

Emergent behavior in particle-laden microfluidic systems informs strategies for improving cell and particle separations

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Supplementary Information

Statistical Model for the Probability of Nucleation Events

We describe hydrodynamic coupling between particles through a statistical model in which the net force acting on the i^{th} particle consists of an external contribution ($\equiv F_i$) and a contribution from interactions with other particles ($\equiv f_i$). The interaction force on particle i can be expressed in terms of coupling coefficients with neighboring particles as $f_i \equiv \sum_j M_{ij} F_j$, where $M_{ii} = 0$. This is analogous to equation (1b) in the main article, with the matrix \mathbf{M} replacing the hydrodynamic Green's function. Rather than use \mathbf{G}_{HD} explicitly to determine the elements of \mathbf{M} , we make two simplifying assumptions. First, since the far-field hydrodynamic Green's function in a shallow channel has the form of a dipole¹, its contribution to the matrix \mathbf{M} vanishes for a uniform suspension of particles averaged over all angles θ . This justifies truncating particle interactions beyond a certain distance R_0 , characterized by the transition from near- to far-field coupling. The second simplifying assumption we make is that the θ -averaged elements of \mathbf{M} ($m \equiv \langle M_{ij} \rangle_\theta$) follow a Poisson distribution with a mean value of k^1 : $p(m) = ke^{-km}$. If we assume that, on average, each particle experiences the same external force ($F_1 = F_2 = \dots = F$), we can replace m with f/F in this distribution, as we have done in the main article (where the external force is taken to be electrostatic, so that we write $F = F_{\text{ES}}$).

To make more physical sense of the parameter k , we examine the implications of taking a Poisson distribution for $p(m)$. Specifically, this is consistent with a mobility that depends logarithmically on the distance between particles, $\langle M_{ij} \rangle_\theta = -(2/k)\log(r_{ij}/R_0)$. As before, R_0 is the distance at which we truncate the θ -averaged hydrodynamic interactions (i.e. $m = 0$ for $r > R_0$). In addition, the maximum coupling between particles occurs when particles are in contact ($r = 2a$). Defining the corresponding mobility as m_0 gives $k = -(2/m_0)\log(2a/R_0)$. The value of m_0 should be comparable to but less than one, requiring for self-consistency that k be significantly larger than one, and thus that R_0 be at least an order of magnitude larger than the particle radius, a . It seems reasonable to expect R_0 to be a few times the depth of the channel (h), based on the far-field approximation developed in reference 1. Accordingly, as a general guideline, we take $R_0 \approx 2h$, such that $k \sim 4\log[h/a]$. This results in the expression for $p(f)$ presented in the main article. For our typical geometry and particle size, this gives $k \sim 12$.

As discussed in the main article, for a uniform distribution of particles at concentration c , the number of neighbors, N , within a radius R_0 ($\approx 2h$) follows a binomial distribution that is accurately approximated by $p[N] = [(4\pi h^3 c)^N / N!] e^{-4\pi h^3 c}$. The distribution of the interaction force, summed over the contribution of N particles, is then given by $p_N(f) = [p(f)]^N = k[(kf/F)^{N-1} / (N-1)!] e^{-kf/F}$. To determine the probability of having a net mobility f_{net} for a given overall particle concentration c , we sum $p[N] \times p_N(f)$ over N :

$$p(f_{\text{net}}) = e^{-4\pi h^3 c} \left[\delta(f_{\text{net}}/F_{\text{ES}}) + \frac{(4\pi h^3 c \log(h/a)) e^{-4\log(h/a)(f_{\text{net}}/F_{\text{ES}})}}{\sqrt{\pi h^3 c \log(h/a)(f_{\text{net}}/F_{\text{ES}})}} \left[I_1 \left\{ 8\sqrt{\pi h^3 c \log(h/a)(f_{\text{net}}/F_{\text{ES}})} \right\} \right] \right] \quad (\text{S1})$$

Here, I_1 is a modified Bessel function of the first kind ($v = 1$), and the impulse, $\delta(...)$, accounts for the case where $N = 0$. For convenience, from this point on we replace $4\pi h^3 c$ with c ; this dimensionless concentration is equivalent to $\langle N \rangle$, the expected number of interactions a particle shares with its neighbors.

The probability of a particle being retained can be calculated by integrating equation S1 over f_{net} from $F_{\text{HD}}-F_{\text{ES}}$ to infinity, where the external force acting on the particle is F_{ES} , and F_{HD} represents the force that must be overcome to retain a particle at infinite dilution ($c = 0$; note that although the true hydrodynamic

force will vary with particle concentration, the external hydrodynamic force F_{HD} is an independent parameter that can be tuned, for example, by varying the flowrate through the device). As a result, the probability of a particle being retained (*i.e.* a nucleation event) can be calculated from:

$$p_{nuc}(c, F_{ES}, F_{HD}) = 1 - e^{-c} \left[u(F_{HD}/F_{ES} - 1) + c^{1/2} \int_0^{4\sqrt{\log(h/a)(F_{HD}/F_{ES}-1)}} e^{-(x/2)^2} I_1\{c^{1/2}x\} dx \right] \quad (\text{S2})$$

Here, we have introduced the change of variable $x = 4[\log(h/a)f_{net}/F_{ES}]^{1/2}$, and $u(\dots)$ represents the unit step function. The quantity $(F_{HD} - F_{ES})$ describes the degree to which the applied force is sub-threshold; if $(F_{HD} - F_{ES})$ is negative, the external force is strong enough to retain a single particle, resulting in $p_{nuc} = 1$ for all c .

