Electronic Supplementary Material (ESI) for Soft Matter

Supplementary Information

"The Sticky Elastica"

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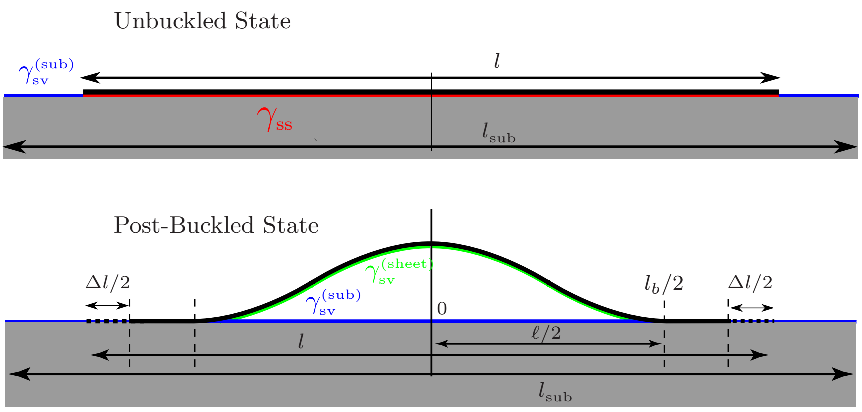
1. Surface Energies



The surface energies per unit area associated with the different interfaces are $\gamma_{\rm ss}$ (substrate–sheet), $\gamma_{\rm sv}^{\rm (sub)}$ (substrate–vapour) and $\gamma_{\rm sv}^{\rm (sheet)}$ (sheet–vapour). For the unbuckled state (see figure 1) the total energy is

$$E^{(0)} = \gamma_{\rm ss} l + \gamma_{\rm sv}^{\rm (sub)}(l_{\rm sub} - l), \tag{1.1}$$

where l and $l_{\rm sub}$ are the lengths of the sheet and substrate, respectively.

We compare this energy with that of a sheet whose ends are first confined by bringing them a distance Δl together. In this case both l and $l_{\rm sub}$ remain constant but sheet–vapour and substrate–vapour surfaces of length l_b are created, and a substrate–sheet surface over the same distance is lost. The total post-buckling surface energy is therefore written as

$$E^{(1)} = \gamma_{\rm ss}(l - l_b) + \gamma_{\rm sv}^{\rm (sheet)} l_b + \gamma_{\rm sv}^{\rm (sub)} \left[l_{\rm sub} - (l - l_b)\right] \tag{1.2}$$

We take $E^{(0)}$ as the zero energy state, so that the change in surface energy for the buckled configuration is:

$$\Delta E = E^{(1)} - E^{(0)} = -l_b(\gamma_{\rm ss} - \gamma_{\rm sv}^{\rm (sheet)} - \gamma_{\rm sv}^{\rm (sub)}) \equiv l_b \Delta\gamma, \tag{1.3}$$

where $\Delta\gamma = \gamma_{\rm sv}^{\rm (sheet)} + \gamma_{\rm sv}^{\rm (sub)} - \gamma_{\rm ss}$, which we expect to be positive since we anticipate that deadhering costs energy.

For the case of a *compressible* substrate, in which the ends of the sheet remain fixed with respect to the substrate, it is important to note that an amount of substrate–vapour surface of length Δl is lost in compression. In this case, the energy change from flat to buckled states is given by

$$\Delta E = l_b \Delta\gamma - \Delta l \gamma_{\mathrm{sv}}^{(\mathrm{sub})}. \tag{1.4}$$

For the approach presented in the main text to be appropriate, therefore, we must have that $\Delta l / l_b \ll \Delta\gamma/\gamma_{\mathrm{sv}}^{(\mathrm{sub})}$. We anticipate that for adhesion dominated situations, we should have $\gamma_{\mathrm{ss}} \gg \gamma_{\mathrm{sv}}^{(\mathrm{sub})}, \gamma_{\mathrm{sv}}^{(\mathrm{sheet})}$ and hence this requirement is satisfied whilst also allowing $\Delta l / l_b = O(1)$. In this scenario, it is therefore legitimate to take $\Delta E \simeq l_b \Delta\gamma \simeq l_b \gamma_{\mathrm{ss}}$.

