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Analysis of Natural Convection in a Fluid Heated From Below (the Rayleigh-Bénard problem)

CHEN 304
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Non-isothermal Systems

Example: Natural Convection

Non-uniform temperature causes density change within volume

- → Can create unstable system
- → Flow

Oceans, Atmosphere, Earth's core
→ Lava Lamps

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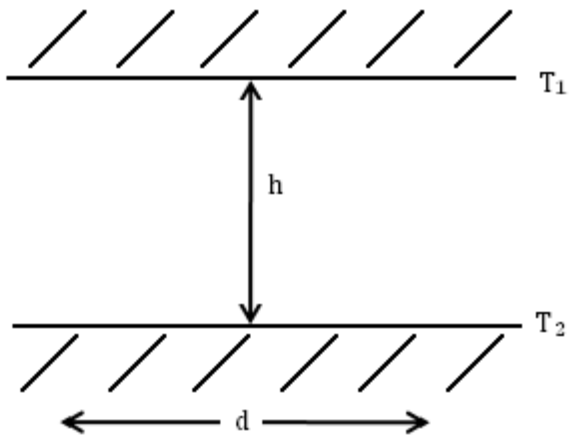
Rayleigh-Bénard Problem

- Fluid confined between two horizontal surfaces at different temperatures



If $T_2 > T_1$, fluid is less dense near bottom
→ creates a "top heavy" arrangement → unstable

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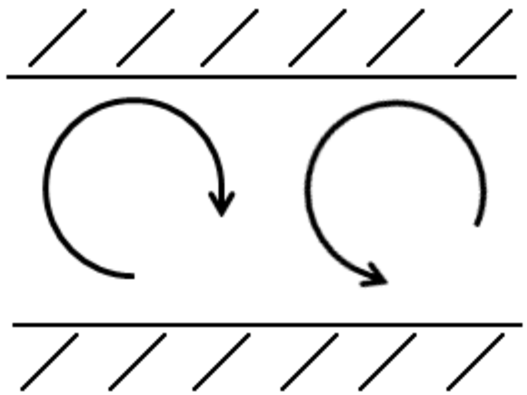
Rayleigh-Bénard Problem

- Under the right conditions, a continuous circulatory flow can be generated



Convection cells with characteristic size and spacing

Conditions for flow should depend on

1) Geometry (h, d)
2) Fluid properties
3) Temperature difference between top and bottom

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How to analyze?

- Conservation of Mass and Momentum
  - (Navier-Stokes)
- Conservation of Energy
  - (accounts for thermal effects, CHEN 323)

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How to analyze?

- Energy equation adds another system of partial differential equations (PDE) that are coupled with the PDE's in Navier-Stokes
- Very difficult to solve
- Must use Computational Fluid Dynamics CFD

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Conservation Equations

Conservation of Mass

$$\nabla \cdot v = 0$$

Conservation of Momentum (Navier-Stokes)

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P + \rho g + \mu \nabla^2 v$$

Conservation of Energy (CHEN 323)

$$\rho\, C_p\left(\frac{\partial T}{\partial t} + v \cdot \nabla T\right) = -k\nabla^2 T$$

- Equations are coupled through velocity
- Also linked by temperature
- Temperature differences are the driving force for fluid motion

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Analyze

- First step: think about body force term
- Changes in density with respect to the base state causes variable in hydrostatic pressure that can become unstable ("top heavy")
- Base state
  - No fluid motion (v = 0)
  - $\rho = \rho_o$

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ANALYZE

- Substitute base state into Navier-Stokes

Hydrostatic Equation

$$0 = -\nabla P_h + \rho_o \mathbf{g}$$

Thus we can express

$$P = P_h + P_d$$

"hydrostatic" pressure that would exist if $\rho = \rho_o$ everywhere

"dynamic" pressure due to density variation in temperature gradient

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ANALYZE

$$\nabla P = \nabla P_h + \nabla P_d$$

$$\nabla P_h = \rho_o \mathrm{g}$$

$$\begin{aligned}\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) &= -(\nabla P_h + \nabla P_d) + \rho \mathrm{g} + \mu \nabla^2 v \\ &= -(\rho_o \mathrm{g} + \nabla P_d) + \rho \mathrm{g} + \mu \nabla^2 v \\ &= -\nabla P_d + (\rho - \rho_o)\mathrm{g} + \mu \nabla^2 v\end{aligned}$$

- Now, we know that changing T also changes ρ (and other fluid properties).
- How do we express mathematically?

