Appendix A. Finite deformation of thick-walled cylinders under internal stress

This section reviews the axisymmetric deformation of a cylindrical void in an elastomer, a mode of deformation which we call breathing.[16-20] As illustrated in Fig. 4, in the undeformed state, the elastomer is a tube, of internal radius A and external radius B; a material particle is named after its distance R from the center of the tube. In the deformed state, the internal radius becomes a, the external radius becomes b, and the material particle R moves to a place a distance r from the center of the tube.

The elastomer is taken to deform under the plane-strain conditions, so that the axial stretch is $\lambda_z = 1$. The elastomer is taken to be incompressible, so that the area of the anulus between a and r in the deformed state equals that between A and R in the undeformed state, namely,

$$r^2 - a^2 = R^2 - A^2 . \tag{A1}$$

When the radius of the void a in the deformed state is known, the field of deformation $r(R)$ is determined by (A1). Consequently, the tube may be regarded as a system of a single degree of freedom, a. By definition the circumferential stretch is

$$\lambda_\theta = r / R . \tag{A2}$$

The assumption of incompressibility relates the radial stretch to the circumferential stretch as $\lambda_r \lambda_\theta = 1$, so that

$$\lambda_r = R / r . \tag{A3}$$

The elastomer is taken to obey the neo-Hookean model. The radial component of the true stress σ_r and the circumferential component of the true stress σ_θ relate to the stretches as

$$\sigma_r = G\lambda_r^2 - \pi, \quad \sigma_\theta = G\lambda_\theta^2 - \pi , \tag{A4}$$

where G is the shear modulus of the elastomer, and π is the Lagrange multiplier to enforce the constraint of incompressibility.

The stresses satisfy mechanical equilibrium:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 , \tag{A5}$$

along with the boundary condition $\sigma_r = 0$ when $r = b$. Integrating the first-order ordinary differential equation (A5), and using (A1)-(A4), we obtain that

$$\frac{\sigma_r}{G} = \frac{A^2 - a^2}{2}\left(\frac{1}{r^2} - \frac{1}{b^2}\right) + \log\frac{Rb}{Br} . \tag{A6}$$

The shear modulus G and the radii of the undeformed tube A and B are known parameters. Once the internal radius a of the deformed state is prescribed, (A1) and (A6) specify the field of deformation and stress. In particular, the traction applied on the surface of the void, σ, is given by σ_r when $r = a$. The relation between σ and a is plotted in Fig. 4 for $B/A \to \infty$, and in Fig. 10 for several finite values of B/A. Similar approach can be applied to a spherical void.

Appendix B. Linear perturbation from a state of finite deformation

This section reviews the linear perturbation analysis. [22-28] Finite deformation of a body is governed by the following equations. Each material particle in the body is named after the coordinate $\mathbf{X}$ of the particle when the body is in the undeformed state. In the deformed state, the particle $\mathbf{X}$ moves to a place of coordinate $\mathbf{x}$. The deformation of the body is described by the field $\mathbf{x}(\mathbf{X})$. The deformation gradient is

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K}. \tag{B1}$$

A material model is specified by the energy function $W(\mathbf{F})$. The field of nominal stress $s_{iK}(\mathbf{X})$ relates to the deformation gradient as

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}. \tag{B2}$$

In equilibrium, the nominal stress satisfies that

$$\frac{\partial s_{iK}}{\partial X_K} = 0 \tag{B3}$$

inside the body, and

$$s_{iK} N_K = T_i \tag{B4}$$

on the surface of the body. Here $\mathbf{N}$ is the unit vector normal to the surface of the body in the undeformed state, and $\mathbf{T}$ is the force applied on the surface of the deformed body per unit area in the undeformed state. Equations (B1)-(B4) define a boundary-value problem that determine the field of finite deformation $\mathbf{x}(\mathbf{X})$.

Perturb a state of finite deformation with a field of infinitesimal strain, and each field variable associated with the perturbation is distinguished with a dot. Thus, (B1)-(B4) become

$$\dot{F}_{iK} = \frac{\partial \dot{x}_i(\mathbf{X})}{\partial X_K}, \tag{B5}$$

$$\dot{s}_{iK} = \frac{\partial^2 W(\mathbf{F})}{\partial F_{iK} \partial F_{jL}} \dot{F}_{jL} , \tag{B6}$$

$$\frac{\partial \dot{s}_{iK}}{\partial X_K} = 0 , \tag{B7}$$

$$\dot{s}_{iK} N_K = \dot{T}_i , \tag{B8}$$

This set of equations is linear in the field of perturbation. To apply this set of equations to an elastomer, we need to specify the energy function $W(\mathbf{F})$, and the traction on the surface $\mathbf{T}$, as discussed respectively in the following two paragraphs.