2. The relative importance of Gravity and Adhesion

The heavy elastica equation may be written as (Vella *et al.*, 2009)

$$B\theta_{ss} = -T\sin\theta + \rho g h (l_b/2 - s)\cos\theta. \tag{2.1}$$

We non-dimensionalize lengths by $\ell_{ec} = (B/\Delta\gamma)^{1/2}$ (i.e. $L = l_b/\ell_{ec},\ S = s/\ell_{ec}$) and rewrite (2.1) as

$$\theta_{SS} = -\tau\sin\theta + \frac{\rho g h l_b}{\Delta\gamma}\left(\tfrac{1}{2} - S/L\right)\cos\theta, \tag{2.2}$$

where $\tau = T/\Delta\gamma$. We further define the parameter ϵ as

$$\epsilon \equiv \rho g h l_b / \Delta\gamma \tag{2.3}$$

ϵ is thus a measure for the relative strengths of gravitation and adhesion. In systems where $\epsilon \ll 1$, the last term in (2.1) may be neglected so that we recover the elastica equation

$$\theta_{SS} = -\tau \sin\theta, \tag{2.4}$$

which corresponds to (2) in the main text.

The values for ϵ in the experiments range from $\epsilon_1 \approx 0.03$ (for the adhesive tape, $\rho_1 \approx 650$ kg/m^3, $\Delta\gamma_1 = 0.25$, $l_{b1} = 2.3$ cm, $h_1 = 52$ μm) to $\epsilon_4 \approx 0.15$ (for non-adhesive tape, $\rho_4 \approx 600$ kg/m^3, $\Delta\gamma_4 = 0.13$, $l_{b4} = 5.0$ cm, $h_4 = 70$ μm) and thus largely satisfy the requirement $\epsilon \ll 1$. [It is clear that the finite weight of the sheet does have a small effect, since the loss of symmetry for highly compressed states (bottom right of figure 2 in the main text) cannot be explained without additional terms in the elastica equation.] For flexible electronic applications ϵ will be significantly smaller than 1.

3. The elastocapillary curvature condition

In this section we present a derivation of the boundary condition (3) of the main paper. This derivation is based on the approach taken by Majidi (2007).

We write the total free energy, (1) of the main manuscript, as:

$$U = \int_{-l_b/2}^{l_b/2} \left(\tfrac{1}{2}B\theta_s^2 + \Delta\gamma\right) \mathrm{d}s - \alpha\left[l_b - \Delta l - \int_{-l_b/2}^{l_b/2} \cos\theta \ \mathrm{d}s\right].$$

As in the main text, this energy can be conveniently non-dimensionalized by taking

$$a = \alpha/\Delta\gamma, \quad \mathcal{U} = U/l_b\Delta\gamma, \quad S = s/\ell_{ec}, \quad L = l_b/\ell_{ec}. \tag{3.1}$$

We further require symmetry around 0, such that $\int_{-L/2}^{L/2} \mathrm{d}S \to 2\int_0^{L/2} \mathrm{d}S$. The energy then takes the dimensionless form

$$\mathcal{U} = 2\int_0^{L/2} \left(\tfrac{1}{2}\theta_S^2 + 1\right) \mathrm{d}S + a\Big[L - \Delta L - 2\int_0^{L/2} \cos\theta \; \mathrm{d}S.\Big] \tag{3.2}$$

We are interested in finding the shape θ, blister size L and Lagrange multiplier a that minimize the total energy $\mathcal{U}$. By letting

$$\theta \to \theta + \delta\theta, \quad L \to L + \delta L, \quad a \to a + \delta a, \tag{3.3}$$

we find, at first order that

$$\delta\mathcal{U} = 2\left[\delta\theta \; \theta_S\right]_0^{(L+\delta L)/2} + 2\int_0^{L/2} \delta\theta\Big\{ -\theta_{SS} + a\sin\theta\Big\} \; \mathrm{d}S \tag{3.4}$$
$$+ \delta a\Big\{L - \Delta L - 2\int_0^{L/2} \cos\theta \; \mathrm{d}S\Big\} + \delta L \; a + 2\int_{L/2}^{(L+\delta L)/2} \left(\tfrac{1}{2}\,\theta_S^2 + 1 - a\cos\theta\right) \mathrm{d}S.$$

The first term reduces to $2\left[\delta\theta \; \theta_S\right]_0^{(L+\delta L)/2} = -\delta L \; \theta_S(L/2)^2$, giving after simplifying:

$$\delta\mathcal{U} = 2\int_0^{L/2} \delta\theta\left\{ -\theta_{SS} + a\sin\theta\right\} \mathrm{d}S + \delta a\left\{L - \Delta L - 2\int_0^{L/2} \cos\theta\right\} \mathrm{d}S$$
$$+ \delta L\left\{1 - \frac{1}{2}\theta_S(L/2)^2\right\}. \tag{3.5}$$

