Electronic Supplementary Material (ESI) for Soft Matter

Electronic Supplementary Information
Cooperative Self-Propulsion of Active and Passive Rotors

October 10, 2011

1 Averaged flow field of a single active rotor

1.1 Dynamics of a single active rotor

We first establish the instantaneous position of the rotor when a/ℓ is not neglected. The instantaneous velocity $\mathbf{v}_{\text{C}}$ of a rotor is given by Eq. (9a) in the paper. In the $(\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\nu}}^{\perp})$ frame, it reads

$$\mathbf{v}_{\text{C}} = \frac{F}{\zeta}\left[\left(1 - \frac{3a}{2\ell} + \frac{a^3}{2\ell^3}\right)\cos\phi\ \hat{\boldsymbol{\nu}} + \left(1 - \frac{3a}{4\ell} - \frac{a^3}{4\ell^3}\right)\sin\phi\ \hat{\boldsymbol{\nu}}^{\perp}\right] \equiv \text{v}_0\hat{\mathbf{v}} \tag{1}$$

where we have introduced the velocity magnitude $\text{v}_0 = \|\mathbf{v}_{\text{C}}\|$ and the associated unit vector $\hat{\mathbf{v}} = \cos\phi'\ \hat{\boldsymbol{\nu}} + \sin\phi'\ \hat{\boldsymbol{\nu}}^{\perp}$ with

$$\phi' = \arctan\left(\tan\ \phi\ \frac{1 - \frac{3a}{4\ell} - \frac{a^3}{4\ell^3}}{1 - \frac{3a}{2\ell} + \frac{a^3}{2\ell^3}}\right) \xrightarrow[a\to 0]{} \phi \tag{2}$$

The instantaneous position of the rotor on its circular trajectory can then be written as

$$\mathbf{r}_{\text{C}}(t) = -\epsilon R_0 \hat{\mathbf{v}}^{\perp}(t) \tag{3}$$

where $\hat{\mathbf{v}}^{\perp} = -\sin\phi'\hat{\boldsymbol{\nu}} + \cos\phi'\hat{\boldsymbol{\nu}}^{\perp}$ and $\epsilon = \boldsymbol{\omega}\cdot\hat{\mathbf{z}}/\omega_0$ is $+1$ if the rotation is counterclockwise and -1 if it is clockwise, and

$$R_0 = \frac{\text{v}_0}{\omega_0} = \frac{4a^2}{3\ell\sin\phi}\sqrt{\left(1 - \frac{3a}{2\ell} + \frac{a^3}{2\ell^3}\right)^2\cos^2\phi + \left(1 - \frac{3a}{4\ell} - \frac{a^3}{4\ell^3}\right)^2\sin^2\phi} \tag{4}$$

1.2 Flow field

The flow field of a rotor at a distance $\mathbf{r}$ from the center of its circular trajectory at any given time is

$$u_i(\mathbf{r},t) = -\frac{1}{8\pi\eta}\Big[\ell\hat{\nu}_k\mathcal{H}_{ijk}\big(\mathbf{r}-\mathbf{r}_{\text{C}}(t)\big) + \frac{\ell^2}{2}\hat{\nu}_k\hat{\nu}_l\partial_{kl}\mathcal{O}_{ij}\big(\mathbf{r}-\mathbf{r}_{\text{C}}(t)\big) + \frac{\ell^3}{6}\hat{\nu}_k\hat{\nu}_l\hat{\nu}_m\partial_{klm}\mathcal{O}_{ij}\big(\mathbf{r}-\mathbf{r}_{\text{C}}(t)\big)\Big]F_j + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{5}$$

Assuming without loss of generality that the rotor rotates clockwise and expanding in R_0/r, we can rewrite the flow field as (the derivatives of the Oseen tensor are now implicitely taken in $\mathbf{r}$)

$$\begin{aligned} u_i(\mathbf{r},t) = -\frac{1}{8\pi\eta}\Big[&\ell\hat{\nu}_k\big(\mathcal{H}_{ijk} - R_0\hat{v}_l^{\perp}\partial_l\mathcal{H}_{ijk} + \frac{R_0^2}{2}\hat{\nu}_k\hat{v}_l^{\perp}\hat{v}_m^{\perp}\partial_{lm}\mathcal{H}_{ijk}\big) \\ &+\frac{\ell^2}{2}\hat{\nu}_k\hat{\nu}_l\big(\partial_{kl}\mathcal{O}_{ij} - R_0\hat{v}_l^{\perp}\partial_{klm}\mathcal{O}_{ij}\big) + \frac{\ell^3}{6}\hat{\nu}_k\hat{\nu}_l\hat{\nu}_m\partial_{klm}\mathcal{O}_{ij}\Big]F_j + \mathcal{O}\left(\frac{1}{r^5}\right) \end{aligned} \tag{6}$$