The elastomer is taken to be incompressible, so that

$$\det \mathbf{F} = 1 . \tag{B9}$$

The elastomer is modeled with the neo-Hookean energy function:

$$W(\mathbf{F}) = \frac{G}{2} F_{iK} F_{iK} - \pi(\det \mathbf{F} - 1). \tag{B10}$$

where G is the shear modulus, and $\pi(\mathbf{X})$ the Lagrange multiplier to enforce the constraint of incompressibility. Inserting (B10) into (B2), we obtain the stress-strain relation:

$$s_{iK} = G F_{iK} - \pi H_{iK} . \tag{B11}$$

In reaching (B11), we have used an identity for any matrix $\mathbf{F}$:

$$\frac{\partial \det \mathbf{F}}{\partial F_{iK}} = H_{iK} \det \mathbf{F} , \tag{B12}$$

where $H_{iK} F_{jK} = \delta_{ij}$ and $H_{iK} F_{iL} = \delta_{KL}$. Perturbing $H_{iK} F_{jK} = \delta_{ij}$, we obtain that $\dot{H}_{iK} = -H_{iL} H_{jK} \dot{F}_{jL}$. The perturbation of the condition of incompressibility (B9) is

$$H_{iK} \dot{F}_{iK} = 0 . \tag{B13}$$

The perturbation of the stress-strain relation (B11) is

$$\dot{s}_{iK} = G \dot{F}_{iK} + \pi H_{iL} H_{jK} \dot{F}_{jL} - \dot{\pi} H_{iK} . \tag{B14}$$

For the problem described in this paper, the internal tension σ is prescribed, so that the boundary condition is

$$s_{iK} N_K dA = \sigma n_i da \, , \tag{B15}$$

The deformation changes a material element of area $\mathbf{N}dA$ in the undeformed body to an element of area $\mathbf{n}da$ in the deformed body. Here $\mathbf{N}$ is the unit vector normal to the element in the undeformed body, and $\mathbf{n}$ is the unit vector normal to the element in the deformed body. Recall a geometric identity

$$F_{iK} n_i da = \det(\mathbf{F}) N_K dA \, . \tag{B16}$$

When the field of finite deformation is perturbed by a field of infinitesimal strain, the internal tension σ is kept constant. Consequently, the perturbation of the boundary-condition (B15) is

$$\dot{s}_{iK} N_K = -\sigma H_{iL} H_{jM} N_M \dot{F}_{jL} \, . \tag{B17}$$

Equations (B5), (B7), (B13), (B14) and (B17) constitute an eigenvalue problem of spectrum of solutions. Each solution represents a mode of perturbation: the associated eigenvalue σ represents a critical internal tension, and the associated eigenfields $\dot{\mathbf{x}}(\mathbf{X})$ and $\dot{\pi}(\mathbf{X})$ represent the incremental fields superimposed on those of the finite deformation.

In the above, the independent variable is the coordinate $\mathbf{X}$ of the material particle in the undeformed body. The independent variable can also be the coordinate of the material particle in the deformed body, prior to the perturbation. The above equations can be rewritten by a change of variable. Write the displacement associated with the perturbation as

$$\mathbf{u}(\mathbf{x}) = \dot{\mathbf{x}}(\mathbf{X}) . \tag{B18}$$

By a change of variable, the perturbation of the deformation gradient (B5) becomes

$$L_{ij} = \frac{\partial u_i(\mathbf{x})}{\partial x_j} \, . \tag{B19}$$

where $L_{ij}(\mathbf{x}) = H_{jK} \dot{F}_{iK}$. The condition of incompressibility (B13) becomes

$$L_{ii} = 0\ . \tag{B20}$$

The perturbed stress-strain relation (B14) becomes

$$\Sigma_{ij} = GF_{jK}F_{pK}L_{ip} + \pi L_{ji} - \dot{\pi}\delta_{ij}\ , \tag{B21}$$

where $\Sigma_{ij}(\mathbf{x}) = \dot{s}_{iK}F_{jK}$. The condition of mechanical equilibrium becomes

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = 0 \tag{B22}$$

inside the body, and

$$\Sigma_{ij}n_j = -\sigma L_{ji}n_j \tag{B23}$$

on the surface of the body.

