Flow rate analysis of a surface tension driven passive micropump – supplementary information

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APPENDIX 1
Trigonometric relationships in a spherical cap

The article states that the motion is broken down into two phases: A first phase during which only the contact angle varies and a second phase during which only the wetted radius recedes. The differential equations are found by expressing all the variables of the system, such as the size of the spherical cap and the flow in function of only H, the height of the cap, and constant parameters. As a spherical cap has only 2 degrees of freedom, if a variable is constant it is possible to write all the parameters of the system in function of on variable.

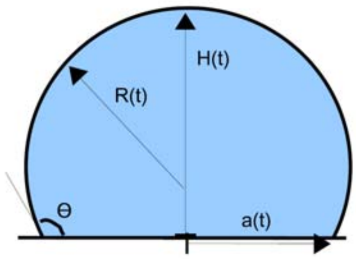

Figure 1 Spherical cap of Radius R and height H.

Lengths

In phase 1, the wetted diameter a is constant, thus using the Pythagorus theorem we write the radius R:

$$R(t) = \frac{H(t)^2 + a^2}{2H(t)} \tag{1}$$

In phase 2, the contact angle Θ is constant and we write the height H, using basic trigonometry:

$$H(t) = (1 - cos\theta)R(t) \tag{2}$$

It is also possible to write the radius in function of the width and the contact angle:

$$a(t) = R(t)sin\theta \tag{3}$$

Volumes

In phase 1, the wetted diameter a is constant, thus using the Pythagorus theorem we write the volume V:

$$V = \frac{\pi}{6}H(t)(3a^2 + H(t)^2) \tag{4}$$

In phase 2, the contact angle Θ is constant and we write the volume V, using basic trigonometry:

$$V = \frac{\pi(2 - 3cos\theta + cos^3\theta)}{3(1 - cos\theta)^3}H^3 = g(\theta)H^3 \tag{5}$$

It is also possible to write the volume in function of the width and the contact angle:

$$V = \frac{\pi(2 - 3cos\theta + cos^3\theta)}{3}\frac{a^3}{sin^3\theta} = f(\theta)\frac{a^3}{sin^3\theta} \tag{6}$$

Flow rate calculations

The Laplace Law for a drop of radius R and surface tension γ is written:

$$P_{inlet\ drop} = \frac{2\gamma}{R(t)} \tag{7}$$

The pressure drop in a rectangular profiled channel of length L_0, width L_D, height L_h and flow rate Q, with $\lambda = L_D/L_h$, for a liquid of viscosity η is written (Washburn law):

$$\Delta P_{in\ channel} = \frac{8\eta L_0 g(\lambda)}{L_D^3 L_h}Q(t) = K_w Q(t) \tag{8}$$

$$with\ g(\lambda) = \begin{cases} \frac{3}{2}\ if\ \lambda > 4.45 \\ \frac{(1+\lambda)^2}{\lambda^2}\ if\ \lambda < 4.45 \end{cases} \tag{9}$$

From (7) and (8) and we obtain the basic differential equation characterizing the motion. Relations (1) (2) allow to write R in function of H wheather the contact angle is constant or the wetted radius is constant:

$$\frac{2\gamma}{R(t)} = K_w Q(t) \tag{10}$$

Moreover the flow rate Q is expressed using the volume of the drop V and (4) if the wetted radius is constant or (5) if the contact angle is constant:

$$Q = -\frac{dV}{dt} = -\frac{\pi}{2}\frac{dH}{dt}(H(t)^2 + a^2) \tag{11}$$

$$Q = = -3g(\theta)\frac{dH}{dt}H(t)^2 \tag{12}$$

First Phase

The drop placed on the substrate starts with a static contact angle Θ_{stat}. The constant parameter in this phase is the wetted radius $a(t)=R_w$. By injecting (1) and (11) into (10), we find a differential equation in H, the height of the drop:

$$\frac{dH}{dt} = K_\lambda \frac{H}{(H^2 + R_w^2)^2} \tag{13}$$

$$with\ K_\lambda = \frac{\pi K_w}{8\gamma} \tag{14}$$

By integrating (13) with the boundary condition being the initial height of the drop H_0 at time $t_0=0$ we can write a function t(H):

$$t - t_0 = -K_\lambda[\frac{H^4 - H_0^4}{4} + R_w^2(H^2 - H_0^2) + R_w^4 ln(\frac{H}{H_0})] \quad (15)$$

Second phase
During the second phase the contact angle does not vary anymore, and the wetted radius recedes. The constant parameter in this phase is the contact angle $\Theta=\Theta_{dyn}$. By injecting (2) and (12) into (10), we find a differential equation in H:

$$\frac{dH}{dt} = \frac{K_\theta}{H^3} \quad (16)$$

$$with\ K_\theta = \frac{2\gamma(1-cos\theta_{dyn})g(\theta_{dyn})}{\pi K_w} \quad (17)$$

We integrate (16) with the the boundary conditions $H_1 = R_w tan(\Theta_{dyn}/2)$ at a time t_1, and find a function H(t):

$$H(t) = [-4K_\theta(t-t_1) + H_1^4]^{1/4} \quad (18)$$

APPENDIX 2
Refill

Let us suppose that the drop has emptied for a set duration and has now a volume V_1, a contact angle Θ_1 and a pressure P_1. The purpose is to refill the drop to continue the motion in phase 1, as it presents interesting characteristics. A simple refill can be done provided the added volume allows the wetted radius of the drop to remain constant. Furthermore a more refined refill can be done for specific conditions of the contact angle, allowing the pressure to be continuous.

Normal Refill
Once refilled the drop cannot have a contact angle larger than Θ_{stat} corresponding to a volume V_{stat}, this limits the volume of the refill V_{add}. The condition is written:

$$V_{add} < V_{stat} - V_1 \quad (19)$$

$$V_{add} < a^3(\frac{f(\theta_{stat})}{sin^3\theta_{stat}} - \frac{f(\theta_1)}{sin^3\theta_1}) \quad (20)$$

Isobar Refill
The new drop, consisting of the old drop and a fresh refill, has a volume V_0, a contact angle Θ_0 and a pressure P_0. The volume of the fresh added drop is V_{add}, therefore we write using (6):

$$V_{add} = V_0 - V_1 \quad (21)$$

$$V_{add} = a^3(\frac{f(\theta_0)}{sin^3\theta_0} - \frac{f(\theta_1)}{sin^3\theta_1}) \quad (22)$$

To ensure an identical pressure in the final drop and the initial drop, the following condition derived from (3) and (7) must be verified:

$$P_0 = P_1 \quad (23)$$

$$\frac{2\gamma sin\theta_0}{a} = \frac{2\gamma sin\theta_1}{a} \quad (24)$$

$$sin\theta_0 = sin\theta_1 \quad (25)$$

$$thus\ \theta_0 = \pi - \theta_1 \quad (26)$$

This can happen only if Θ_0>90 and Θ_1<90, which means that only in the cases of a hydrophobic substrate can this special refill be effectuated. In this case we find the volume to add with (22) and (26):

$$V_{add} = \frac{2\pi a^3}{3sin^3\theta_1}(3cos\theta_1 - cos^3\theta_1) \quad (27)$$