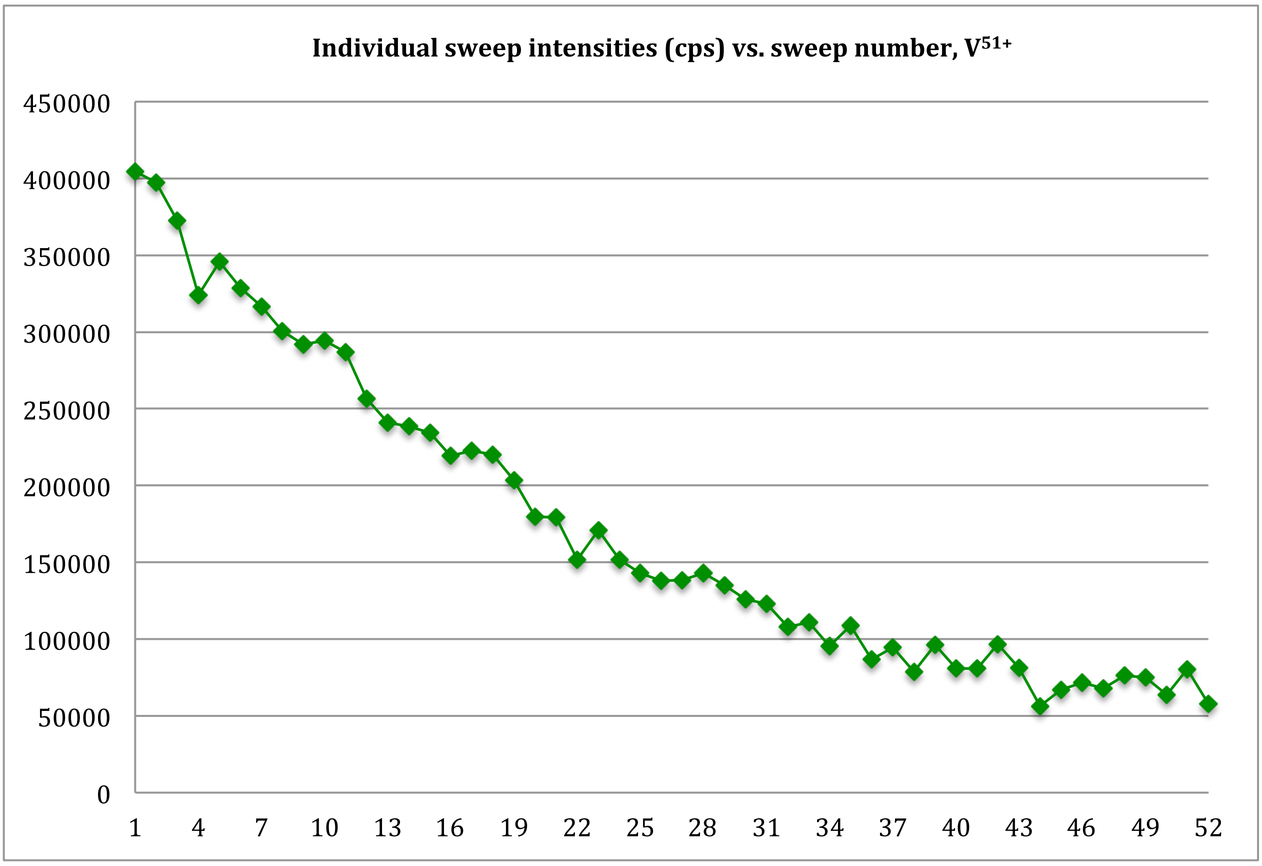
$V^{51+}$ signal, spot ablation of the standard glass NIST612 at a repetition rate of 15 Hz, beam size of 75 mkm and on-sample energy density of 4.5 J/cm$^2$

