Electronic Supplementary Material (ESI) for Soft Matter

Supplementary information :

Viscoelastic SFSR spectra

1 Introduction

The problem is to find approximations for the integral

$$S(\omega) = -\frac{C}{\omega}\int_0^\infty k^3 \operatorname{Im}(\chi(\omega;k))\, e^{-\frac{k^2R^2}{c}} dk \tag{1}$$

where

$$\frac{1}{\chi(\omega;k)} = \sigma k^2 + \frac{G^2k^3}{\omega^2\rho}\left[4\sqrt{1-\frac{\omega^2\rho}{k^2G}} - \left(2-\frac{\omega^2\rho}{k^2G}\right)^2\right] \tag{2}$$

C is a constant for the experiment and the notations $G = G' + iG'' = |G|\exp(i\delta)$ are used. It is convenient to introduce the reduced variables

$$\omega^* = c^{3/4}\sqrt{\frac{\sigma}{\rho R^3}} \tag{3}$$

$$|G|^* = c^{1/2}\frac{\sigma}{R} \tag{4}$$

$$\Omega = \frac{\omega}{\omega^*} \tag{5}$$

$$\Gamma = \frac{|G|}{|G|^*} \tag{6}$$

and to analyse the behavior of the integral (1) in the plane, (Ω, Γ) as these parameters vary easily over orders of magnitude while the surface tension σ does not vary much.

As intermediate parameters, one define

$$\varepsilon = \frac{\Omega}{\Gamma^{1/2}} \tag{7}$$

$$\beta = \frac{\Gamma^{3/2}}{\Omega} \tag{8}$$

Then

$$S(\omega) = -C\frac{\omega\rho R^2}{\sigma|G|}\operatorname{Im}\int_0^\infty \frac{xe^{-\varepsilon^2x^2}}{1+\beta\psi(x)}dx \tag{9}$$

where

$$\psi(x) = xe^{2i\delta}\left[4\sqrt{1-\frac{e^{-i\delta}}{x^2}} - \left(2-\frac{e^{-i\delta}}{x^2}\right)^2\right] \tag{10}$$

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2 Discussion

The problem is to find approximations of

$$J = \int_0^\infty \frac{x e^{-\varepsilon^2 x^2}}{1+\beta\psi(x)} dx \tag{11}$$

for different values of the parameters ε and β. As the imaginary part of J is sought, the leading non purely integer contribution to J is noted J' (more generally the prime symbol is used in the following to express the leading non purely integer term). One may notice the asymptotic behaviors :

$$\psi(x) \simeq -x^{-3} + 4e^{i\delta}x^{-1} + O\left(x^0\right) \text{ for } x \to 0 \tag{12}$$

$$\simeq 2e^{i\delta}x^{-1} + O\left(x^{-3}\right) \text{ for } x \to \infty \tag{13}$$

(in these developments, the first non-real term is kept).
The rigorous uniform asymptotics analysis of equ. (11) with respect to both parameters ε and β is rather cumbersome. The description given below is based on the following remark : except in a very narrow region near $x = 1$, one can approximate the function $\psi(x)$ by the above asymptotic behavior (14) and (15), respectively for $x < 1$ and $x > 1$:

$$\psi(x) \simeq -x^{-3} + 4e^{i\delta}x^{-1} \text{ for } x < 1 \tag{14}$$

$$\simeq 2e^{i\delta}x^{-1} \text{ for } x > 1 \tag{15}$$

Hence, to evaluate the quantity $1 + \beta\psi(x)$, it is convenient to define x^* as the solution [1] of $|\psi(x^*)| = \beta^{-1}$. For $\beta \ll 1$, one find $x^* \simeq \beta^{1/3}$ and for $\beta \gg 1$, $x^* \simeq 2\beta$. In both cases,

$$\beta\,|\psi(x)| \ll 1 \text{ for } x \gg x^* \tag{16}$$

$$\beta\,|\psi(x)| \gg 1 \text{ for } x \ll x^* \tag{17}$$

Moreover, the Gaussian factor imposes the practical integration region in equ. (11) : only the points $0 < x < \varepsilon^{-1}$ contributes to the integral. Then, one can discuss several situations according to the values of the parameters.

2.1 $\varepsilon \gg 1$ or $\Gamma \ll \Omega^2$

In the case $\varepsilon \gg 1$, the development (14) is relevant to represent ψ as the contributions arising from $x > 1$ are negligible. Then $x^* \simeq \beta^{1/3}$. It remains to compare both scales $\beta^{1/3}$ and ε^{-1}

1. When the sample is rather elastic, other solutions may exist near $x = 1$. They correspond to the Rayleigh elastic waves. They are not expected to give important contribution to the spectrum.