For an extremum of $\mathcal{U}$ we require that $\delta\mathcal{U}/\delta\theta = \delta\mathcal{U}/\delta a = \delta\mathcal{U}/\delta L = 0$. The first of these results in the classical *Elastica equation*

$$\theta_{SS} = a\sin\theta. \tag{3.6}$$

From this we will define the dimensionless stress in the sheet as $\tau = -a$ (see main text). The second term presents the *inextensibility constraint*

$$L - \Delta L = \int_{-L/2}^{L/2} \cos\theta \; \mathrm{d}S, \tag{3.7}$$

whereas the δL variation yields the *delamination boundary condition*

$$\theta_S(L/2) = \sqrt{2}, \tag{3.8}$$

which is (3) of the main text.

4. Asymptotic Considerations

We are interested in the behaviour of blister heights and widths in the small compression limit, $\Delta L \ll 1$. We proceed by first finding expressions in terms of θ_0 up to second order. Subsequently, the relations $\delta(\Delta L)$, $\lambda(\Delta L)$ are established by deriving the asymptotic dependence $\theta_0(\Delta L)$.

(a) Dimensions of Blister, as Function of Maximum Angle θ_0

The height of the blister is given by

$$\delta = \int_{-L/2}^{0} \sin\theta \; \mathrm{d}S = 2^{3/2}(1-\cos\theta_0), \tag{4.1}$$

which may be expanded to give

$$\delta(\theta_0) \simeq \sqrt{2}\theta_0^2 - \theta_0^4/6\sqrt{2}. \tag{4.2}$$

The width of the blister is given by

$$\lambda = 2\int_0^{L/2} \cos\theta \; \mathrm{d}S. \tag{4.3}$$

We take $\mathrm{d}S = \mathrm{d}\theta/\theta_S$ and make use of (5) to rewrite this as

$$\lambda = 2^{3/2}(1-\cos\theta_0)^{1/2} \underbrace{\int_0^{\theta_0} \frac{\cos\theta}{\sqrt{\cos\theta - \cos\theta_0}}\mathrm{d}\theta}_{I(\theta_0)} . \tag{4.4}$$

Substituting $u = \sin\theta/\sin\theta_0$, $\mathrm{d}u = \mathrm{d}\theta(\sin\theta_0/(1-u^2\sin^2\theta_0)^{1/2})$ and approximating

$$(1-u^2\sin^2\theta_0)^{1/2} \simeq 1 - \frac{1}{2}u^2\sin^2\theta_0 - \frac{1}{8}u^4\sin^4\theta_0, \tag{4.5}$$

$$\cos\theta_0 \simeq 1 - \frac{1}{2}\sin^2\theta_0 - \frac{1}{8}\sin^4\theta_0, \tag{4.6}$$

results in the following integral:

$$I(\theta_0) \simeq \sqrt{2}\int_0^1 \frac{1}{\sqrt{1-u^2}\sqrt{1+\left(\frac{\sin\theta_0}{2}\right)^2(1+u^2)}}\,\mathrm{d}u. \tag{4.7}$$

Approximating $1/[1+(\frac{\sin\theta_0}{2})^2(1+u^2)]^{1/2} \simeq 1 - \frac{1}{2}(\frac{\sin\theta_0}{2})^2(1+u^2)$ and integrating gives $I(\theta_0) = (\pi/\sqrt{2}) - (3\pi/16\sqrt{2})\sin^2\theta_0$. Expanding $\sin^2\theta_0$ around 0 yields:

$$I(\theta_0) = \frac{\pi}{\sqrt{2}} - \frac{3\pi}{16\sqrt{2}}\theta_0^2 + \ldots \, . \tag{4.8}$$

Substituting this result into (4.4) gives the final answer for the blister width:

$$\lambda(\theta_0) = \sqrt{2}\pi\theta_0 - \frac{11\pi}{24\sqrt{2}}\theta_0^3 + \ldots \, . \tag{4.9}$$