**ABBE TEST DOES NOT DETECT A TREND IN THE SERIES OF INTENSITY DIFFERENCES**

| Original intensity data and their differencing | | | | | Abbe test for the series of intensity differences | |
|---|---|---|---|---|---|---|
| time | sweep number | intensity (cps) | number of intensity difference | intensity differences $I_{2i-1}-I_{2i}$ | difference of difference $(I_{2i-1}-I_{2i})-(I_{2i-1}-I_{2i})_{mean}$ | square of difference of difference |
| 81.7 | 1 | 404352 | 1 | 7014 | 31981743.81 | 1707342400 |
| 84.2 | 2 | 397338 | 2 | 48334 | 1271975078 | 986085604 |
| 86.6 | 3 | 372250 | 3 | 16932 | 18171119.54 | 1327104 |
| 89.0 | 4 | 323916 | 4 | 15780 | 9676825.385 | 344547844 |
| 91.4 | 5 | 345618 | 5 | -2782 | 238740829.4 | 1095212836 |
| 93.8 | 6 | 328686 | 6 | 30312 | 311266966.8 | 768065796 |
| 96.2 | 7 | 316432 | 7 | 2598 | 101429882.9 | 148644864 |
| 98.6 | 8 | 300652 | 8 | 14790 | 4497621.346 | 147549609 |
| 101.0 | 9 | 291746 | 9 | 2643 | 100525496.3 | 450458176 |
| 103.4 | 10 | 294528 | 10 | 23867 | 125389820.4 | 14227984 |
| 105.9 | 11 | 286732 | 11 | 27639 | 224093702.9 | 72097081 |
| 108.3 | 12 | 256420 | 12 | 19148 | 41974326.15 | 195300625 |
| 110.7 | 13 | 241105 | 13 | 5173 | 56193619.9 | 92833225 |
| 113.1 | 14 | 238507 | 14 | -4462 | 293479397.1 | 184538640.3 |
| 115.5 | 15 | 234223 | 15 | 9122.5 | 12579367.36 | 33907329 |
| 117.9 | 16 | 219433 | 16 | 14945.5 | 5181357.837 | 720801 |
| 120.3 | 17 | 222644 | 17 | 15794.5 | 9767247.664 | 39325441 |
| 122.7 | 18 | 220001 | 18 | 22065.5 | 88289694.76 | 42074682.25 |
| 125.1 | 19 | 203440 | 19 | 15579 | 8466701.019 | 257049 |
| 127.5 | 20 | 179573 | 20 | 15072 | 5773253.769 | 958645444 |
| 130.0 | 21 | 179473 | 21 | -15890 | 815630211.3 | 1664313616 |
| 132.4 | 22 | 151834 | 22 | 24906 | 149738285.9 | 864565812.3 |
| 134.8 | 23 | 170880 | 23 | -4497.5 | 294696975.4 | 13601344 |
| 137.2 | 24 | 151732 | 24 | -8185.5 | 434920196.5 | 374451525.6 |
| 139.6 | 25 | 143203 | 25 | 11165.25 | 2261987.077 | 124807998.1 |
| 142.0 | 26 | 138030 | 26 | 22337 | 93465575.98 | |
| 144.4 | 27 | 138335 | | **12669** | **4750167285** | **10324902830** |
| 146.8 | 28 | 142797 | | | | |
| 149.2 | 29 | 135138 | | | | |
| 151.6 | 30 | 126015.5 | | | | |
| 154.1 | 31 | 122972 | | | | |
| 156.5 | 32 | 108026.5 | | | | |
| 158.9 | 33 | 111065.5 | | | | |
| 161.3 | 34 | 95271 | | | | |
| 163.7 | 35 | 108685 | | | | |
| 166.1 | 36 | 86619.5 | | | | |
| 168.5 | 37 | 94412 | | | | |
| 170.9 | 38 | 78833 | | | | |
| 173.3 | 39 | 96081.5 | | | | |
| 175.7 | 40 | 81009.5 | | | | |
| 178.2 | 41 | 80753.5 | | | | |
| 180.6 | 42 | 96643.5 | | | | |
| 183.0 | 43 | 81310 | | | | |
| 185.4 | 44 | 56404 | | | | |
| 187.8 | 45 | 67209 | | | | |
| 190.2 | 46 | 71706.5 | | | | |
| 192.6 | 47 | 68069 | | | | |
| 195.0 | 48 | 76254.5 | | | | |
| 197.4 | 49 | 74940.5 | | | | |
| 199.8 | 50 | 63775.25 | | | | |
| 202.3 | 51 | 80254 | | | | |
| 204.7 | 52 | 57917 | | | | |

| | |
|---|---|
| *r* calculated | **1.087** |
| *r* critical | **0.689** |
| A trend is present if *r* calculated is below this value (95% cnf. level) | |


Intensity differences (cps) for successive sweeps vs. sweep number, $V^{51+}$
60000
50000
40000
30000
20000
10000
0
-10000
-20000
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26


Individual sweep intensities (cps) vs. sweep number, $V^{51+}$
450000
400000
350000
300000
250000
200000
150000
100000
50000
0
1 4 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52