Electronic Supplementary Material (ESI) for Soft Matter

We then average over a period of rotation, i.e. over the orientation of the rotor (see appendix A):

$$\mathbf{u}^0(\mathbf{r}) \equiv \big\langle \mathbf{u}(\mathbf{r},t)\big\rangle = -\frac{1}{8\pi\eta}\Big\langle \ell F_j \hat{\nu}_k \mathcal{H}_{ijk} + \frac{R_0^2\ell}{2} F_j \hat{\nu}_k \hat{v}_l^\perp \hat{v}_m^\perp \partial_{lm}\mathcal{H}_{ijk} - \frac{R_0\ell^2}{2} F_j \hat{\nu}_k \hat{\nu}_l \hat{v}_l^\perp \partial_{klm}\mathcal{O}_{ij} + \frac{\ell^3}{6} F_j \hat{\nu}_k \hat{\nu}_l \hat{\nu}_m \partial_{klm}\mathcal{O}_{ij}\Big\rangle + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{7}$$

$$= \frac{\ell F\cos\phi}{16\pi\eta}\,\frac{\hat{\mathbf{r}}}{r^2} + \frac{3\ell^3 F}{128\pi\eta}\,\frac{\hat{\mathbf{r}}}{r^4}\Bigg\{\cos\phi\;\hat{\mathbf{r}} + 2\sin\phi\;\hat{\mathbf{r}}^\perp + \frac{R_0}{\ell}\Big[\big(2\sin\delta + \sin(2\phi+\delta)\big)\hat{\mathbf{r}} + \big(4\cos\delta - 2\cos(2\phi+\delta)\big)\hat{\mathbf{r}}^\perp\Big] + \frac{R_0^2}{\ell^2}\Big[\big(2\cos\phi - \cos(\phi+2\delta)\big)\hat{\mathbf{r}} + 2\sin(\phi+2\delta)\hat{\mathbf{r}}^\perp\Big]\Bigg\} + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{8}$$

where $\delta = \phi' - \phi = \mathcal{O}\left(\frac{a}{\ell}\right)$ is the angle between $\mathbf{F}$ and $\mathbf{v}_0$. Note that the quadrupolar terms all vanish when averaged over a rotation due to the fact that they are proportional to the product of three vectors $\hat{\boldsymbol{\nu}}$ that all change sign when the rotor is rotated by π. It is evident from Eq. (4) that, provided ϕ is finite, $R_0/\ell \sim \mathcal{O}(a/\ell)^2$. Hence to leading order in a/ℓ the averaged flow is given by

$$\langle \mathbf{u}(\mathbf{r})\rangle = \frac{\ell F\cos\phi}{16\pi\eta r^2}\hat{\mathbf{r}} + \frac{\ell^3 F}{16\pi\eta r^4}\left(\frac{3}{8}\cos\phi\;\hat{\mathbf{r}} + \frac{3}{4}\sin\phi\;\hat{\mathbf{z}}\times\hat{\mathbf{r}}\right). \tag{9}$$

The first term on the right hand side of Eq. (9) is the dipolar contribution which, as expected, is purely radial for active rotors. The second term describes contributions to octupolar order and contains both radial and azimuthal terms. The corrections due to the finite value of the radius R_0 of the circular trajectory of the center of the rotor give corrections of order $(a/\ell)^2$ and higher to the coefficients of both the radial and azimuthal octupolar flow fields, but do not change their qualitative behavior.