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2.1.1 $\beta\varepsilon^3 \gg 1$ or $\Omega > 1$

Here, $x^* \gg \varepsilon^{-1}$, then the term 1 is everywhere negligible as compared to $\beta\psi(x)$. Then, the integral for J can be simplified :

$$J \simeq -\frac{1}{\beta}\int_0^\infty \frac{x^4 e^{-\varepsilon^2 x^2}}{1-4x^2 e^{i\delta}}dx \tag{18}$$

$$\simeq -\frac{1}{\beta}\int_0^\infty x^4\left(1+4x^2e^{i\delta}\right)e^{-\varepsilon^2x^2}dx \tag{19}$$

and thus the leading non real term is

$$J' \simeq -\frac{15\sqrt{\pi}}{4}e^{i\delta}\varepsilon^{-7}\beta^{-1} \tag{20}$$

2.1.2 $\beta\varepsilon^3 \ll 1$ or $\Omega < 1$

Here, the complete denominator term has to be used

$$J \simeq \frac{1}{\beta}\int_0^\infty \frac{x^4 e^{-\varepsilon^2 x^2}}{\frac{x^3}{\beta}-1+4x^2 e^{i\delta}}dx \tag{21}$$

$$\simeq \frac{\beta^{2/3}}{3}\int_0^\infty \frac{v^{2/3}\exp\left(-\varepsilon^2\beta^{2/3}v^{2/3}\right)}{v-1+4\beta^{2/3}e^{i\delta}v^{2/3}}dv \tag{22}$$

As in this region $\beta \ll 1$, a pole at $v \simeq 1$ exists just below the integration path. It gives rise to the leading non-real contribution to the integral :

$$J' \simeq \frac{i\pi}{3}\beta^{2/3}\exp\left(-\varepsilon^2\beta^{2/3}\right) \tag{23}$$

$$\simeq \frac{i\pi}{3}\beta^{2/3} \tag{24}$$

2.2 $\varepsilon \ll 1$ or $\Gamma \gg \Omega^2$

A decomposition of the integral (11) can be written as :

$$J = \int_0^1 \frac{x\,dx}{1+\beta\left(-x^{-3}+4e^{i\delta}x^{-1}\right)} + \int_1^\infty \frac{xe^{-\varepsilon^2x^2}dx}{1+2\beta e^{i\delta}x^{-1}} \tag{25}$$

$$= J_- + J_+ \tag{26}$$

(the Gaussian term is dropped in the first integral as it is nearly equal to 1). To simplify further the problem, one may compare x^* and 1 :

2.2.1 $\beta \ll 1$ or $\Gamma \ll \Omega^{2/3}$

When $\beta \ll 1$, one have $0 < x^* < 1$ and the calculation of the first integral is similar to equ. (21),

$$J'_- \simeq \frac{i\pi}{3}\beta^{2/3} \tag{27}$$

In the second integral, $2\beta x^{-1} \ll 1$,

$$J_+ \simeq \int_1^\infty x\left(1 - 2\beta e^{i\delta}x^{-1}\right)e^{-\varepsilon^2 x^2}dx \tag{28}$$

and thus

$$J'_+ \simeq -2\beta e^{i\delta}\int_1^\infty e^{-\varepsilon^2 x^2}dx \tag{29}$$

$$\simeq -\sqrt{\pi}\beta\varepsilon^{-1}e^{i\delta} \tag{30}$$

Finally,

$$J' \simeq \frac{i\pi}{3}\beta^{2/3} - \sqrt{\pi}\beta\varepsilon^{-1}e^{i\delta} \tag{31}$$

Amplitude of both terms is similar when $\beta \simeq \varepsilon^3$, i.e. when $\Gamma = \Omega^{4/3}$, then

$$J' \simeq \frac{i\pi}{3}\beta^{2/3} \text{ for } \Gamma < \Omega^{4/3} \tag{32}$$

$$\simeq -\sqrt{\pi}\beta\varepsilon^{-1}e^{i\delta} \text{ for } \Gamma > \Omega^{4/3} \tag{33}$$

2.2.2 $\beta \gg 1$ or $\Gamma \gg \Omega^{2/3}$

Here, $x^* = 2\beta \gg 1$. To evaluate J_-, one can consider that the term 1 in the denominator is negligible, just as in eq. (18) :

$$J_- \simeq -\frac{1}{\beta}\int_0^1 \frac{x^4}{1 - 4x^2 e^{i\delta}}dx \tag{34}$$

and thus

$$J_- = O\left(\beta^{-1}\right) \tag{35}$$

After splitting J_+ into two terms

$$J_+ = \int_1^{x^*} \frac{xe^{-\varepsilon^2 x^2}dx}{1 + 2\beta e^{i\delta}x^{-1}} + \int_{x^*}^\infty \frac{xe^{-\varepsilon^2 x^2}dx}{1 + 2\beta e^{i\delta}x^{-1}} \tag{36}$$