(b) Dimensions of Blister as Functions of Compression ΔL

The compression is known to be related to the maximum angle by

$$\Delta L = 4\int_0^{L/4}(1-\cos\theta)\mathrm{d}S = 2^{3/2}(1-\cos\theta_0)^{1/2}\int_0^{\theta_0}\frac{1-\cos\theta}{\sqrt{\cos\theta-\cos\theta_0}}\mathrm{d}\theta, \tag{4.10}$$

where we have again made use of $\mathrm{d}S = \mathrm{d}\theta/\theta_S$. This can be shown to integrate to

$$\Delta L = 2^{5/2}(1-\cos\theta_0)\left[F\left(\frac{\theta_0}{2}, \csc^2\frac{\theta_0}{2}\right) - E\left(\frac{\theta_0}{2}, \csc^2\frac{\theta_0}{2}\right)\right], \tag{4.11}$$

where $F(...)$ and $E(...)$ are the Incomplete Elliptic Integrals of the First and Second kind, respectively, which are defined as

$$F(\phi, q) = \int_0^{\phi} \left(1 - q\sin^2\theta\right)^{-1/2} \mathrm{d}\theta,$$
$$E(\phi, q) = \int_0^{\phi} \left(1 - q\sin^2\theta\right)^{1/2} \mathrm{d}\theta.$$

Using procedures analogous to the one in the previous section we find the following asymptotic approximations for the elliptical integrals:

$$F\left(\frac{\theta_0}{2}, \csc^2\frac{\theta_0}{2}\right) \simeq \frac{\pi}{4}\left(\theta_0 + \frac{\theta_0^3}{48} + \frac{23}{15360}\theta_0^5 + ...\right), \tag{4.12}$$

$$E\left(\frac{\theta_0}{2}, \csc^2\frac{\theta_0}{2}\right) \simeq \frac{\pi}{8}\left(\theta_0 - \frac{\theta_0^3}{96} - \frac{7}{15360}\theta_0^5 - ...\right). \tag{4.13}$$

Substituting this into (4.11) and simplifying gives to second order:

$$\Delta L \simeq \frac{\pi}{2\sqrt{2}}\left(\theta_0^3 - \frac{1}{32}\theta_0^5\right). \tag{4.14}$$

Being interested in the dependence $\theta_0 = \theta_0(\Delta L)$, we assume a power series of the form:

$$\theta_0 = \left(\frac{2\sqrt{2}}{\pi}\Delta L\right)^{1/3} + \alpha_2 \Delta L^{2/3} + \alpha_3 \Delta L + ... , \tag{4.15}$$

and choose the coefficients α_2, α_3 to satisfy (4.14). We find $\alpha_2 = 0, \alpha_3 = 1/24\pi$ to give

$$\theta_0(\Delta L) \simeq \sqrt{2}\left(\frac{\Delta L}{\pi}\right)^{1/3} + \frac{1}{24\sqrt{2}}\left(\frac{\Delta L}{\pi}\right). \tag{4.16}$$

Substituting this into the asymptotic expressions for $\lambda(\theta_0)$ and $\delta(\theta_0)$ produces the following results (see (9) and (10)):

$$\delta = 2\sqrt{2}\left(\frac{\Delta L}{\pi}\right)^{2/3} - \frac{1}{2\sqrt{2}}\left(\frac{\Delta L}{\pi}\right)^{4/3} + ... , \tag{4.17}$$

$$\lambda = 2\pi^{2/3}\Delta L^{1/3} - \frac{7}{8}\Delta L + ... \tag{4.18}$$

(c) *Typical Curvature vs Aspect Ratio*

We are further interested in the typical curvature as a function of the aspect ratio. To this end we write, from (4.2) and (4.9):

$$\frac{\delta}{\lambda} = \frac{\theta_0}{\pi} + \frac{7}{48\pi}\theta_0^3 + ... \tag{4.19}$$

and the typical curvature is written as

$$\frac{\delta}{\lambda^2} = \frac{1}{\sqrt{2}\pi^2} + \frac{3}{8\sqrt{2}\pi^2}\theta_0^2 + ... \tag{4.20}$$

Analogously to above we pose the power series expansion

$$\theta_0 = \pi\delta/\lambda + \alpha_2(\delta/\lambda)^2 + \alpha_3(\delta/\lambda)^3 + ... \tag{4.21}$$

and choose the coefficients α_2, α_3 etc. to satisfy (4.19). We find that $\alpha_2 = 0$, $\alpha_3 = -(7/48)\pi^3$. Substituting (4.21) into (4.20) finally gives (11)

$$\frac{\delta}{\lambda^2} = \frac{1}{\sqrt{2}\pi^2} + \frac{3}{8\sqrt{2}}\left(\frac{\delta}{\lambda}\right)^2 - ...$$

References

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