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BOUSSINESQ APPROXIMATION

1) Variations in fluid properties with temperature are neglected, except for density
2) Temperature dependence of density is only important in buoyancy term
3) Here, density changes linearly with T

<u>Valid for small temperature variations</u>

$$\rho = \rho_o[1 - \beta(T - T_o)]$$

At base state

β - thermal expansion coefficient (fluid property)

BOUSSINESQ APPROXIMATION

- Substitute into N-S

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P_d + \{[\cancel{\rho_o} - \rho_o \beta (T - T_o)] - \cancel{\rho_o}\}g + \mu \nabla^2 v$$

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P_d + \rho_o \beta (T - T_o)g + \mu \nabla^2 v$$

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NON-DIMENSIONALIZE

$$v^* = \frac{v}{V}$$

$$t^* = \frac{t}{L/V}$$

$$T^* = \frac{T - T_o}{T_2 - T_o} = \theta$$

$$\nabla^* = L\nabla$$

$$P^* = \frac{P}{\left(\frac{\mu V}{L}\right)}$$

$$\nabla^{*2} = L^2\nabla^2$$

SUBSTITUTE

$$\rho_o \frac{V^2}{L}\left(\frac{\partial v^*}{\partial t^*} + v^* \cdot \nabla^* v^*\right) = -\frac{\mu V}{L^2}\nabla^* P^* - \rho_o \beta\Big(\underbrace{[\theta(T_2 - T_o) + \cancel{T_o}]}_{T} - \cancel{T_o}\Big)\mathrm{g} + \frac{\mu V}{L^2}\nabla^{*2} v^*$$

$$\left(\frac{\partial v^*}{\partial t^*} + v^* \cdot \nabla^* v^*\right) = -\underbrace{\frac{\mu}{\rho_o V L}}_{\frac{1}{\mathrm{Re}}}\nabla^* P^* - \underbrace{\frac{\cancel{\rho_o}\beta(T_2 - T_o)\mathrm{L}}{\cancel{\rho_o}V^2}}_{\text{Buoyancy term}}\theta \mathrm{g} + \underbrace{\frac{\mu}{\rho_o V L}}_{\frac{1}{\mathrm{Re}}}\nabla^{*2} v^*$$

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Buoyancy

- Let's look at the buoyancy term

- If gravity acts in the –z direction

$$\mathrm{g} = -\,\mathrm{g}\,\hat{\mathrm{e}}_z$$

$$\text{Buoyancy term} = +\frac{\mathrm{g}\beta(T_2 - T_o)\mathrm{L}}{V^2}\theta\hat{\mathrm{e}}_z$$

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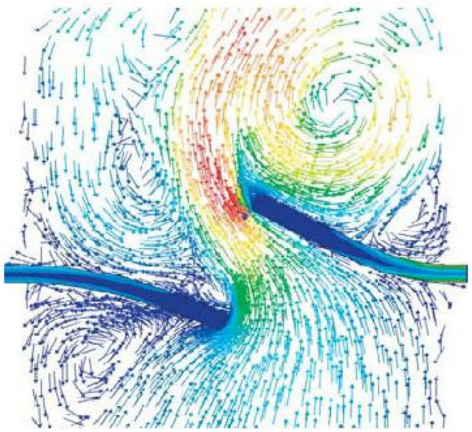
Fluid Properties

Can we identify a characteristic velocity V?



$$V = \frac{L}{T}$$

Choose timescale of momentum diffusion

$$T = \frac{L^2}{\nu}$$

$\frac{\mu}{\rho}$ Kinematic viscosity
units = L^2/T

SUBSTITUTE

$$\text{Buoyancy term} = +\frac{g\beta(T_2 - T_o)L^3}{\nu^2}\theta\hat{e}_z$$

This gives us the following dimensionless equation:

$$\left(\frac{\partial v^*}{\partial t^*} + v^* \cdot \nabla^* v^*\right) = -\frac{1}{Re}\nabla^* P^* + Gr\theta\hat{e}_z + \frac{1}{Re}\nabla^{*2} v^*$$

where

$$Gr = \frac{g\beta(T_2 - T_o)L^3}{\nu^2}$$ Grashof number

- Multiply both sides of the equation by Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\text{momentum diffusion}}{\text{thermal diffusion}}$$

$$\alpha = \frac{k}{\rho C_p}$$

This gives a group

$$\mathrm{GrPr} = \mathrm{Ra} = \frac{g\beta(T_2 - T_o)\mathrm{L}^3}{\nu\ \alpha}$$