a simplification of the integrals takes place when leading terms are kept :

$$J_+ \simeq \frac{e^{-i\delta}}{2\beta}\int_1^{2\beta} x^2 e^{-\varepsilon^2 x^2} dx + \int_{2\beta}^{\infty} x\left(1 - 2\beta e^{i\delta} x^{-1}\right) e^{-\varepsilon^2 x^2} dx \tag{37}$$

$$J'_+ \simeq 4\beta^2 \left[e^{-i\delta}\int_0^1 t^2 e^{-z^2 t^2} dt - e^{i\delta}\int_1^{\infty} e^{-z^2 t^2} dt\right] \tag{38}$$

$$\tag{39}$$

where $z = 2\beta\varepsilon = 2\Gamma$ and the lower limit for the first integral has been extended down to 0 without changing appreciably its value. Now, two situations are to be discussed :

- For $\Gamma \gg 1$ the first integral dominates. Watson lemma leads to the result

$$J'_+ \simeq \frac{\sqrt{\pi} e^{-i\delta}}{8\beta\varepsilon^3} \tag{40}$$

- For $\Gamma \ll 1$ the second integral dominates and gives

$$J'_+ \simeq -\frac{\sqrt{\pi}\beta e^{i\delta}}{\varepsilon} \tag{41}$$

In both cases the J_- contribution is negligible, thus

$$J' \simeq \frac{\sqrt{\pi} e^{-i\delta}}{8\beta\varepsilon^3} \text{ for } \Gamma \gg 1 \tag{42}$$

$$\simeq -\frac{\sqrt{\pi}\beta e^{i\delta}}{\varepsilon} \text{ for } \Gamma \ll 1 \tag{43}$$

3 Results

Results obtained for the different regions can be collected in the plane (Ω, Γ). The figure below gives a summary of the results in this plane.

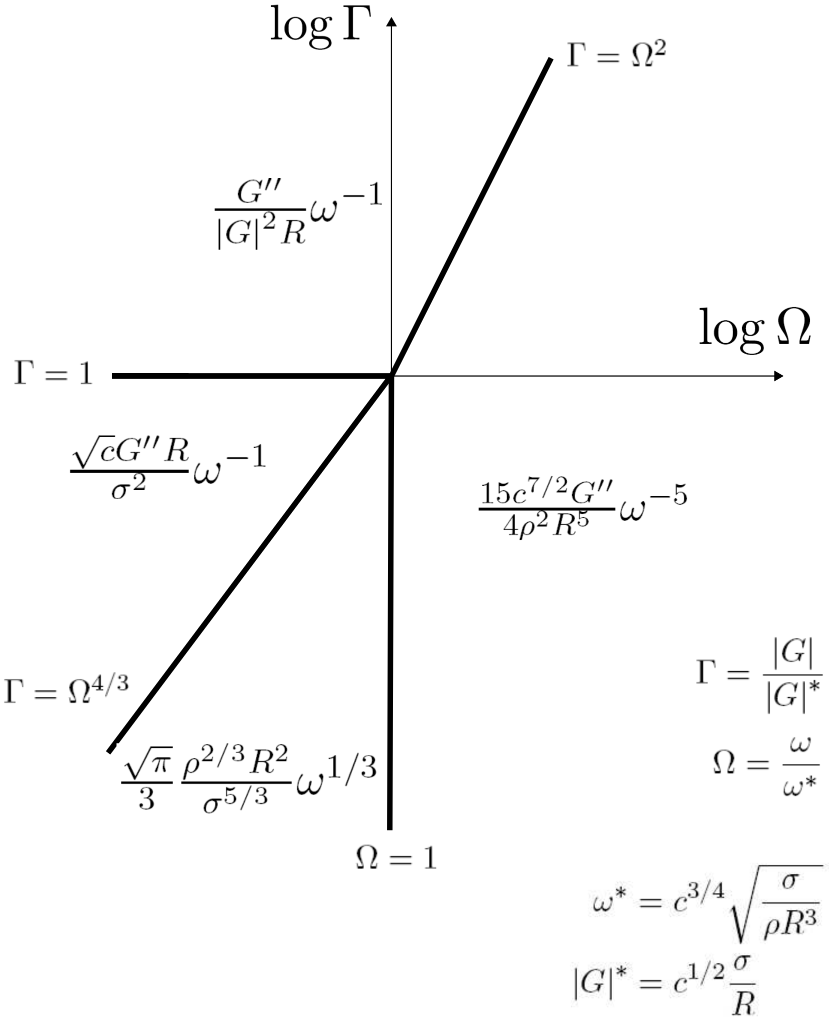

FIGURE 1 – Asymptotic behaviors for the SFSR spectrum in the plane $(\log \Gamma,\ \log \Omega)$.