Although the integral in equation (S2) cannot be evaluated in closed form to our knowledge, p_{nuc} can be approximated as a Hill function, as discussed in the main article. Specifically, if $c^{1/2}x$ is small, then it simplifies to $p_{nuc} \approx 1 - e^{-c}[1 + c(1 - e^{-4\log(h/a)(F_{HD}/F_{ES}-1)})] \times u(F_{HD}/F_{ES} - 1)$. Alternatively, for larger values of $c^{1/2}x$, p_{nuc} can be approximated by a general sigmoidal function in the form of a Hill equation, analogous to cooperative binding. To determine the effective value of the constants K_D and n , we expand around the concentration at half saturation, using $p_{nuc}(K_D) = 1/2$ and $p'_{nuc}(K_D) = n / 4K_D$. An approximation valid for all concentrations and forces is $K_D \approx 4\log(h/a)(F_{HD}/F_{ES} - 1) + \log 2$ and $n \approx 2[(4/\pi)\log(h/a)(F_{HD}/F_{ES} - 1) + (\log 2)^2]^{1/2}$, producing equation (5) in the main article when units are restored to K_D by dividing by $4\pi h^3 c$. Figure S1 compares a plot of the nucleation probability evaluated numerically from equation (S2) with the sigmoidal approximation of equations (4-5) in the main article.

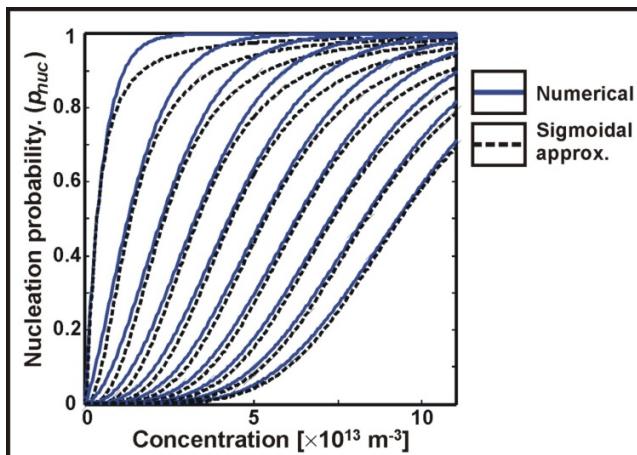


Figure S1: Comparison of nucleation probability as a function of concentration, calculated numerically from equation (S2), or from the sigmoidal approximation of equations (4-5) in the main article.

Nucleation in Heterogeneous Suspensions

The statistical model we presented for particle nucleation (equations 4 in the main article) can be generalized and applied to heterogeneous mixtures by accounting for the interaction force of each component of the mixture separately and then combining each of these contributions into a total interaction force. Specifically, we generalize equation S1 to apply to heterogeneous mixtures by defining the distributions $p_i(f_{int})$ for particles of type i contributing to the net interaction force, f_{int} :

$$p_i(f_{int}) = e^{-4\pi h^3 c_i} \left[\delta(f_{int}/F_{ES}^i) + \frac{4\pi h^3 c_i k e^{-k f_{int}/F_{ES}^i}}{\sqrt{4\pi h^3 c_i k (f_{int}/F_{ES}^i)}} \left[I_1\left\{2\sqrt{4\pi h^3 c_i k (f_{int}/F_{ES}^i)}\right\}\right] \right] \quad (\text{S3})$$

Here, f_{int} denotes the net interaction force, while F_{ES}^i denotes the external electrostatic force applied to the i^{th} particle. We can then find the probability distribution for the total interaction force acting on a particle by convolving the distributions associated with each component of the mixture: $p(f_{int}) = p_1(f_{int}) * p_2(f_{int}) *$

... $*p_n(f_{int})$ (here, n denotes the number of components in the mixture). To calculate the fraction of a given particle type (j) that is retained, we integrate over f_{int} from $(F_{HD}^j - F_{ES}^j)$ to infinity:

$$p_{nuc}^j(c_j, f_j) = \int_{F_{HD}^j - F_{ES}^j}^{\infty} p(f') df' \quad (\text{S4})$$

Equation S4 (with the appropriately defined $p(f)$) is analogous to the result for a homogeneous suspension presented in the main article, generalized for a heterogeneous mixture of particles.

1. Bhattacharya, S., Blawzdziewicz, J. & Wajnryb, E. Far-field approximation for hydrodynamics interactions in parallel-wall geometry. *Journal of Computational Physics* 212, 718-738 (2006).