2 Averaged interaction

We now evaluate the hydrodynamic interaction between two rotors, i.e. the first term on the right hand side of the equation of motion (Eq. (17) in the paper)

$$\partial_t \mathbf{r}_1 = \left[\left(1 + \frac{a^2}{6}\nabla^2\right)\mathbf{u}_2\right]_{\mathbf{r}_{C1}} - \frac{1}{\zeta}g(r_{12})\hat{\mathbf{r}}_{12}\,. \tag{10}$$

where $\mathbf{r}_{12} = \mathbf{r}_{C1} - \mathbf{r}_{C2}$. As pointed out in appendix A, to octupolar order the expression for the period averaged interaction can be obtained by neglecting the motion of the rotors due to interactions, i.e. by assuming that each rotor goes around a circle as if it were isolated. The relative position of the two rotors is then given by $\mathbf{r}_{12} = \mathbf{r} - \epsilon_1 R_0 \hat{\mathbf{v}}_1^\perp + \epsilon_2 R_0 \hat{\mathbf{v}}_2^\perp$ where $\mathbf{r} = \langle \mathbf{r}_{12}\rangle$ is the distance between the centers of the circular trajectories of the two rotors (see Fig. 1) and $\epsilon_\alpha = \pm 1$ is the vorticity of the rotor α.

2.1 Averaged flow field experienced by a second rotor

Assuming without loss of generality that $\epsilon_1 = 1$ and noting $\epsilon_2 \equiv \epsilon$, the flow field appearing in Eq. (10) becomes

$$\mathbf{u}(\mathbf{r},t) = \mathbf{u}\big(\mathbf{r} - R_0\hat{\mathbf{v}}_1^\perp(t) + \epsilon R_0 \hat{\mathbf{v}}_2^\perp(t)\big) \tag{11}$$

Expanding in R_0/r yields Eq. (7) provided that we now note $\hat{\mathbf{v}}(t) = \hat{\mathbf{v}}_1(t) - \epsilon\hat{\mathbf{v}}_2(t)$. Again, all the quadrupolar terms vanish over a rotation because they can be written as a sum of terms contracted with three $\boldsymbol{\nu}$'s that all change sign after a π rotation. Note that even though these terms can now depend on

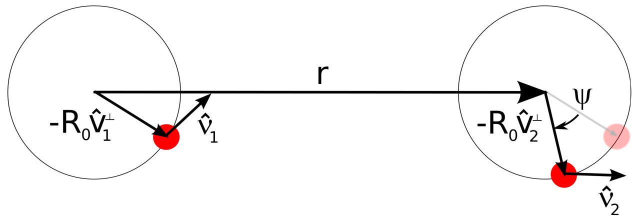

Figure 1: Schematic of the relative position of two rotors over a period of rotation, showing the angle ψ between $\hat{\boldsymbol{\nu}}_1$ and $\hat{\boldsymbol{\nu}}_2$.

both $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$, the result is not affected since $\boldsymbol{\nu}_1$'s and $\mathbf{v}_2$'s rotate at the same rate ω_0. After averaging, we get the equivalent of Eq. (8) for the period averaged flow field felt by the rotor 1:

$$\big\langle \mathbf{u}(\mathbf{r},t)\big\rangle = \mathbf{u}^0(\mathbf{r}) + \frac{3\ell^3 F}{128\pi\eta}\,\frac{\hat{\mathbf{r}}}{r^4}\left(\frac{R_0}{\ell}\Big[Q^{\|}(\psi,\phi,\delta)\hat{\mathbf{r}} + Q^{\perp}(\psi,\phi,\delta)\hat{\mathbf{r}}^{\perp}\Big] + \frac{R_0^2}{\ell^2}\Big[D^{\|}(\psi,\phi,\delta)\hat{\mathbf{r}} + D^{\perp}(\psi,\phi,\delta)\hat{\mathbf{r}}^{\perp}\Big]\right) + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{12}$$

$$\begin{aligned} Q^{\|}(\psi,\phi,\delta) &= \frac{1-3\epsilon}{2}\sin\big[\psi+\epsilon(2\phi+\delta)\big] + \frac{5\epsilon-9}{2}\sin(\psi+\epsilon\delta) \\ Q^{\perp}(\psi,\phi,\delta) &= (5\epsilon-3)\cos\big[\psi+\epsilon(2\phi+\delta)\big] + (2-6\epsilon)\cos(\psi+\epsilon\delta) \end{aligned} \tag{13}$$

