

Appendix A

The Hessian matrix and its eigenvalues

Near a stationary point (minimum, maximum or saddle), which we take as the origin of coordinates, the free energy F of a foam can be approximated by

$$F = F_0 + \frac{1}{2} x^T H x, \quad (\text{A.1})$$

where F_0 is the free energy at the stationary point, x is a column matrix whose entries x_i ($i=1,2,\dots,n$) are the node displacements relative to the stationary point, and H is the matrix of second derivatives of F , called the Hessian matrix of F , evaluated at the stationary point.⁵⁸ The nature of the stationary point is determined by the eigenvalues of H ¹⁵:

- If all eigenvalues of H are positive, the stationary point is a relative (or local) minimum.
- If all eigenvalues of H are negative, the stationary point is a relative (or local) maximum.
- If H has both positive and negative eigenvalues, the stationary point is a saddle point.

We can gain further insight into the meaning of the eigenvalues (and eigenvectors) of H , which are provided by the Surface Evolver. Start by noting that H is a symmetric matrix: it can then be shown⁵⁹ that (i) all its eigenvalues are real numbers; and (ii) eigenvectors associated with different eigenvalues are orthogonal. It is then possible to construct an orthonormal coordinate system from the eigenvectors of H , called the principal axes system. By the principal axes theorem⁵⁸, in the principal axes coordinate system the quadratic form $x^T H x$ has the form

$$\sum l_i x_i'^2, \quad (\text{A.2})$$

where l_i are the eigenvalues of H and x_i' are the displacements of the foam nodes in the new coordinate system ($i=1,2,\dots,n$). Clearly, l_k is the (quadratic) rate of change of F along the direction of the k th eigenvector of H : if $l_k > 0$ (< 0) then F has a minimum (maximum) along this direction. If $l_k = 0$ no conclusions can be drawn: F would need to be expanded to higher order in x .

There are many detailed technical considerations in the numerical implementation of the Hessian analysis.^{see 29} In the 2d case, for which the latest version of the Evolver uses the constraint

of circular arcs for the edges in the energy minimisation, this may be maintained in perturbing the structure to calculate the Hessian eigenvalues. This restriction to a small basis set (3 degrees of freedom per vertex and one per side) is exact for the lowest eigenvalue, by the variational principle.

Examples of the lowest eigenvalues of the Hessian matrix are provided in Fig. A1 and A2, respectively, for a 3d catenoid soap film and for a 2d flower cluster.

Additional references

58 K. Brakke, *Phil. Trans. R. Soc. Lond. A*, 1995, **354**, 2143-2157.

59 H. Anton and C. Rorres, *Elementary Linear Algebra - Applications Version*, 8th ed., John Wiley, New York, 2000.