Rayleigh number

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- Gr, Ra express the ratio of the destabilizing forces
  - (ΔT, g, h)
- Create density differences opposed by the restoring forces
  - viscous and thermal diffusion that act to smooth out these differences

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EUREKA

- Makes sense!
- Increasing g, ΔT, h makes it unstable
- Increasing ν, α makes it more stable
- This shows us how the basic parameters influence the flow

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ENERGY BALANCE

- Non-dimensionalize the energy balance

$$\rho C_p\left\{\frac{\mathrm{V}}{\mathrm{L}}\frac{\partial}{\partial t^*}[\theta(T_2-T_o)+T_o]+\frac{\mathrm{V}}{\mathrm{L}}\nabla^* v^*[\theta(T_2-T_o)+T_o]\right\} = \frac{k}{L^2}\nabla^{*2}[\theta(T_2-T_o)+T_o]$$

$$\left(\frac{\partial\theta}{\partial t^*}+v^*\cdot\nabla^*\theta\right)=\frac{k}{LV\rho C_p}\nabla^{*2}\theta$$

multiply by $\frac{\mu}{\mu}$

$$\left(\frac{\partial\theta}{\partial t^*}+v^*\cdot\nabla^*\theta\right)=\frac{\mu}{LV\rho}\cdot\frac{k}{\mu C_p}\nabla^{*2}\theta$$

$$\left(\frac{\partial\theta}{\partial t^*}+v^*\cdot\nabla^*\theta\right)=\frac{1}{\mathrm{Re}}\cdot\frac{1}{\mathrm{Pr}}\nabla^{*2}\theta$$

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EQUATIONS

- Now we have a system of 3 coupled PDE's to solve

$$\nabla^* \cdot v^* = 0$$

$$\left(\frac{\partial v^*}{\partial t^*} + v^* \cdot \nabla^* v^*\right) = -\frac{1}{\mathrm{Re}}\nabla^* P^* + \mathrm{Gr}\theta\hat{\mathrm{e}}_z + \frac{1}{\mathrm{Re}}\nabla^{*2} v^*$$

$$\frac{\partial \theta}{\partial t^*} + v^* \cdot \nabla^* \theta = \frac{1}{\mathrm{Re}} \cdot \frac{1}{\mathrm{Pr}}\nabla^{*2}\theta$$

where

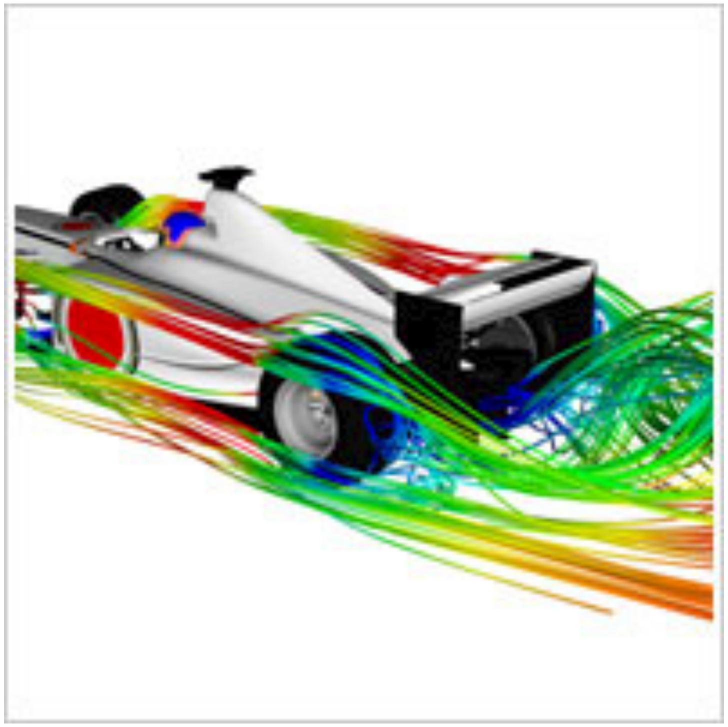
$$\mathrm{Re} = \frac{\rho V L}{\mu}$$

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

$$\mathrm{Gr} = \frac{\mathrm{g}\beta(T_2 - T_o)\mathrm{L}^3}{\nu^2}$$

Additionally
Initial conditions and boundary conditions for velocity and temperature

SOLVE?



Computational Fluid dynamics (CFD)