$$\begin{aligned} D^{\|}(\psi,\phi,\delta) &= (12\epsilon-13)\cos\big[2\psi-\phi+2\epsilon(\phi+\delta)\big] + (4\epsilon-2)\cos\big[\psi+\epsilon(\phi+2\delta)\big] \\ &\quad + (2\epsilon-4)\cos\big[\psi-\phi-\delta+\epsilon(2\phi+\delta)\big] - 2\cos(\psi-\phi) + 2\cos\phi \\ D^{\perp}(\psi,\phi,\delta) &= (6\epsilon-4)\sin\big[2\psi-\phi+2\epsilon(\phi+\delta)\big] - 4\sin\big[\psi+\epsilon(\phi+2\delta)\big] \\ &\quad + 4(\epsilon-1)\sin(\psi-\phi) + 2(\epsilon-1)\sin(\psi-3\phi-2\delta) \end{aligned} \tag{14}$$

where ψ is the phase between the two rotors and $\mathbf{u}^0(\mathbf{r})$ is the averaged flow field produced at the (static) center of the trajectory of the rotor 1 by the moving rotor 2, calculated in the previous section and given by Eq. (8). The additional terms are due to the fact that rotor 2 is moving as well and contain all the dependance on the relative angle ψ.

2.2 Faxén law

In order to obtain the interaction from the flow field, we finally need to discuss the consequence of the laplacian term in the equation of motion (10). Since the laplacian commutes with the time average, it can be applied on the averaged flow. Taking the laplacian of a term raises its order by two, so to octupolar order only the stresslet in $\mathbf{u}$ contributes:

$$\left[\frac{a^2}{6}\nabla^2\mathbf{u}^0\right]_{\mathbf{r}_{12}} = \frac{a^2}{6}\nabla^2\left(\frac{\ell F}{16\pi\eta r^2}\cos\phi\,\hat{\mathbf{r}}\right) = \frac{\ell a^2 F\cos\phi}{16\pi\eta}\,\frac{\hat{\mathbf{r}}}{r^4} \tag{15}$$

2.3 Averaged interaction

Finally, the period averaged interaction $\mathbf{U}(\mathbf{r}) = \left\langle \left[\left(1+\frac{a^2}{6}\nabla^2\right)\mathbf{u}\right]_{\mathbf{r}_{12}}\right\rangle$ is given by

$$\begin{aligned}\mathbf{U}(\mathbf{r}) = \frac{\ell F\cos\phi}{16\pi\eta}\ \frac{\hat{\mathbf{r}}}{r^2} + \frac{3\ell^3 F}{128\pi\eta}\ \frac{\hat{\mathbf{r}}}{r^4}\Bigg\{ &\cos\phi\ \hat{\mathbf{r}} + 2\sin\phi\ \hat{\mathbf{r}}^{\perp} + \frac{8}{3}\frac{a^2}{\ell^2}\cos\phi\ \hat{\mathbf{r}} \\ &+ \frac{R_0}{\ell}\Big[\big(2\sin\delta + \sin(2\phi+\delta) + Q^{\parallel}(\psi,\phi,\delta)\big)\hat{\mathbf{r}} + \big(4\cos\delta - 2\cos(2\phi+\delta) + Q^{\perp}(\psi,\phi,\delta)\big)\hat{\mathbf{r}}^{\perp}\Big] \\ &\quad + \frac{R_0^2}{\ell^2}\Big[\big(2\cos\phi - \cos(\phi+2\delta) + D^{\parallel}(\psi,\phi,\delta)\big)\hat{\mathbf{r}} + \big(2\sin(\phi+2\delta) + D^{\parallel}(\psi,\phi,\delta)\big)\hat{\mathbf{r}}^{\perp}\Big]\Bigg\} + \mathcal{O}\left(\frac{1}{r^5}\right)\end{aligned} \tag{16}$$

where $Q^{\parallel,\perp}$ and $D^{\parallel,\perp}$ are given by Eqs. (13) and (14).

To leading order in a/ℓ, the interaction between the two rotors does not differ from the averaged flow field given by Eq. (9). Note that regardless of the a/ℓ assumption, the Faxén correction is purely radial and proportional to the dipolar term but with a faster decay, and therefore does not change the qualitative behavior of the interaction. The corrections due to $Q^{\parallel,\perp}$ and $D^{\parallel,\perp}$, on the other hand, introduce a dependance on the relative orientation of the two rotors of the octupolar interaction. These corrections are negligible in our model but could become important for other types of rotors.