The above general formulation can be specialized to the case when the field of finite deformation is of cylindrical symmetry and under the plane-strain conditions. The incremental displacement due to perturbation has the radial component $u_r(r,\theta)$ and the circumferential component $u_\theta(r,\theta)$. The gradient of incremental displacement is given by

$$L_{rr} = \frac{\partial u_r}{\partial r},\ \ L_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta},\ \ L_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r},\ \ L_{\theta r} = \frac{\partial u_\theta}{\partial r}\ . \tag{B24}$$

The condition of incompressibility becomes

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0\ . \tag{B25}$$

The finite deformation takes the form $F_{rr} = \lambda_r$, $F_{\theta\theta} = \lambda_\theta$, $F_{r\theta} = F_{\theta r} = 0$, so that the incremental stress-strain relation takes the form

$$\begin{aligned} \Sigma_{rr} &= \left(G\lambda_r^2 + \pi\right)L_{rr} - \dot{\pi} \\ \Sigma_{\theta\theta} &= \left(G\lambda_\theta^2 + \pi\right)L_{\theta\theta} - \dot{\pi} \\ \Sigma_{r\theta} &= G\lambda_\theta^2 L_{r\theta} + \pi L_{\theta r} \\ \Sigma_{\theta r} &= G\lambda_r^2 L_{\theta r} + \pi L_{r\theta} \end{aligned} \tag{B26}$$

The incremental equilibrium equations are

$$\begin{aligned}&\frac{\partial \Sigma_{rr}}{\partial r}+\frac{\partial \Sigma_{r\theta}}{r\partial \theta}+\frac{\Sigma_{rr}-\Sigma_{\theta\theta}}{r}=0\\&\frac{\partial \Sigma_{\theta r}}{\partial r}+\frac{\partial \Sigma_{\theta\theta}}{r\partial \theta}+\frac{\Sigma_{\theta r}+\Sigma_{r\theta}}{r}=0\end{aligned} \qquad \text{(B27)}$$

The boundary conditions are

$$\begin{aligned}\Sigma_{rr}&=-\sigma L_{rr}\\ \Sigma_{\theta r}&=-\sigma L_{r\theta}\end{aligned} \qquad \text{(B28)}$$

We set

$$\begin{aligned}u_r(r,\theta)&=f(r)\cos(m\theta),\\ u_\theta(r,\theta)&=g(r)\sin(m\theta),\\ \dot{\pi}(r,\theta)&=k(r)\cos(m\theta),\end{aligned} \qquad \text{(B29)}$$

where $f(r)$, $g(r)$ and $k(r)$ are real functions, and m is a real number. Substituting (B29) into (B24)-(B27), we obtain that

$$\begin{aligned}&Gr^3R^6 f^{IV}-2r^2R^4(R^2+2r^2)Gf'''+rR^2[(C^2(4+m^2)+2C(m^2-1)r^2+(2m^2-5)r^4)G+r^3R^2\pi']f''\\&+[(-C^3(4+m^2)-C^2(m^2+8)r^2+C(4m^2-1)r^4+(2m^2+1)r^6)G+r^3R^4(2\pi'+r\pi'')]f'+\\&(1-m^2)r^3[(C\ (1+m^2)+(m^2-1)r^2)G+R^4\pi'']f=0,\end{aligned} \qquad \text{(B30)}$$

with $C=A^2-a^2$. Moreover, substitution of eqn. (B29) into the boundary condition (B28) yields

$$\begin{aligned}&(m^2-1)f+rf'+r^2f''=0,\\&(1-m^2)r^2R^2(rG+R^2\pi')f+R^2[((3m^2-1)r^4+4m^2r^2C+2(1+m^2)C^2)G+R^2r^2\pi']f'\\&-2R^4r(R^2+r^2)Gf''-R^6r^2Gf'''=0.\end{aligned} \qquad \text{(B31)}$$

The ordinary differential equation (B30), along with the boundary conditions (B31), is solved numerically using the compound matrix method. [42]